6.013 - Electromagnetics and Applications Fall 2005

Lecture 15 - Dielectric Waveguides

Prof. Markus Zahn November 3, 2005

I. TM Solutions

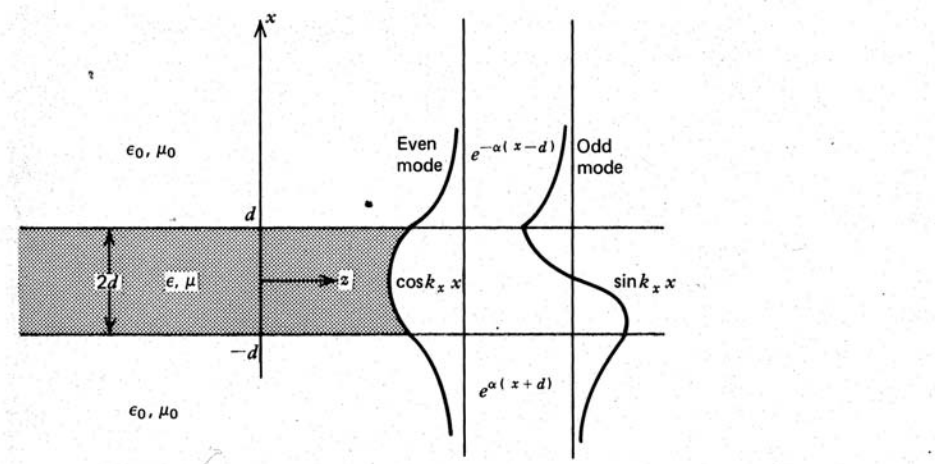

Figure 8-30 TE and TM modes can also propagate along dielectric structures. The fields can be essentially confined to the dielectric over a frequency range if the speed of the wave in the dielectric is less than that outside. It is convenient to separate the solutions into even and odd modes.

We are considering solutions with

$$\frac{\partial}{\partial y} = 0$$

$$E_z(x,z,t) = \begin{cases} \text{Re}\left[A_2 e^{-\alpha(x-d)} e^{j(\omega t - k_z z)}\right] & x \geq d \\ \text{Re}\left[\left(A_1 \sin(k_x x) + B_1 \cos(k_x x)\right) e^{j(\omega t - k_z z)}\right] & |x| \leq d \\ \text{Re}\left[A_3 e^{+\alpha(x+d)} e^{j(\omega t - k_z z)}\right] & x \leq -d \end{cases}$$

$$k_x^2 + k_z^2 = \omega^2 \epsilon \mu \qquad \text{(in dielectric)}$$
$$-\alpha^2 + k_z^2 = \omega^2 \epsilon_0 \mu_0 \qquad \text{(in free space)}$$

For propagation in the dielectric and evanescence in free space

$$k_z^2 = \omega^2 \epsilon \mu - k_x^2 = \omega^2 \epsilon_0 \mu_0 + \alpha^2$$
$$k_z^2 < \omega^2 \epsilon \mu, k_z^2 > \omega^2 \epsilon_0 \mu_0 \Rightarrow \omega^2 \epsilon_0 \mu_0 < k_z^2 < \omega^2 \epsilon \mu$$

A. Odd Solutions $[E_z(x,z,t) = -E_z(-x,z,t)]$

$$\hat{E}_z(x) = \begin{cases} A_2 e^{-\alpha(x-d)} & x \geq d \\ A_1 \sin(k_x x) & |x| \leq d \\ -A_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\nabla \cdot \bar{E} \Rightarrow \frac{\partial \hat{E}_x}{\partial x} - jk_z \hat{E}_z = 0$$

$$\Rightarrow \hat{E}_x = \begin{cases} -\frac{jk_z}{\alpha} A_2 e^{-\alpha(x-d)} & x > d \\ -\frac{jk_z}{k_x} A_1 \cos(k_x x) & |x| < d \\ -\frac{jk_z}{\alpha} A_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \hat{H}_x = 0, \hat{H}_z = 0$$

$$\hat{H}_y = -\frac{1}{j\omega\mu}\left(-jk_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x}\right)$$

$$= \begin{cases} -\frac{j\omega\epsilon_0}{\alpha} A_2 e^{-\alpha(x-d)} & x \geq d \\ -\frac{j\omega\epsilon}{k_x} A_1 \cos(k_x x) & |x| \leq d \\ -\frac{j\omega\epsilon_0}{\alpha} A_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

Boundary Conditions

$$E_z(x = d_+) = E_z(x = d_-) \Rightarrow A_1 \sin(k_x d) = A_2$$

$$H_y(x = d_+) = H_y(x = d_-) \Rightarrow +\frac{\cancel{j\omega}\epsilon_0}{\alpha} A_2 = +\frac{\cancel{j\omega}\epsilon}{k_x} A_1 \cos(k_x d)$$

$$\frac{A_1}{A_2} = \frac{1}{\sin(k_x d)} = \frac{k_x \epsilon_0}{\alpha \epsilon \cos(k_x d)} \Rightarrow \alpha = \frac{\epsilon_0 k_x}{\epsilon} \tan(k_x d)$$

Critical condition for a guided wave occurs when $\alpha = 0$. At this point

$$k_x d = n\pi, k_z^2 = \omega^2 \epsilon_0 \mu_0$$

$$\left(\frac{n\pi}{d}\right)^2 + \omega^2 \epsilon_0 \mu_0 = \omega^2 \epsilon \mu \Rightarrow \omega^2 = \frac{(n\pi/d)^2}{\epsilon\mu - \epsilon_0\mu_0}, n = 1, 2, 3, \ldots$$

