6.013 - Electromagnetics and Applications	Fall 2005
Lecture 12+13 - Transient Waves on Transmission Lines	
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I. Wave equation (Lossless)

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} \qquad \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial z^2} \qquad c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\epsilon\mu}}$$

Solution: $v(z,t) = V_+\left(t - \frac{z}{c}\right) + V_-\left(t + \frac{z}{c}\right)$

Proof: Let $\alpha = t - \frac{z}{c} \Rightarrow \frac{\partial \alpha}{\partial t} = 1, \frac{\partial \alpha}{\partial z} = -\frac{1}{c}$

$$\text{Superposition: } v_+(z,t) = V_+\left(t - \frac{z}{c}\right)$$

$$\frac{\partial v_+}{\partial t} = \frac{dv_+}{d\alpha}\frac{\partial \alpha}{\partial t} = \frac{dv_+}{d\alpha}$$

$$\frac{\partial^2 v_+}{\partial t^2} = \frac{d^2 v_+}{d\alpha^2}\frac{\partial \alpha}{\partial t} = \frac{d^2 v_+}{d\alpha^2}$$

$$\frac{\partial v_+}{\partial z} = \frac{dv_+}{d\alpha}\frac{\partial \alpha}{\partial z} = -\frac{1}{c}\frac{dv_+}{d\alpha}$$

$$\frac{\partial^2 v_+}{\partial z^2} = -\frac{1}{c}\frac{d^2 v_+}{d\alpha^2}\frac{\partial \alpha}{\partial z} = \frac{1}{c^2}\frac{d^2 v_+}{d\alpha^2}$$

$$\frac{\partial^2 v_+}{\partial t^2} = \frac{d^2 v_+}{d\alpha^2} = c^2\frac{\partial^2 v}{\partial z^2} = c^2\left(\frac{1}{c^2}\frac{d^2 v_+}{d\alpha^2}\right) = \frac{d^2 v_+}{d\alpha^2}$$

Negative z directed waves: Let $\beta = t + \frac{z}{c} \Rightarrow \frac{\partial \beta}{\partial t} = 1, \frac{\partial \beta}{\partial z} = \frac{1}{c}$

$$\frac{\partial v_-}{\partial t} = \frac{dv_-}{d\beta}\frac{\partial \beta}{\partial t} = \frac{dv_-}{d\beta}$$

$$\frac{\partial^2 v_-}{\partial t^2} = \frac{d^2 v_-}{d\beta^2}\frac{\partial \beta}{\partial t} = \frac{d^2 v_-}{d\beta^2}$$

$$\frac{\partial v_-}{\partial z} = \frac{dv_-}{d\beta}\frac{\partial \beta}{\partial z} = \frac{1}{c}\frac{dv_-}{d\beta}$$

$$\frac{\partial^2 v_-}{\partial z^2} = \frac{1}{c}\frac{d^2 v_-}{d\beta^2}\frac{\partial \beta}{\partial z} = \frac{1}{c^2}\frac{d^2 v_-}{d\beta^2}$$

$$\frac{\partial^2 v_-}{\partial t^2} = \frac{d^2 v_-}{d\beta^2} = c^2\frac{\partial^2 v_-}{\partial z^2} = c^2\left(\frac{1}{c^2}\frac{d^2 v_-}{d\beta^2}\right) = \frac{d^2 v_-}{d\beta^2}$$

II. Solution for current $i(z,t)$

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t} \qquad \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} \quad \Rightarrow \quad \frac{\partial^2 i}{\partial t^2} = c^2\frac{\partial^2 i}{\partial z^2}$$

Solution: $i(z,t) = I_+\left(t - \frac{z}{c}\right) + I_-\left(t + \frac{z}{c}\right)$

$v(z,t) = V_+\left(t - \frac{z}{c}\right) + V_-\left(t + \frac{z}{c}\right)$

$+z$ solution: $\alpha = t - \frac{z}{c}$, $\frac{\partial\alpha}{\partial t} = 1$, $\frac{\partial\alpha}{\partial z} = -\frac{1}{c}$

$$\begin{aligned}\frac{\partial v_+}{\partial z} = -L\frac{\partial i_+}{\partial t} \Rightarrow \frac{dv_+}{d\alpha}\frac{\partial\alpha}{\partial z} &= -\frac{1}{c}\frac{dv_+}{d\alpha} \\ &= -L\frac{di_+}{d\alpha}\frac{\partial\alpha}{\partial t} \\ &= -L\frac{di_+}{d\alpha}\end{aligned}$$

$$\frac{dv_+}{d\alpha} = Lc\frac{di_+}{d\alpha} = \frac{L}{\sqrt{LC}}\frac{di_+}{d\alpha} = \sqrt{\frac{L}{C}}\frac{di_+}{d\alpha} = Z_0\frac{di_+}{d\alpha}$$

$$v_+ = i_+Z_0 \Rightarrow I_+\left(t - \frac{z}{c}\right) = Y_0V_+\left(t - \frac{z}{c}\right)$$

