

Transform Representation of Signals and LTI Systems

As you have seen in your prior studies of signals and systems, and as emphasized in the review in Chapter 2, transforms play a central role in characterizing and representing signals and LTI systems in both continuous and discrete time. In this chapter we discuss some specific aspects of transform representations that will play an important role in later chapters. These aspects include the interpretation of Fourier transform phase through the concept of group delay, and methods — referred to as spectral factorization — for obtaining a Fourier representation (magnitude and phase) when only the Fourier transform magnitude is known.

3.1 FOURIER TRANSFORM MAGNITUDE AND PHASE

The Fourier transform of a signal or the frequency response of an LTI system is in general a complex-valued function. A magnitude-phase representation of a Fourier transform $X(j\omega)$ takes the form

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}. \quad (3.1)$$

In eq. (3.1), $|X(j\omega)|$ denotes the (non-negative) magnitude and $\angle X(j\omega)$ denotes the (real-valued) phase. For example, if $X(j\omega)$ is the sinc function, $\sin(\omega)/\omega$, then $|X(j\omega)|$ is the absolute value of this function, while $\angle X(j\omega)$ is 0 in frequency ranges where the sinc is positive, and π in frequency ranges where the sinc is negative. An alternative representation is an amplitude-phase representation

$$A(\omega)e^{j\angle_A X(j\omega)} \quad (3.2)$$

in which $A(\omega) = \pm|X(j\omega)|$ is real but can be positive for some frequencies and negative for others. Correspondingly, $\angle_A X(j\omega) = \angle X(j\omega)$ when $A(\omega) = +|X(j\omega)|$, and $\angle_A X(j\omega) = \angle X(j\omega) \pm \pi$ when $A(\omega) = -|X(j\omega)|$. This representation is often preferred when its use can eliminate discontinuities of π radians in the phase as $A(\omega)$ changes sign. In the case of the sinc function above, for instance, we can pick $A(\omega) = \sin(\omega)/\omega$ and $\angle_A = 0$. It is generally convenient in the following discussion for us to assume that the transform under discussion has no zeros on the $j\omega$ -axis, so that we can take $A(\omega) = |X(j\omega)|$ for all ω (or, if we wish, $A(\omega) = -|X(j\omega)|$ for all ω). A similar discussion applies also, of course, in discrete-time.

In either a magnitude-phase representation or an amplitude-phase representation, the phase is ambiguous, as any integer multiple of 2π can be added at any frequency

without changing $X(j\omega)$ in (3.1) or (3.2). A typical phase computation resolves this ambiguity by generating the phase modulo 2π , i.e., as the phase passes through $+\pi$ it “wraps around” to $-\pi$ (or from $-\pi$ wraps around to $+\pi$). In Section 3.2 we will find it convenient to resolve this ambiguity by choosing the phase to be a continuous function of frequency. This is referred to as the unwrapped phase, since the discontinuities at $\pm\pi$ are unwrapped to obtain a continuous phase curve. The unwrapped phase is obtained from $\angle X(j\omega)$ by adding steps of height equal to $\pm\pi$ or $\pm 2\pi$ wherever needed, in order to produce a continuous function of ω . The steps of height $\pm\pi$ are added at points where $X(j\omega)$ passes through 0, to absorb sign changes as needed; the steps of height $\pm 2\pi$ are added wherever else is needed, invoking the fact that such steps make no difference to $X(j\omega)$, as is evident from (3.1). We shall proceed as though $\angle X(j\omega)$ is indeed continuous (and differentiable) at the points of interest, understanding that continuity can indeed be obtained in all cases of interest to us by adding in the appropriate steps of height $\pm\pi$ or $\pm 2\pi$.

Typically, our intuition for the time-domain effects of frequency response magnitude or amplitude on a signal is rather well-developed. For example, if the Fourier transform magnitude is significantly attenuated at high frequencies, then we expect the signal to vary slowly and without sharp discontinuities. On the other hand, a signal in which the low frequencies are attenuated will tend to vary rapidly and without slowly varying trends.

In contrast, visualizing the effect on a signal of the phase of the frequency response of a system is more subtle, but equally important. We begin the discussion by first considering several specific examples which are helpful in then considering the more general case. Throughout this discussion we will consider the system to be an all-pass system with unity gain, i.e. the amplitude of the frequency response $A(j\omega) = 1$ (continuous time) or $A(e^{j\Omega}) = 1$ (discrete time) so that we can focus entirely on the effect of the phase. The unwrapped phase associated with the frequency response will be denoted as $\angle_A H(j\omega)$ (continuous time) and $\angle_A H(e^{j\Omega})$ (discrete time).

EXAMPLE 3.1 Linear Phase

Consider an all-pass system with frequency response

$$H(j\omega) = e^{-j\alpha\omega} \quad (3.3)$$

i.e. in an amplitude/phase representation $A(j\omega) = 1$ and $\angle_A H(j\omega) = -\alpha\omega$. The unwrapped phase for this example is linear with respect to ω , with slope of $-\alpha$. For input $x(t)$ with Fourier transform $X(j\omega)$, the Fourier transform of the output is $Y(j\omega) = X(j\omega)e^{-j\alpha\omega}$ and correspondingly the output $y(t)$ is $x(t - \alpha)$. In words, linear phase with a slope of $-\alpha$ corresponds to a time delay of α (or a time advance if α is negative).

For a discrete time system with

$$H(e^{j\Omega}) = e^{-j\alpha\Omega} \quad |\Omega| < \pi \quad (3.4)$$

the phase is again linear with slope $-\alpha$. When α is an integer, the time domain interpretation of the effect on an input sequence $x[n]$ is again straightforward and is

a simple delay (α positive) or advance (α negative) of $|\alpha|$. When α is not an integer, the effect is still commonly referred to as “a delay of α ”, but the interpretation is more subtle. If we think of $x[n]$ as being the result of sampling a band-limited, continuous-time signal $x(t)$ with sampling period T , the output $y[n]$ will be the result of sampling the signal $y(t) = x(t - \alpha T)$ with sampling period T . In fact we saw this result in Example 2.4 of chapter 2 for the specific case of a half-sample delay, i.e. $\alpha = \frac{1}{2}$.