For real frequencies $(\omega^2 > 0), \epsilon\mu > \epsilon_0\mu_0$

B. Even Solutions $[E_z(x,z,t) = +E_z(-x,z,t)]$

$$\hat{E}_z(x) = \begin{cases} B_2 e^{-\alpha(x-d)} & x \geq d \\ B_1 \cos(k_x x) & |x| \leq d \\ B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\frac{\partial \hat{E}_x}{\partial x} - jk_z \hat{E}_z = 0$$

$$\Rightarrow \hat{E}_x = \begin{cases} -\frac{jk_z}{\alpha} B_2 e^{-\alpha(x-d)} & x > d \\ \frac{jk_z}{k_x} B_1 \sin(k_x x) & |x| < d \\ \frac{jk_z}{\alpha} B_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\hat{H}_y = \begin{cases} -\frac{j\omega\epsilon_0}{\alpha} B_2 e^{-\alpha(x-d)} & x \geq d \\ \frac{j\omega\epsilon}{k_x} B_1 \sin(k_x x) & |x| \leq d \\ \frac{j\omega\epsilon_0}{\alpha} B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

Boundary Conditions

$$E_z(x = d_+) = E_z(x = d_-) \Rightarrow B_2 = B_1 \cos(k_x d)$$

$$H_y(x = d_+) = H_y(x = d_-) \Rightarrow -\frac{j\omega\epsilon_0}{\alpha} B_2 = \frac{j\omega\epsilon}{k_x} B_1 \sin(k_x d)$$

$$\frac{B_2}{B_1} = \cos(k_x d) = -\frac{\epsilon\alpha}{\epsilon_0 k_x} \sin(k_x d) \Rightarrow \alpha = -\frac{\epsilon_0 k_x}{\epsilon} \cot(k_x d)$$

$$\text{Critical Condition: } \alpha = 0 \Rightarrow k_x d = (2n+1)\frac{\pi}{2}, k_z^2 = \omega^2 \epsilon_0 \mu_0$$

$$\omega^2 = \frac{\left[\frac{(2n+1)\pi}{2d}\right]^2}{\epsilon\mu - \epsilon_0\mu_0} \qquad n = 0, 1, 2, \ldots$$

II. TE Solutions

A. Odd Solutions

$$\hat{H}_z = \begin{cases} A_2 e^{-\alpha(x-d)} & x \geq d \\ A_1 \sin(k_x x) & |x| \leq d \\ -A_2 e^{\alpha(x-d)} & x \leq -d \end{cases}$$

$$\nabla \cdot \bar{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_z}{\partial z} \Rightarrow \frac{\partial \hat{H}_x}{\partial x} - jk_z \hat{H}_z = 0$$

$$\hat{H}_x = \begin{cases} -\frac{jk_z}{\alpha} A_2 e^{-\alpha(x-d)} & x > d \\ -\frac{jk_z}{k_x} A_1 \cos(k_x x) & |x| < d \\ -\frac{jk_z}{\alpha} A_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \hat{E}_x = 0, \hat{E}_z = 0$$

$$j\omega\epsilon \hat{E}_y = -jk_z \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x}$$

$$\hat{E}_y = \begin{cases} \frac{j\omega\mu_0}{\alpha} A_2 e^{-\alpha(x-d)} & x \geq d \\ \frac{j\omega\mu}{k_x} A_1 \cos(k_x x) & |x| \leq d \\ \frac{j\omega\mu_0}{\alpha} A_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

Boundary Conditions

$$\hat{E}_y(x = d_+) = \hat{E}_y(x = d_-) \Rightarrow \frac{\cancel{j\omega}\mu_0}{\alpha} A_2 = \frac{\cancel{j\omega}\mu}{k_x} A_1 \cos(k_x d)$$

$$\hat{H}_z(x = d_+) = \hat{H}_z(x = d_-) \Rightarrow A_2 = A_1 \sin(k_x d)$$

$$\frac{A_2}{A_1} = \sin(k_x d) = \frac{\mu\alpha}{\mu_0 k_x} \cos(k_x d) \Rightarrow \alpha = \frac{\mu_0 k_x}{\mu} \tan(k_x d)$$

B. Even Solutions

$$\hat{H}_z = \begin{cases} B_2 e^{-\alpha(x-d)} & x \geq d \\ B_1 \cos(k_x x) & |x| \leq d \\ B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\hat{H}_x = \begin{cases} -\frac{jk_z}{\alpha} B_2 e^{-\alpha(x-d)} & x > d \\ \frac{jk_z}{k_x} B_1 \sin(k_x x) & |x| < d \\ \frac{jk_z}{\alpha} B_2 e^{\alpha(x+d)} & x < -d \end{cases}$$

$$\hat{E}_y = \begin{cases} \frac{j\omega\mu_0}{\alpha} B_2 e^{-\alpha(x-d)} & x \geq d \\ -\frac{j\omega\mu}{k_x} B_1 \sin(k_x x) & |x| \leq d \\ -\frac{j\omega\mu_0}{\alpha} B_2 e^{\alpha(x+d)} & x \leq -d \end{cases}$$

$$\hat{E}_y(x = d_+) = \hat{E}_y(x = d_-) \Rightarrow \frac{\cancel{j\omega}\mu_0}{\alpha} B_2 = -\frac{\cancel{j\omega}\mu}{k_x} B_1 \sin(k_x d)$$

$$\hat{H}_z(x = d_+) = \hat{H}_z(x = d_-) \Rightarrow B_2 = B_1 \cos(k_x d)$$

$$\frac{B_2}{B_1} = \cos(k_x d) = -\frac{\alpha\mu}{\mu_0 k_x} \sin(k_x d) \Rightarrow \alpha = -\frac{\mu_0 k_x}{\mu} \cot(k_x d)$$