$$Y_0 = \sqrt{\frac{C}{L}} = \frac{1}{Z_0}$$

$-z$ solution: $\beta = t + \frac{z}{c}$, $\frac{\partial\beta}{\partial t} = 1$, $\frac{\partial\beta}{\partial z} = \frac{1}{c}$

$$\begin{aligned}\frac{\partial v_-}{\partial z} = -L\frac{\partial i_-}{\partial t} \Rightarrow \frac{dv_-}{d\beta}\frac{\partial\beta}{\partial z} &= \frac{1}{c}\frac{dv_-}{d\beta} \\ &= -L\frac{di_-}{d\beta}\frac{\partial\beta}{\partial t} \\ &= -L\frac{di_-}{d\beta}\end{aligned}$$

$$\frac{dv_-}{d\beta} = -Lc\frac{di_-}{d\beta} = -\frac{L}{\sqrt{LC}}\frac{di_-}{d\beta} = -\sqrt{\frac{L}{C}}\frac{di_-}{d\beta} = -Z_0\frac{di_-}{d\beta}$$

$$v_- = -i_-Z_0 \Rightarrow I_+\left(t + \frac{z}{c}\right) = -Y_0V_-\left(t + \frac{z}{c}\right)$$

$$\begin{aligned}v(z,t) &= V_+\left(t - \frac{z}{c}\right) + V_-\left(t + \frac{z}{c}\right) \\ i(z,t) &= Y_0\left[V_+\left(t - \frac{z}{c}\right) - V_-\left(t + \frac{z}{c}\right)\right]\end{aligned}$$

III. Transmission Line Transient Waves

A. Transients on Infinitely Long Transmission Lines

1. Initial Conditions

$$v(z, t=0) = V_+\left(-\frac{z}{c}\right) + V_-\left(\frac{z}{c}\right) = 0$$

$$i(z, t=0) = Y_0\left[V_+\left(-\frac{z}{c}\right) - V_-\left(\frac{z}{c}\right)\right] = 0$$

$$V_+\left(-\frac{z}{c}\right) = 0, V_-\left(\frac{z}{c}\right) = 0$$

$$z > 0, t > 0 \Rightarrow t + \frac{z}{c} > 0 \Rightarrow V_-\left(t + \frac{z}{c}\right) = 0$$

$$t - \frac{z}{c} > 0 \text{ if } t > \frac{z}{c} \text{ to allow } V_+\left(t - \frac{z}{c}\right) \neq 0$$

$$\text{With } V_-\left(t+\frac{z}{c}\right)=0 \Rightarrow v(z,t)=V_+\left(t-\frac{z}{c}\right) \Rightarrow \frac{v(z,t)}{i(z,t)}=Z_0$$

$$i(z,t)=Y_0V_+\left(t-\frac{z}{c}\right)$$

2. Traveling Wave Solution

$$v(z=0,t)=V(t)=V_+(t)$$

$$v(z=0,t)=\frac{Z_0}{Z_0+R_S}V(t)=V_+(t)$$

$$i(z=0,t)=Y_0V_+(t)=\frac{V(t)}{R_S+Z_0}$$

$$v(z,t)=\frac{Z_0}{Z_0+R_S}V\left(t-\frac{z}{c}\right)$$

$$i(z,t)=\frac{1}{R_S+Z_0}V\left(t-\frac{z}{c}\right)$$

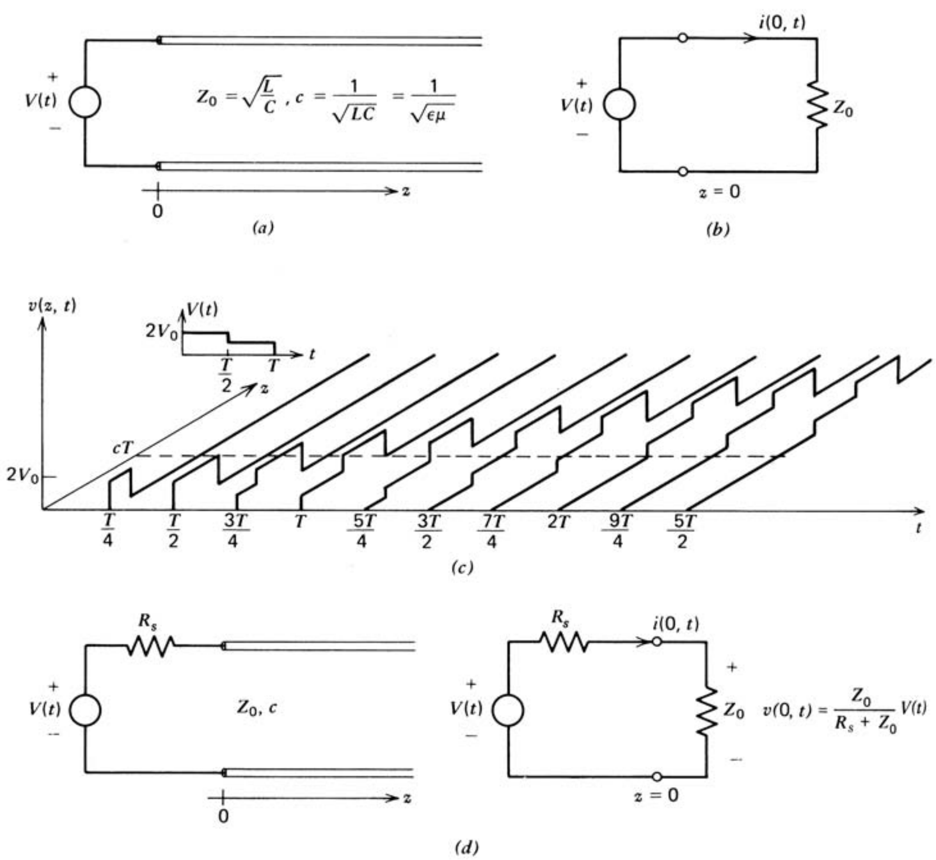
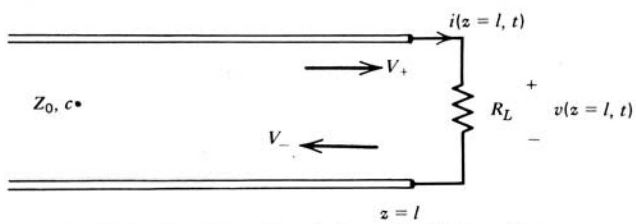