EXAMPLE 3.2 Constant Phase Shift

As a second example, we again consider an all-pass system with $A(j\omega) = 1$ and unwrapped phase

$$\angle_A H(j\omega) = \begin{cases} -\phi_0 & \text{for } \omega > 0 \\ +\phi_0 & \text{for } \omega < 0 \end{cases}$$

as indicated in Figure 3.1

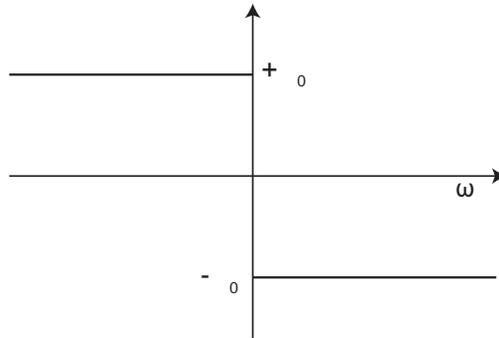


FIGURE 3.1 Phase plot of all-pass system with constant phase shift, ϕ_0 .

Note that the phase is required to be an odd function of ω if we assume that the system impulse response is real valued. In this example, we consider $x(t)$ to be of the form

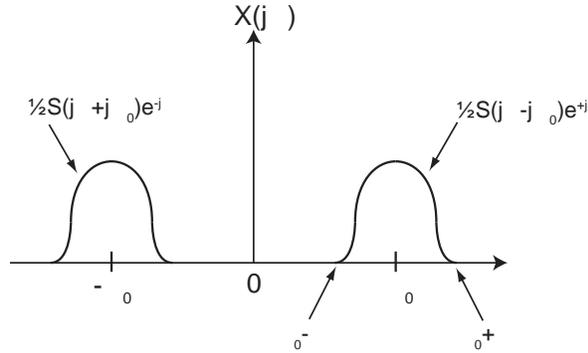
$$x(t) = s(t) \cos(\omega_0 t + \theta) \quad (3.5)$$

i.e. an amplitude-modulated signal at a carrier frequency of ω_0 . Consequently, $X(j\omega)$ can be expressed as

$$X(j\omega) = \frac{1}{2} S(j\omega - j\omega_0) e^{j\theta} + \frac{1}{2} S(j\omega + j\omega_0) e^{-j\theta} \quad (3.6)$$

where $S(j\omega)$ denotes the Fourier transform of $s(t)$.

For this example, we also assume that $S(j\omega)$ is bandlimited to $|\omega| < \Delta$, with Δ sufficiently small so that the term $S(j\omega - j\omega_0) e^{j\theta}$ is zero for $\omega < 0$ and the term $S(j\omega + j\omega_0) e^{-j\theta}$ is zero for $\omega > 0$, i.e. that $(\omega_0 - \Delta) > 0$. The associated spectrum of $x(t)$ is depicted in Figure 3.2.

FIGURE 3.2 Spectrum of $x(t)$ with $s(t)$ narrowband

With these assumptions on $x(t)$, it is relatively straightforward to determine the output $y(t)$. Specifically, the system frequency response $H(j\omega)$ is

$$H(j\omega) = \begin{cases} e^{-j\phi_0} & \omega > 0 \\ e^{+j\phi_0} & \omega < 0 \end{cases} \quad (3.7)$$

Since the term $S(j\omega - j\omega_0)e^{j\theta}$ in eq. (3.6) is non-zero only for $\omega > 0$, it is simply multiplied by $e^{-j\phi_0}$, and similarly the term $S(j\omega + j\omega_0)e^{-j\theta}$ is multiplied only by $e^{+j\phi_0}$. Consequently, the output frequency response, $Y(j\omega)$, is given by

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \frac{1}{2}S(j\omega - j\omega_0)e^{+j\theta}e^{-j\phi_0} + \frac{1}{2}S(j\omega + j\omega_0)e^{-j\theta}e^{+j\phi_0} \end{aligned} \quad (3.8)$$

which we recognize as a simple phase shift by ϕ_0 of the carrier in eq. (3.5), i.e. replacing θ in eq. (3.6) by $\theta - \phi_0$. Consequently,

$$y(t) = s(t) \cos(\omega_0 t + \theta - \phi_0) \quad (3.9)$$

This change in phase of the carrier can also be expressed in terms of a time delay for the carrier by rewriting eq. (3.9) as

$$y(t) = s(t) \cos \left[\omega_0 \left(t - \frac{\phi_0}{\omega_0} \right) + \theta \right] \quad (3.10)$$

3.2 GROUP DELAY AND THE EFFECT OF NONLINEAR PHASE

In Example 3.1, we saw that a phase characteristic that is linear with frequency corresponds in the time domain to a time shift. In this section we consider the

effect of a nonlinear phase characteristic. We again assume the system is an all-pass system with frequency response

$$H(j\omega) = A(j\omega)e^{j\angle_A[H(j\omega)]} \quad (3.11)$$

with $A(j\omega) = 1$. A general nonlinear unwrapped phase characteristic is depicted in Figure 3.3

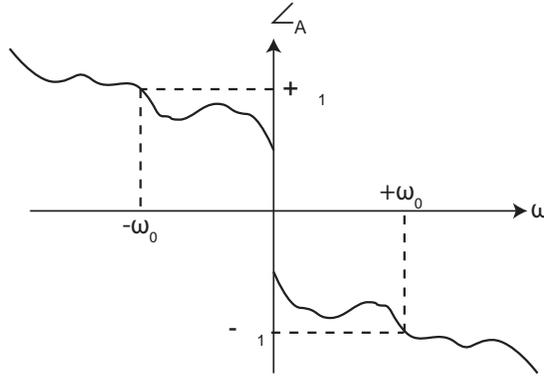


FIGURE 3.3 Nonlinear Unwrapped Phase Characteristic

As we did in Example 3.2, we again assume that $x(t)$ is narrowband of the form of equation (3.5) and as depicted in Figure 3.2. We next assume that Δ in Figure 3.2 is sufficiently small so that in the vicinity of $\pm\omega_0$, $\angle_A H(j\omega)$ can be approximated sufficiently well by the zeroth and first order terms of a Taylor's series expansion, i.e.

$$\angle_A H(j\omega) \approx \angle_A H(j\omega_0) + (\omega - \omega_0) \left[\frac{d}{d\omega} \angle_A H(j\omega) \right]_{\omega=\omega_0} \quad (3.12)$$

Defining $\tau_g(\omega)$ as

$$\tau_g(\omega) = -\frac{d}{d\omega} \angle_A H(j\omega) \quad (3.13)$$

our approximation to $\angle_A H(j\omega)$ in a small region around $\omega = \omega_0$ is expressed as

$$\angle_A H(j\omega) \approx \angle_A H(j\omega_0) - (\omega - \omega_0)\tau_g(\omega_0) \quad (3.14)$$

Similarly in a small region around $\omega = -\omega_0$, we make the approximation

$$\angle_A H(j\omega) \approx \angle_A H(j\omega_0) - (\omega + \omega_0)\tau_g(-\omega_0) \quad (3.15)$$

As we will see shortly, the quantity $\tau_g(\omega)$ plays a key role in our interpretation of the effect on a signal of a nonlinear phase characteristic.