Figure 8-6 (*a*) A semi-infinite transmission line excited by a voltage source at $z=0$. (*b*) To the source, the transmission line looks like a resistor Z_0 equal to the characteristic impedance. (*c*) The spatial distribution of the voltage $v(z,t)$ at various times for a staircase pulse of $V(t)$. (*d*) If the voltage source is applied to the transmission line through a series resistance R_s, the voltage across the line at $z=0$ is given by the voltage divider relation.

B. Reflections from Resistive Terminations

1. Reflection Coefficient



$$v(z=l,t) = V_+ + V_- = i(z=l,t)R_L = R_L Y_0 [V_+ - V_-]$$

$$\Rightarrow \Gamma_L = \frac{V_-}{V_+} = \frac{R_L - Z_0}{R_L + Z_0}$$

Figure 8-7 A V_+ wave incident upon the end of a transmission line with a load resistor R_L is reflected as a V_- wave.

$$\begin{aligned}
\text{At } z = l : v(l,t) &= V_+\left(t - \frac{l}{c}\right) + V_-\left(t + \frac{l}{c}\right) \\
&= i(l,t)R_L \\
&= Y_0 R_L \left[V_+\left(t - \frac{l}{c}\right) - V_-\left(t + \frac{l}{c}\right)\right] \\
\Gamma_L &= \frac{V_-\left(t + \frac{l}{c}\right)}{V_+\left(t - \frac{l}{c}\right)} = \frac{R_L - Z_0}{R_L + Z_0}
\end{aligned}$$

Special cases:

a. $R_L = Z_0 \Rightarrow \Gamma_L = 0$ (matched line)

b. $R_L = 0 \Rightarrow \Gamma_L = -1$ (short circuited line)
If $R_L < Z_0, \Gamma_L < 0$

c. $R_L = \infty \Rightarrow \Gamma_L = +1$ (open circuited line)
If $R_L > Z_0, \Gamma_L > 0$

2. Step Voltage
At $z = 0$:

$$\begin{aligned}
&v(z=0,t) + i(0,t)R_S = V_0 \\
&V_+(z=0,t) + V_-(z=0,t) + Y_0 R_S \left[V_+(z=0,t) - V_-(z=0,t)\right] = V_0 \\
&V_+(z=0,t) = \Gamma_S V_-(z=0,t) + \frac{Z_0 V_0}{Z_0 + R_S}, \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0}
\end{aligned}$$

a. Matched Line: $R_L = Z_0, \Gamma_L = 0; R_S = Z_0, \Gamma_S = 0$
$\Gamma_L = 0 \Rightarrow V_-\left(t + \frac{z}{c}\right) = 0, V_+(z=0,t) = \frac{V_0}{2}$, in steady state after time $T = \frac{l}{c}$

b. Short circuited line: $R_L = 0, \Gamma_L = -1, R_S = Z_0, \Gamma_S = 0$
$\Gamma_L = -1 \Rightarrow V_+ = -V_-$. When $V_+\left(t - \frac{z}{c}\right)$ and $V_-\left(t + \frac{z}{c}\right)$ overlap in space, $v(z,t) = 0$. For $t \geq 2T = \frac{2l}{c}, v(z,t) = 0, i(z,t) = \frac{V_0}{Z_0}$.

c. Open circuited line: $R_L = \infty, \Gamma_L = +1, R_S = Z_0, \Gamma_S = 0$
$\Gamma_L = +1 \Rightarrow V_+ = +V_-$. For $t \geq 2T = \frac{2l}{c}, v(z,t) = V_0, i(z,t) = 0$

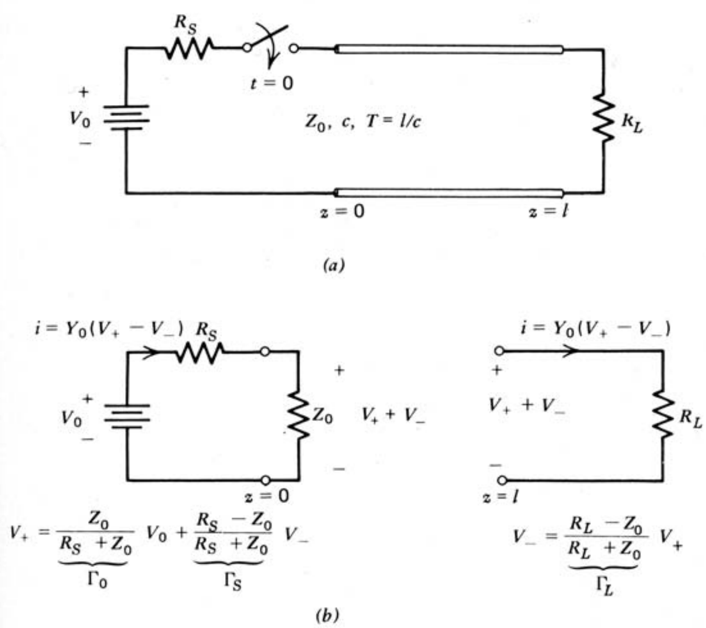


Figure 8-8 (*a*) A dc voltage V_0 is switched onto a resistively loaded transmission line through a source resistance R_s. (*b*) The equivalent circuits at $z = 0$ and $z = l$ allow us to calculate the reflected voltage wave amplitudes in terms of the incident waves.

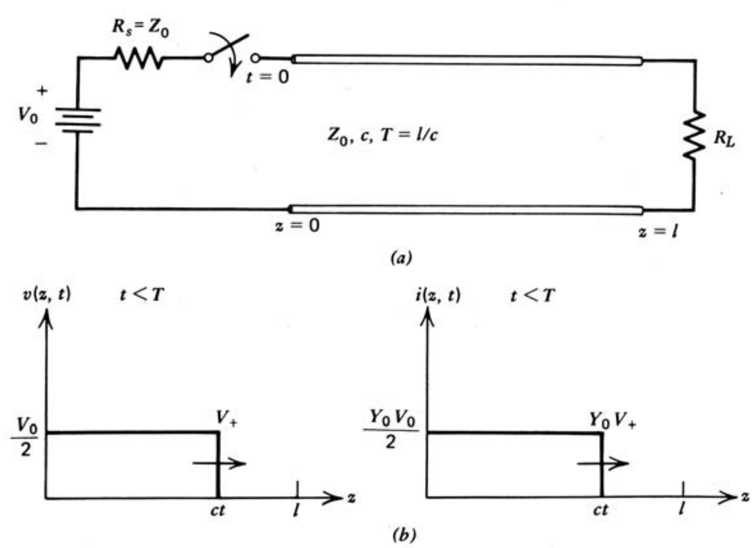


Figure 8-9 (*a*) A dc voltage is switched onto a transmission line with load resistance R_L through a source resistance R_s matched to the line. (*b*) Regardless of the load resistance, half the source voltage propagates down the line towards the load. If the load is also matched to the line ($R_L = Z_0$), there are no reflections and the steady state of $v(z, t \geq T) = V_0/2$, $i(z, t \geq T) = Y_0 V_0/2$ is reached for $t \geq T$. (*c*) If the line is short circuited ($R_L = 0$), then $\Gamma_L = -1$ so that the V_+ and V_- waves cancel for the voltage but add for the current wherever they overlap in space. Since the source end is matched, no further reflections arise at $z = 0$ so that the steady state is reached for $t \geq 2T$. (*d*) If the line is open circuited ($R_L = \infty$) so that $\Gamma_L = +1$, the V_+ and V_- waves add for the voltage but cancel for the current.

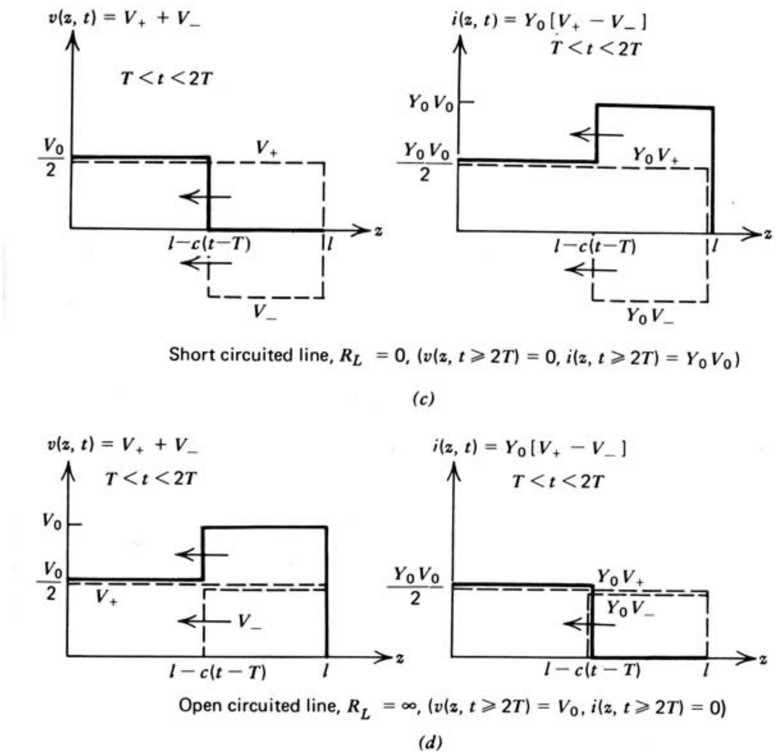


Figure 8-9

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3. Approach to the DC Steady State (neither end matched)