With the Taylor's series approximation of eqs. (3.14) and (3.15) and for input signals with frequency content for which the approximation is valid, we can replace Figure 3.3 with Figure 3.4.

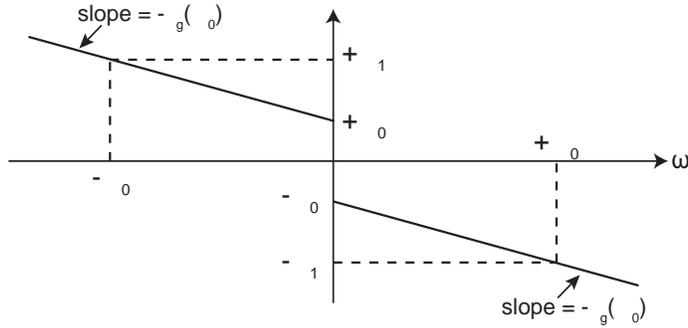


FIGURE 3.4 Taylor's series approximation of nonlinear phase in the vicinity of $\pm\omega_0$

where

$$-\phi_1 = \angle_A H(j\omega_0)$$

and

$$-\phi_0 = \angle_A H(j\omega_0) + \omega_0 \tau_g(\omega_0)$$

Since for LTI systems in cascade, the frequency responses multiply and correspondingly the phases add, we can represent the all-pass frequency response $H(j\omega)$ as the cascade of two all-pass systems, $H_I(j\omega)$ and $H_{II}(j\omega)$, with unwrapped phase as depicted in Figure 3.5.

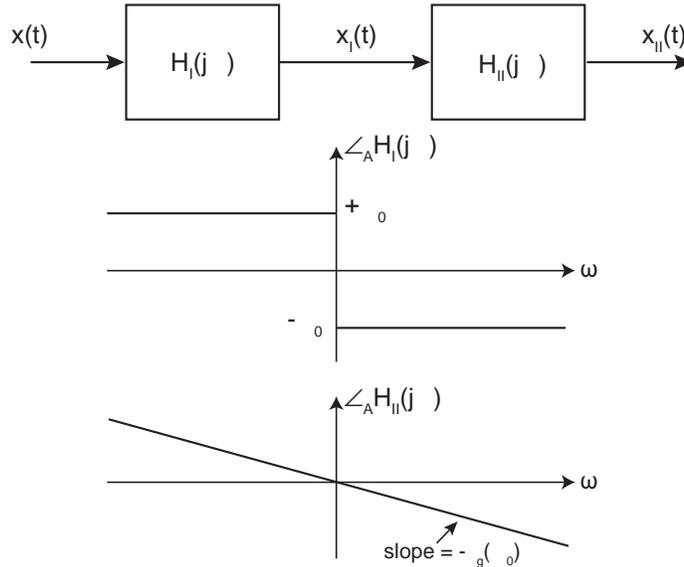


FIGURE 3.5 An all-pass system frequency response, $H(j\omega)$, represented as the cascade of two all-pass systems, $H_I(j\omega)$ and $H_{II}(j\omega)$.

We recognize $H_I(j\omega)$ as corresponding to Example 3.2. Consequently, with $x(t)$ narrowband, we have

$$\begin{aligned} x(t) &= s(t) \cos(\omega_0 t + \theta) \\ x_I(t) &= s(t) \cos \left[\omega_0 \left(t - \frac{\phi_0}{\omega_0} \right) + \theta \right] \end{aligned} \quad (3.16)$$

Next we recognize $H_{II}(j\omega)$ as corresponding to Example 3.1 with $\alpha = \tau_g(\omega_0)$. Consequently,

$$x_{II}(t) = x_I(t - \tau_g(\omega_0)) \quad (3.17)$$

or equivalently

$$x_{II}(t) = s(t - \tau_g(\omega_0)) \cos \left[\omega_0 \left(t - \frac{\phi_0 + \omega_0 \tau_g(\omega_0)}{\omega_0} \right) + \theta \right] \quad (3.18)$$

Since, from Figure 3.4, we see that

$$\phi_1 = \phi_0 + \omega_0 \tau_g(\omega_0)$$

equation (3.18) can be rewritten as

$$x_{II}(t) = s(t - \tau_g(\omega_0)) \cos \left[\omega_0 \left(t - \frac{\phi_1}{\omega_0} \right) + \theta \right] \quad (3.19a)$$

or

$$x_{II}(t) = s(t - \tau_g(\omega_0)) \cos [\omega_0 (t - \tau_p(\omega_0)) + \theta] \quad (3.19b)$$

where τ_p , referred to as the phase delay, is defined as $\tau_p = \frac{\phi_1}{\omega_0}$.

In summary, according to eqs. (3.18) and (3.19a), the time-domain effect of the nonlinear phase for the narrowband group of frequencies around the frequency ω_0 is to delay the narrowband signal by the group delay, $\tau_g(\omega_0)$, and apply an additional phase shift of $\frac{\phi_1}{\omega_0}$ to the carrier. An equivalent, alternate interpretation is that the time-domain envelope of the frequency group is delayed by the group delay and the carrier is delayed by the phase delay.