$$z = 0 : V_+(t) = \Gamma_0 V_0 + \Gamma_S V_-(t), \Gamma_0 = \frac{Z_0}{R_S + Z_0}, \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0}$$

$$z = l : V_-\left(t + \frac{l}{c}\right) = \Gamma_L V_+\left(t - \frac{l}{c}\right), \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

at $z = l$:

$$V_{+n} = \Gamma_0 V_0 + \Gamma_S V_{-(n-1)}$$

$$V_{-(n-1)} = \Gamma_L V_{+(n-1)}$$

$$V_{+n} = \Gamma_0 V_0 + \Gamma_S \Gamma_L V_{+(n-1)} \Rightarrow V_{+n} - \Gamma_S \Gamma_L V_{+(n-1)} = \Gamma_0 V_0$$

Particular Solution:

$$V_{+n} = \text{constant}$$

$$\text{constant}\left(1 - \Gamma_S \Gamma_L\right) = \Gamma_0 V_0$$

$$\text{constant} = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L}$$

Homogeneous Solution:

$$V_{+n} - \Gamma_S \Gamma_L V_{+(n-1)} = 0$$

Try a solution of the form: $V_{+n} = A\lambda^n$

$$A\left(\lambda^n - \Gamma_S \Gamma_L \lambda^{n-1}\right) = 0 \Rightarrow \lambda = \Gamma_S \Gamma_L$$

$$V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} + A\left(\Gamma_S \Gamma_L\right)^n$$

Initial Condition:

$$V_{+1} = \Gamma_0 V_0 = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} + A(\Gamma_S \Gamma_L) \Rightarrow A = -\frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L}$$

$$V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L}\left[1 - (\Gamma_S \Gamma_L)^n\right]$$

$$V_{-(n-1)} = \Gamma_L V_{+(n-1)} \Rightarrow V_{-n} = \Gamma_L V_{+n}$$

$$V_n = V_{+n} + V_{-n} = V_{+n}(1 + \Gamma_L) = \frac{V_0(1 + \Gamma_L)\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L}\left[1 - (\Gamma_S \Gamma_L)^n)\right]$$

$$= \frac{R_L}{R_L + R_S} V_0 \left[1 - (\Gamma_S \Gamma_L)^n\right]$$

$$\lim_{n \to \infty} V_n = \frac{R_L}{R_L + R_S} V_0$$

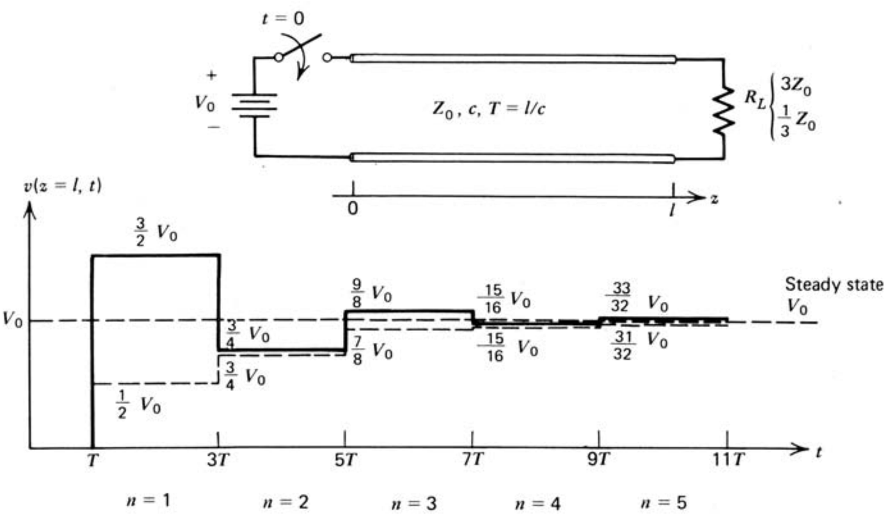

Figure 8-10 The load voltage as a function of time when $R_s = 0$ and $R_L = 3Z_0$ so that $\Gamma_s \Gamma_L = -\frac{1}{2}$ (solid) and with $R_L = \frac{1}{3} Z_0$ so that $\Gamma_s \Gamma_L = \frac{1}{2}$ (dashed). The dc steady state is the same as if the transmission line were considered a pair of perfectly conducting wires in a circuit.

$$I_n = Y_0\left[V_{+n} - V_{-n}\right] = Y_0(1 - \Gamma_L)V_{+n} = \frac{Y_0(1 - \Gamma_L)\Gamma_0 V_0\left[1 - (\Gamma_S \Gamma_L)^n\right]}{1 - \Gamma_S \Gamma_L}$$

$$= \frac{V_0\left[1 - (\Gamma_S \Gamma_L)^n\right]}{R_L + R_S}$$

a. Special Case: $R_S = 0, R_L = 3Z_0$.
$\Gamma_S = -1, \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{2}{4} = \frac{1}{2} \Rightarrow \Gamma_S \Gamma_L = -\frac{1}{2}$

$$z = l \qquad V_n = V_0\left[1 - \left(-\frac{1}{2}\right)^n\right]$$

$$I_n = \frac{V_0}{3Z_0}\left[1 - \left(-\frac{1}{2}\right)^n\right]$$

b. Special Case: $R_S = 0, R_L = \frac{1}{3} Z_0$

$\Gamma_S = -1, \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{-\frac{2}{3}}{\frac{4}{3}} = -\frac{1}{2} \Rightarrow \Gamma_S\Gamma_L = +\frac{1}{2}$

$$z = l \qquad\qquad V_n = V_0\left[1 - \left(\frac{1}{2}\right)^n\right]$$

c. Special Case: $R_S = 0, R_L = \infty$ (open circuit)

$$\begin{aligned} \Gamma_S\Gamma_L &= -1 \\ V_n = \frac{R_L}{R_S + R_L} V_0\left[1 - (\Gamma_S\Gamma_L)^n\right] &= V_0\left(1 - (-1)^n\right) \\ &= \begin{cases} 0 & \text{n even} \\ 2V_0 & \text{n odd} \end{cases} \end{aligned}$$

d. Special Case: $R_S = 0, R_L = 0$ (short circuit)

$$\begin{aligned} \Gamma_S\Gamma_L &= +1 \\ I_n &= \frac{V_0}{R_L + R_S}\left[1 - (\Gamma_S\Gamma_L)^n\right] \quad \text{Indeterminate} \\ \Gamma_S &= \frac{R_S - Z_0}{R_S + Z_0} = \frac{\frac{R_S}{Z_0} - 1}{\frac{R_S}{Z_0} + 1} \approx -\left(1 - \frac{R_S}{Z_0}\right)^2 \approx -\left(1 - 2\frac{R_S}{Z_0}\right) \\ \Gamma_L &= \frac{R_L - Z_0}{R_L + Z_0} \approx -\left(1 - \frac{2R_L}{Z_0}\right) \\ I_N &= \frac{V_0}{R_L + R_S}\left[1 - \left[\left(1 - \frac{2R_L}{Z_0}\right)\left(1 - \frac{2R_S}{Z_0}\right)\right]^n\right] \\ &\approx \frac{V_0}{R_L + R_S}\left[1 - \left(1 - \frac{2(R_L + R_S)}{Z_0}\right)^n\right] \\ &\approx \frac{V_0}{R_L + R_S}\left[1 - 1 + \frac{2n(R_L + R_S)}{Z_0}\right] \\ &\approx \frac{V_0 \cdot 2n}{Z_0} \end{aligned}$$

This approximates an inductor: $V_0 = (Ll)\frac{di}{dt} \Rightarrow i = \frac{V_0}{Ll}t$

e. Special Case: $R_L = \infty$ (open circuit)
$\Gamma_L = 1 \Rightarrow V_n = V_0\left[1 - \Gamma_S^n\right]$
This approximates the transmission line as a capacitor being charged through the resistor R_S:

$$\begin{aligned} v(t) &= V_0\left(1 - e^{-t/\tau}\right) \\ \tau &= R_S C l \end{aligned}$$