The discussion has been carried out thus far for narrowband signals. To extend the discussion to broadband signals, we need only recognize that any broadband signal can be viewed as a superposition of narrowband signals. This representation can in fact be developed formally by recognizing that the system in Figure 3.6 is an identity system, i.e. $r(t) = x(t)$ as long as

$$\sum_{i=0}^{\infty} H_i(j\omega) = 1 \quad (3.20)$$

By choosing the filters $H_i(j\omega)$ to satisfy eq. (3.20) and to be narrowband around center frequencies ω_i , each of the output signals, $y_i(t)$, is a narrowband signal. Consequently the time-domain effect of the phase of $G(j\omega)$ is to apply the group

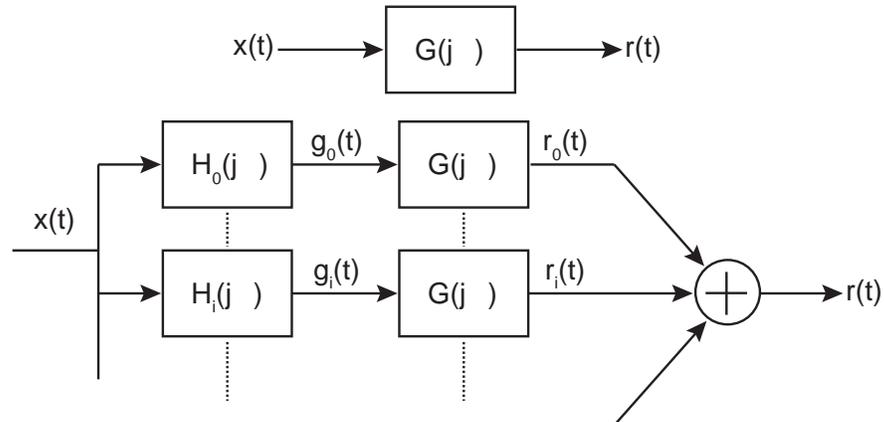


FIGURE 3.6 Continuous-time all-pass system with frequency response amplitude, phase and group delay as shown in Figure 3.7

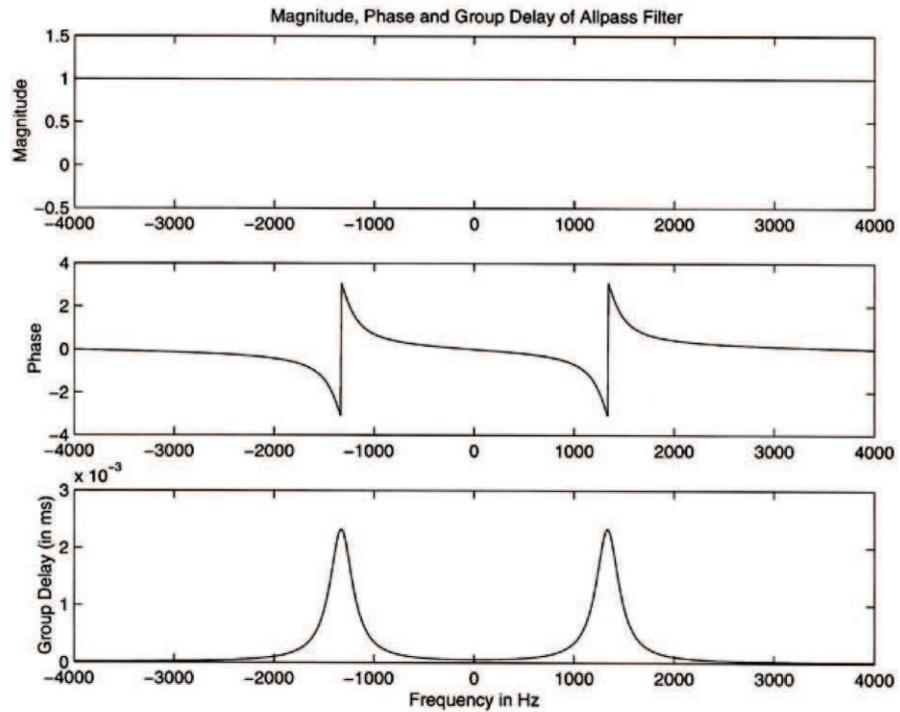


FIGURE 3.7 Magnitude, (nonlinear) phase, and group delay of an all-pass filter.

delay and phase delay to each of the narrowband components (i.e. frequency groups) $y_i(t)$. If the group delay is different at the different center (i.e. carrier) frequencies

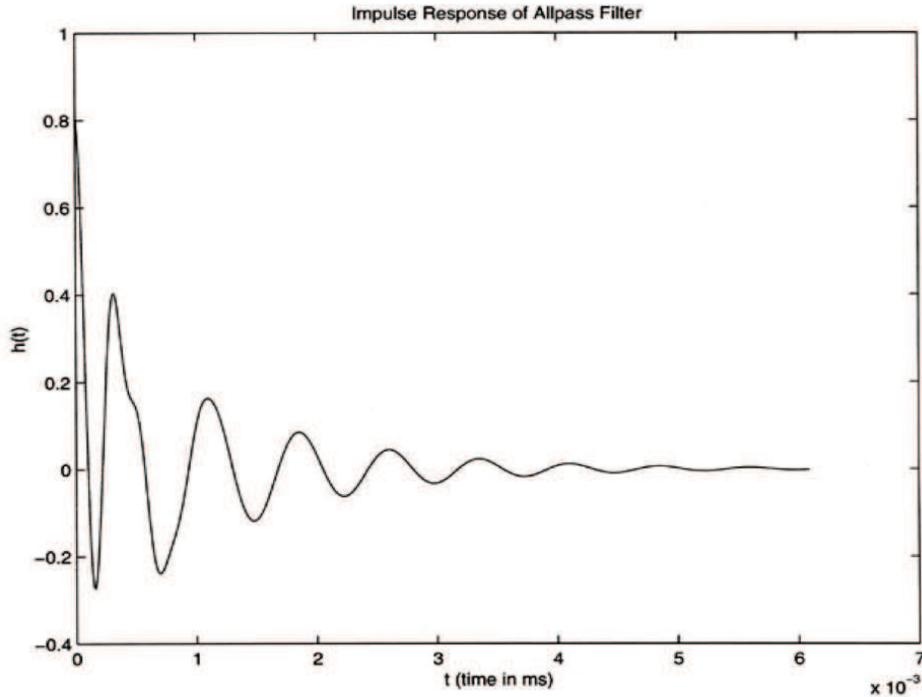


FIGURE 3.8 Impulse response for all-pass filter shown in Figure 3.7

ω_i , then the time domain effect is for different frequency groups to arrive at the output at different times.

As an illustration of this effect, consider $G(j\omega)$ in Figure 3.6 to be the continuous time all-pass system with frequency response amplitude, phase and group delay as shown in Figure 3.7. The corresponding impulse response is shown in Figure 3.8.