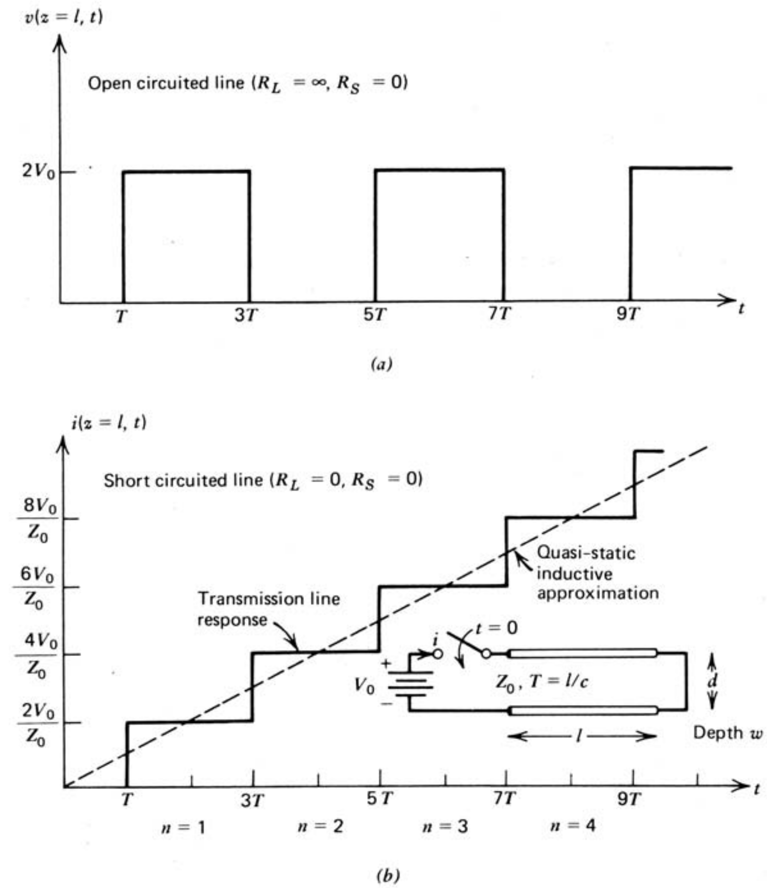

Figure 8-11 The (*a*) open circuit voltage and (*b*) short circuit current at the $z = l$ end of the transmission line for $R_s = 0$. No dc steady state is reached because the system is lossless. If the short circuited transmission line is modeled as an inductor in the quasi-static limit, a step voltage input results in a linearly increasing current (shown dashed). The exact transmission line response is the solid staircase waveform.

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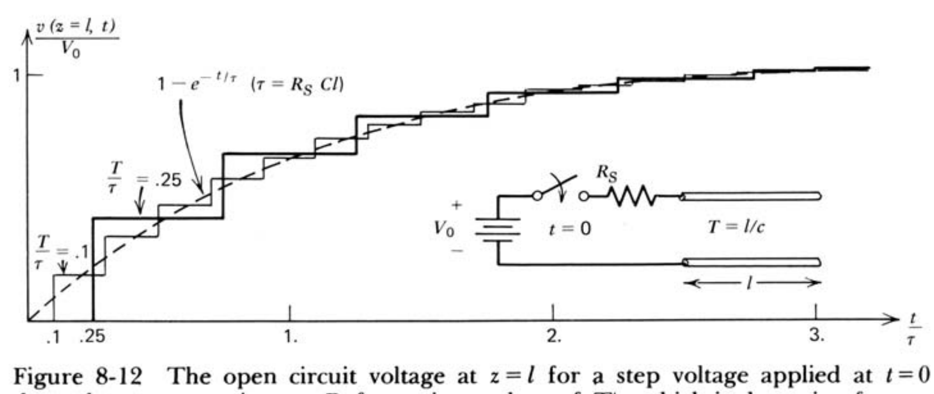

Figure 8-12 The open circuit voltage at $z = l$ for a step voltage applied at $t = 0$ through a source resistance R_s for various values of T/τ, which is the ratio of propagation time $T = l/c$ to quasi-static charging time $\tau = R_s Cl$. The dashed curve shows the exponential rise obtained by a circuit analysis assuming the open circuited transmission line is a capacitor.

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C. Reflections from Arbitrary Terminations

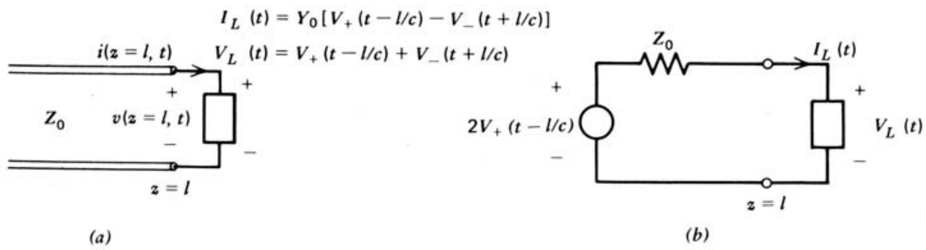

Figure 8-13 A transmission line with an (a) arbitrary load at the $z = l$ end can be analyzed from the equivalent circuit in (b). Since V_+ is known, calculation of the load current or voltage yields the reflected wave V_-.

$$v(z = l, t) = V_L(t) = V_+\left(t - \frac{l}{c}\right) + V_-\left(t + \frac{l}{c}\right)$$

$$i(z = l, t) = I_L(t) = Y_0\left[V_+\left(t - \frac{l}{c}\right) - V_-\left(t + \frac{l}{c}\right)\right]$$

$$\text{Eliminate} \quad V_-\left(t + \frac{l}{c}\right) \Rightarrow 2V_+\left(t - \frac{l}{c}\right) = V_L(t) + I_L(t)Z_0$$