If the phase of $G(j\omega)$ were linear with frequency, the impulse response would simply be a delayed impulse, i.e. all the narrowband components would be delayed by the same amount and correspondingly would add up to a delayed impulse. However, as we see in Figure 3.7, the group delay is not constant since the phase is nonlinear. In particular, frequencies around 1200 Hz are delayed significantly more than around other frequencies. Correspondingly, in Figure 3.8 we see that frequency group appearing late in the impulse response.

A second example is shown in Figure 3.9, in which $G(j\omega)$ is again an all-pass system with nonlinear phase and consequently non-constant group delay. With this example, we would expect to see different delays in the frequency groups around $\omega = 2\pi \cdot 50$, $\omega = 2\pi \cdot 100$, and $\omega = 2\pi \cdot 300$ with the group at $\omega = 2\pi \cdot 50$ having the maximum delay and therefore appearing last in the impulse response.

In both of these examples, the input is highly concentrated in time (i.e. an impulse) and the response is dispersed in time because of the non-constant group delay, i.e.

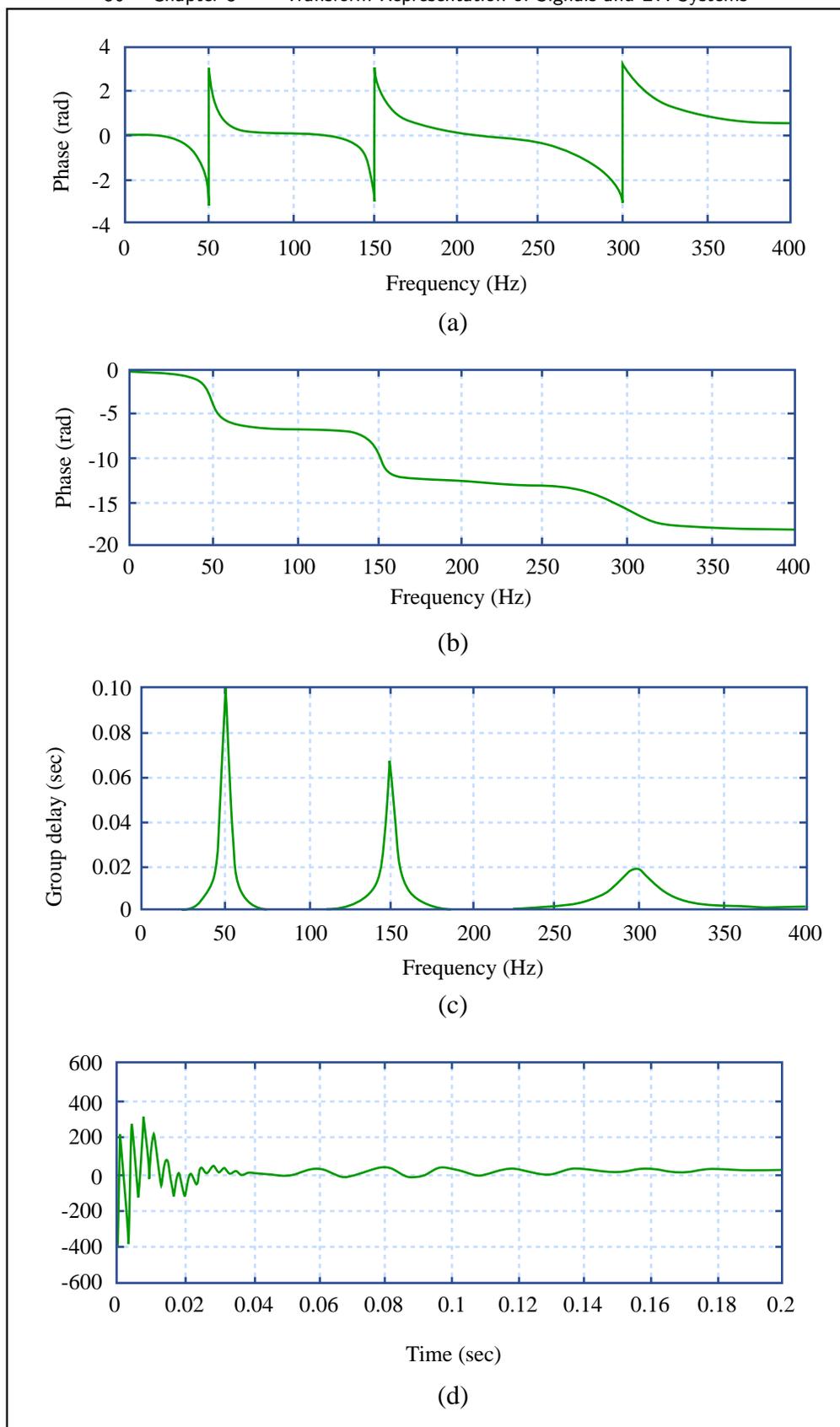


Image by MIT OpenCourseWare, adapted from *Signals and Systems*, Alan Oppenheim and Alan Willsky. Prentice Hall, 1996.

FIGURE 3.9 Phase, group delay, and impulse response for an all-pass system: (a) principal phase; (b) unwrapped phase; (c) group delay; (d) impulse response. (From Oppenheim and Willsky, *Signals and Systems*, Prentice Hall, 1997, Figure 6.5.)

the nonlinear phase. In general, the effect of nonlinear phase is referred to as dispersion. In communication systems and many other application contexts even when a channel has a relatively constant frequency response magnitude characteristic, nonlinear phase can result in significant distortion and other negative consequences because of the resulting time dispersion. For this reason, it is often essential to incorporate phase equalization to compensate for non-constant group-delay.

As a third example, we consider an all-pass system with phase and group delay as shown in Figure 3.10¹. The input for this example is the touch-tone digit “five” which consists of two very narrowband tones at center frequencies 770 and 1336 Hz. The time-domain signal and its two narrowband component signals are shown in Figure 3.11.

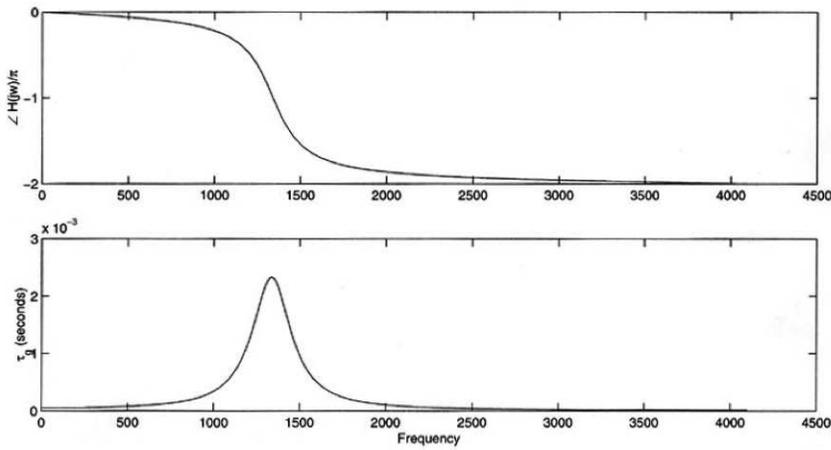


FIGURE 3.10 Phase and group delay for all-pass filter for touch-tone signal example.

The touch-tone signal is processed with multiple passes through the all-pass system of Figure 3.10. From the group delay plot, we expect that, in a single pass through the all-pass filter, the tone at 1336 Hz would be delayed by about 2.5 milliseconds relative to the tone at 770 Hz. After 200 passes, this would accumulate to a relative delay of about 0.5 seconds.

In Figure 3.12, we show the result of multiple passes through filters and the accumulation of the delays.

3.3 ALL-PASS AND MINIMUM-PHASE SYSTEMS

Two particularly interesting classes of stable LTI systems are all-pass systems and minimum-phase systems. We define and discuss them in this section.

¹This example was developed by Prof. Bernard Lesieutre of the University of Wisconsin, Madison, when he taught the course with us at MIT

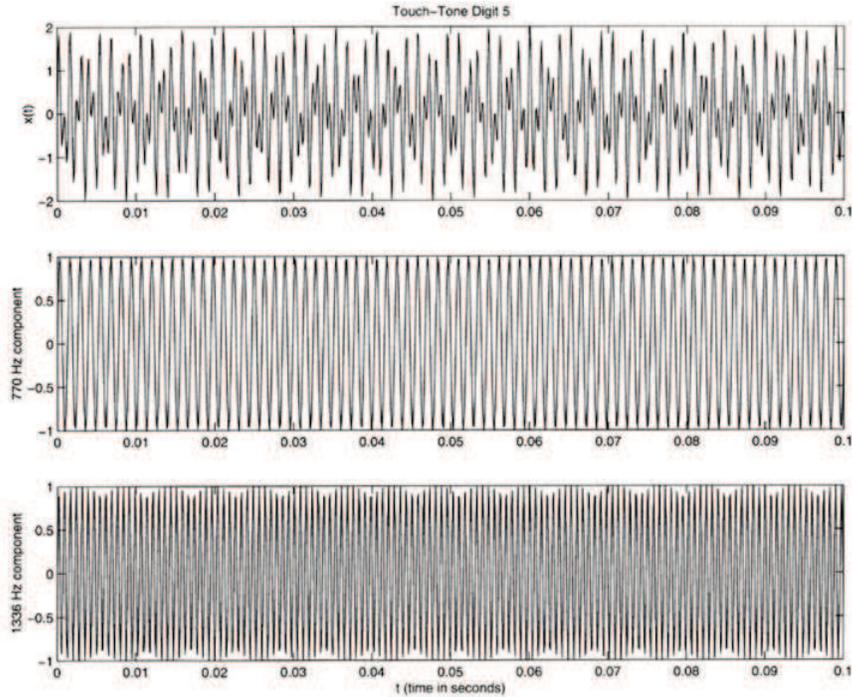


FIGURE 3.11 Touch-tone signal with its two narrowband component signals.

3.3.1 All-Pass Systems

An all-pass system is a stable system for which the magnitude of the frequency response is a constant, independent of frequency. The frequency response in the case of a continuous-time all-pass system is thus of the form

$$H_{ap}(j\omega) = Ae^{j\angle H_{ap}(j\omega)}, \quad (3.21)$$

where A is a constant, not varying with ω . Assuming the associated transfer function $H(s)$ is rational in s , it will correspondingly have the form

$$H_{ap}(s) = A \prod_{k=1}^M \frac{s + a_k^*}{s - a_k}. \quad (3.22)$$

Note that for each pole at $s = +a_k$ this has a zero at the mirror image across the imaginary axis, namely at $s = -a_k^*$; and if a_k is complex and the system impulse response is real-valued, every complex pole and zero will occur in a conjugate pair, so there will also be a pole at $s = +a_k^*$ and a zero at $s = -a_k$. An example of a pole-zero diagram (in the s -plane) for a continuous-time all-pass system is shown in Figure (3.13). It is straightforward to verify that each of the M factors in (3.22) has unit magnitude for $s = j\omega$.

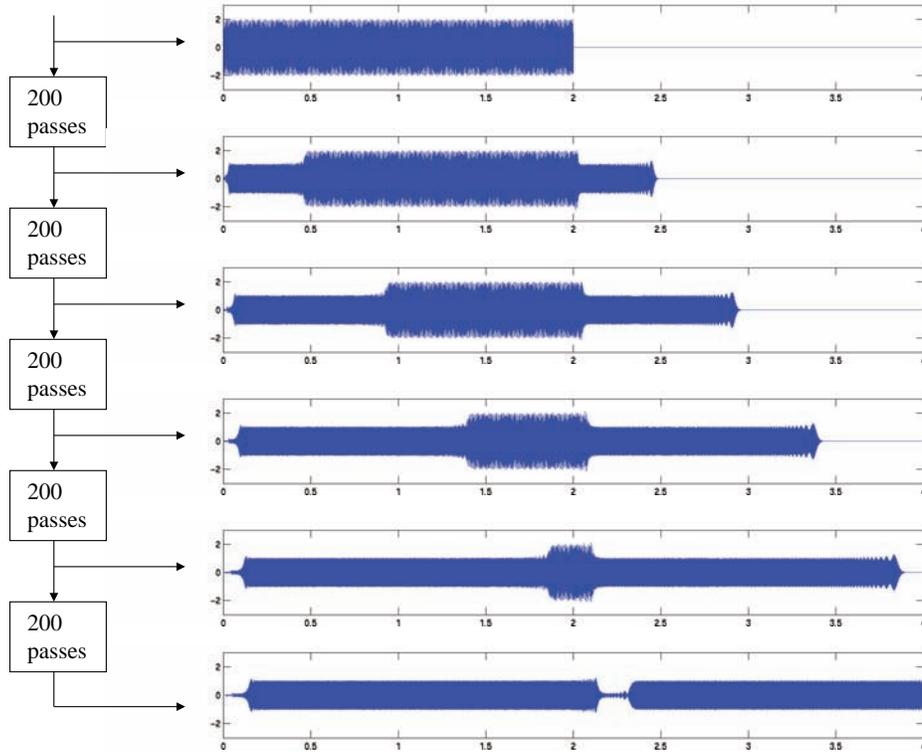


FIGURE 3.12 Effect of passing touchtone signal (Figure 3.11) through multiple passes of an all-pass filter and the accumulation of delays