$$V_-\left(t + \frac{l}{c}\right) = V_L(t) - V_+\left(t - \frac{l}{c}\right)$$

1. Capacitor C_L at $z = l, R_S = Z_0 \Rightarrow V_+ = \frac{V_0}{2}$

$$t > T \qquad V_L(t) = v_c(t), I_L(t) = C_L\frac{dv_c}{dt}$$

$$Z_0C_L\frac{dv_c}{dt} + v_c = 2V_+ = V_0, t > T$$

$$v_c(t) = V_0\left[1 - e^{-(t-T)/(Z_0C_L)}\right], t > T$$

$$T = \frac{l}{c}$$

$$V_- = v_c(t) - V_+$$

$$= -\frac{V_0}{2} + V_0\left[1 - e^{-(t-T)/(Z_0C_L)}\right]$$

$$= \frac{V_0}{2} - V_0e^{-(t-T)/(Z_0C_L)}$$

$$i_c = C_L\frac{dv_c}{dt} = \frac{V_0}{Z_0}e^{-(t-T)/(Z_0C_L)}, t > T$$

2. Inductor L_L at $z = l$

$$L_L\frac{di_L}{dt} + i_LZ_0 = 2V_+ = V_0, t > T$$

$$i_L = \frac{V_0}{Z_0}\left(1 - e^{-(t-T)Z_0/L_L}\right), t > T$$

$$v_L = L_L\frac{di_L}{dt} = V_0e^{-(t-T)Z_0/L_L}, t > T$$

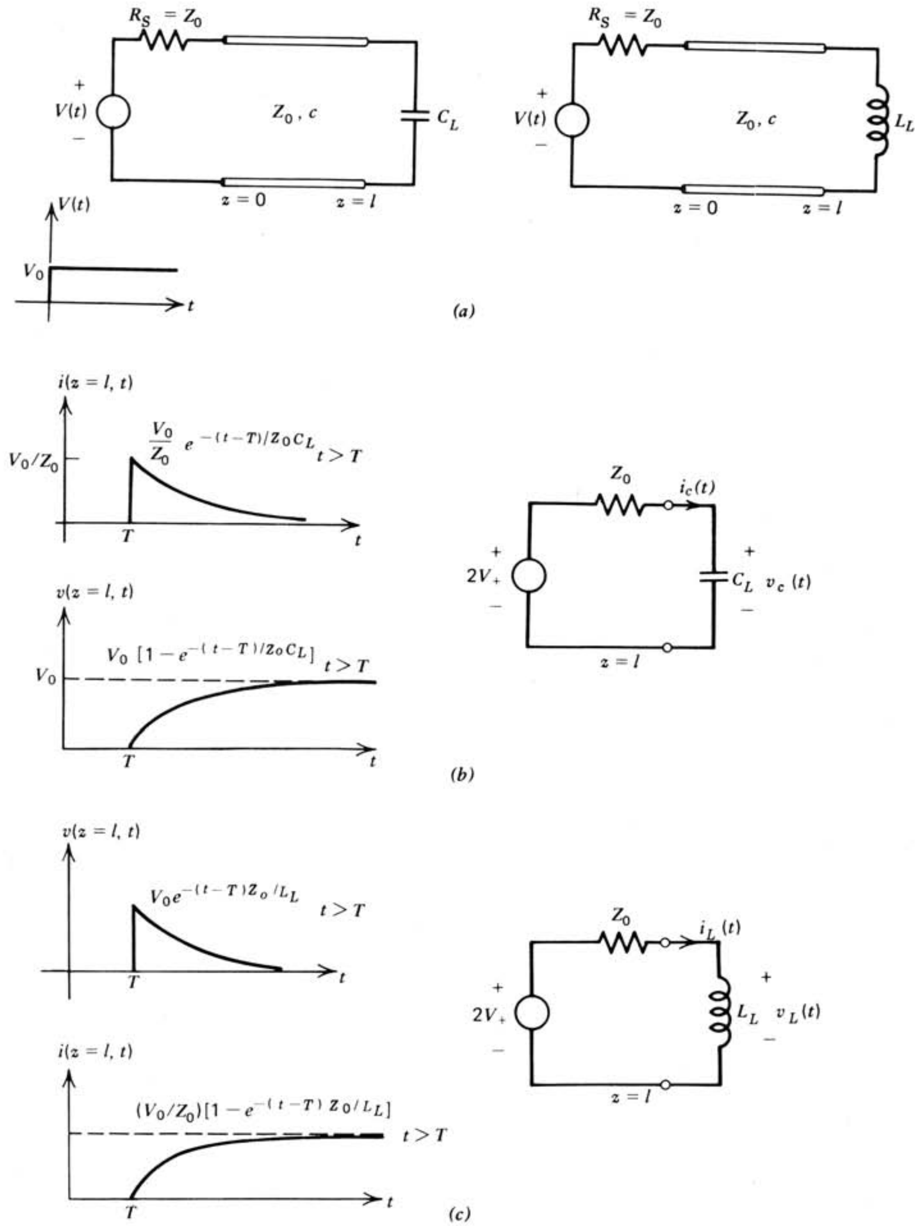

Figure 8-14 (*a*) A step voltage is applied to transmission lines loaded at $z = l$ with a capacitor C_L or inductor L_L. The load voltage and current are calculated from the (*b*) resistive-capacitive or (*c*) resistive-inductive equivalent circuits at $z = l$ to yield exponential waveforms with respective time constants $\tau = Z_0 C_L$ and $\tau = L_L/Z_0$ as the solutions approach the dc steady state. The waveforms begin after the initial V_+ wave arrives at $z = l$ after a time $T = l/c$. There are no further reflections as the source end is matched.