For a discrete-time all-pass system, the frequency response is of the form

$$H_{ap}(e^{j\Omega}) = Ae^{j\angle H_{ap}(e^{j\Omega})}. \quad (3.23)$$

If the associated transfer function $H(z)$ is rational in z , it will have the form

$$H_{ap}(z) = A \prod_{k=1}^M \frac{z^{-1} - b_k^*}{1 - b_k z^{-1}}. \quad (3.24)$$

The poles and zeros in this case occur at conjugate reciprocal locations: for each pole at $z = b_k$ there is a zero at $z = 1/b_k^*$. A zero at $z = 0$ (and associated pole at ∞) is obtained by setting $b_k = \infty$ in the corresponding factor above, after first dividing both the numerator and denominator by b_k ; this results in the corresponding factor in (3.24) being just z . Again, if the impulse response is real-valued then every complex pole and zeros will occur in a conjugate pair, so there will be a pole at $z = b_k^*$ and a zero at $z = 1/b_k$. An example of a pole-zero diagram (in the z plane) for a discrete-time all-pass system is shown in Figure (3.14). It is once more

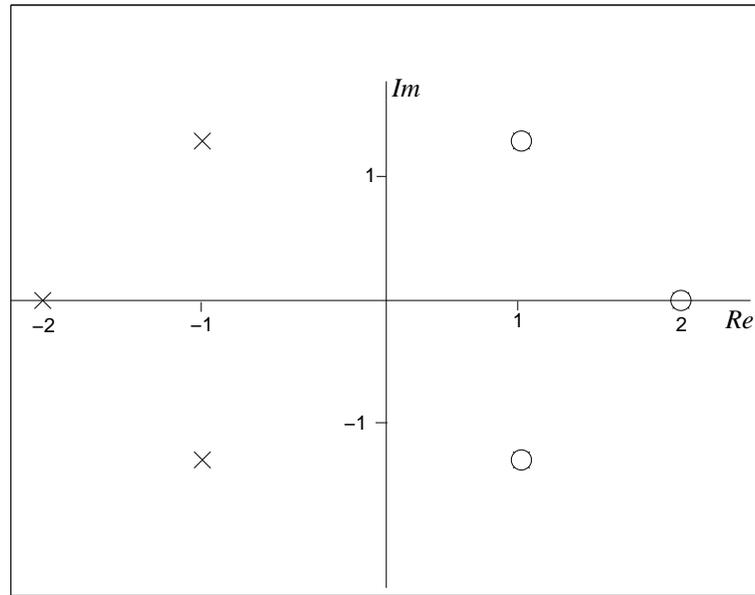


FIGURE 3.13 Typical pole-zero plot for a continuous-time all-pass system.

straightforward to verify that each of the M factors in (3.24) has unit magnitude for $z = e^{j\Omega}$.

The phase of a continuous-time all-pass system will be the sum of the phases associated with each of the M factors in (3.22). Assuming the system is causal (in addition to being stable), then for each of these factors $\text{Re}\{a_k\} < 0$. With some algebra it can be shown that each factor of the form $\frac{s+a_k^*}{s-a_k}$ now has positive group delay at all frequencies, a property that we will make reference to shortly. Similarly, assuming causality (in addition to stability) for the discrete-time all-pass system in (3.24), each factor of the form $\frac{z^{-1}-b_k^*}{1-b_k z^{-1}}$ with $|b_k| < 1$ contributes positive group delay at all frequencies (or zero group delay in the special case of $b_k = 0$). Thus, in both continuous- and discrete-time, the frequency response of a causal all-pass system has constant magnitude and positive group delay at all frequencies.

3.3.2 Minimum-Phase Systems

In discrete-time, a stable system with a rational transfer function is called minimum-phase if its poles and zeros are all inside the unit circle, i.e., have magnitude less than unity. This is equivalent in the DT case to the statement that the system is stable and causal, and has a stable and causal inverse.

A similar definition applies in the case of a stable continuous-time system with a rational transfer function. Such a system is called minimum-phase if its poles and

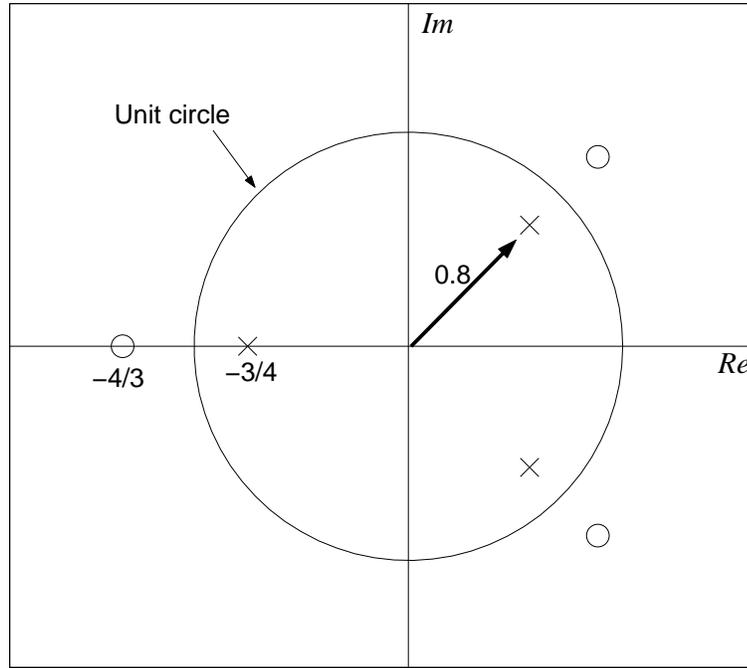


FIGURE 3.14 Typical pole-zero plot for a discrete-time all-pass system.

finite zeros are in the left-half-plane, i.e., have real parts that are negative. The system is therefore necessarily causal. If there are as many finite zeros as there are poles, then a CT minimum-phase system can equivalently be characterized by the statement that both the system and its inverse are stable and causal, just as we had in the DT case. However, it is quite possible — and indeed common — for a CT minimum-phase system to have fewer finite zeros than poles. (Note that a stable CT system must have all its poles at finite locations in the s -plane, since poles at infinity would imply that the output of the system involves derivatives of the input, which is incompatible with stability. Also, whereas in the DT case a zero at infinity is clearly outside the unit circle, in the CT case there is no way to tell if a zero at infinity is in the left half plane or not, so it should be no surprise that the CT definition involves only the finite zeros.)

The use of the term ‘minimum phase’ is historical, and the property should perhaps more appropriately be termed ‘minimum group delay’, for reasons that we will bring out next. To do this, we need a fact that we shall shortly establish: that any causal and stable CT system with a rational transfer function $H_{cs}(s)$ and no zeros on the imaginary axis can be represented as the cascade of a minimum-phase system and an all-pass system,

$$H_{cs}(s) = H_{min}(s)H_{ap}(s). \quad (3.25)$$

Similarly, in the DT case, provided the transfer function $H_{cs}(z)$ has no zeros on

the unit circle, it can be written as

$$H_{cs}(z) = H_{min}(z)H_{ap}(z) . \quad (3.26)$$

The frequency response magnitude of the all-pass factor is constant, independent of frequency, and for convenience let us set this constant to unity. Then from (3.25)

$$|H_{cs}(j\omega)| = |H_{min}(j\omega)| , \quad \text{and} \quad (3.27a)$$

$$\text{grpdelay}[H_{cs}(j\omega)] = \text{grpdelay}[H_{min}(j\omega)] + \text{grpdelay}[H_{ap}(j\omega)] \quad (3.27b)$$

and similar equations hold in the DT case.

We will see in the next section that the minimum-phase term in (3.25) or (3.26) can be uniquely determined from the magnitude of $H_{cs}(j\omega)$, respectively $H_{cs}(e^{j\Omega})$. Consequently all causal, stable systems with the same frequency response magnitude differ only in the choice of the all-pass factor in (3.25) or (3.26). However, we have shown previously that all-pass factors must contribute positive group delay. Therefore we conclude from (3.27b) that among all causal, stable systems with the same CT frequency response magnitude, the one with no all-pass factors in (3.25) will have the minimum group delay. The same result holds in the DT case.

We shall now demonstrate the validity of (3.25); the corresponding result in (3.26) for discrete time follows in a very similar manner. Consider a causal, stable transfer function $H_{cs}(s)$ expressed in the form

$$H_{cs}(s) = A \frac{\prod_{k=1}^{M_1} (s - l_k) \prod_{i=1}^{M_2} (s - r_i)}{\prod_{n=1}^N (s - d_n)} \quad (3.28)$$

where the d_n 's are the poles of the system, the l_k 's are the zeros in the left-half plane and the r_i 's are the zeros in the right-half plane. Since $H_{cs}(s)$ is stable and causal, all of the poles are in the left-half plane and would be associated with the factor $H_{min}(s)$ in (3.25), as would be all of the zeros l_k . We next represent the right-half-plane zeros as

$$\prod_{i=1}^{M_2} (s - r_i) = \prod_{i=1}^{M_2} (s + r_i) \prod_{i=1}^{M_2} \frac{(s - r_i)}{(s + r_i)} \quad (3.29)$$

Since $\text{Re}\{r_i\}$ is positive, the first factor in (3.29) represents left-half-plane zeros. The second factor corresponds to all-pass terms with left-half-plane poles, and with zeros at mirror image locations to the poles. Thus, combining (3.28) and (3.29), $H_{cs}(s)$ has been decomposed according to (3.25) where

$$H_{min}(s) = A \frac{\prod_{k=1}^{M_1} (s - l_k) \prod_{i=1}^{M_2} (s + r_i)}{\prod_{n=1}^N (s - d_n)} \quad (3.30a)$$

$$H_{ap}(s) = \prod_{i=1}^{M_2} \frac{(s - r_i)}{(s + r_i)} \quad (3.30b)$$

EXAMPLE 3.3 Causal, stable system as cascade of minimum-phase and all-pass

Consider a causal, stable system with transfer function

$$H_{cs} = \frac{(s-1)}{(s+2)(s+3)} \quad (3.31)$$

The corresponding minimum-phase and all-pass factors are

$$H_{min}(s) = \frac{(s+1)}{(s+2)(s+3)} \quad (3.32)$$

$$H_{ap}(s) = \frac{s-1}{s+1} \quad (3.33)$$

3.4 SPECTRAL FACTORIZATION

The minimum-phase/all-pass decomposition developed above is useful in a variety of contexts. One that is of particular interest to us in later chapters arises when we are given or have measured the magnitude of the frequency response of a stable system with a rational transfer function $H(s)$ (and real-valued impulse response), and our objective is to recover $H(s)$ from this information. A similar task may be posed in the DT case, but we focus on the CT version here. We are thus given

$$|H(j\omega)|^2 = H(j\omega)H^*(j\omega) \quad (3.34)$$

or, since $H^*(j\omega) = H(-j\omega)$,

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega). \quad (3.35)$$

Now $H(j\omega)$ is $H(s)$ for $s = j\omega$, and therefore

$$|H(j\omega)|^2 = H(s)H(-s) \Big|_{s=j\omega} \quad (3.36)$$

For any numerator or denominator factor $(s-a)$ in $H(s)$, there will be a corresponding factor $(-s-a)$ in $H(s)H(-s)$. Thus $H(s)H(-s)$ will consist of factors in the numerator or denominator of the form $(s-a)(-s-a) = -s^2 + a^2$, and will therefore be a rational function of s^2 . Consequently $|H(j\omega)|^2$ will be a rational function of ω^2 . Thus, if we are given or can express $|H(j\omega)|^2$ as a rational function of ω^2 , we can obtain the product $H(s)H(-s)$ by making the substitution $\omega^2 = -s^2$.

The product $H(s)H(-s)$ will always have its zeros in pairs that are mirrored across the imaginary axis of the s -plane, and similarly for its poles. For any pole or zero of $H(s)H(-s)$ at the real value a , there will be another at the mirror image $-a$, while for any pole or zero at the complex value q , there will be others at q^* , $-q$ and $-q^*$,

forming a complex conjugate pair (q, q^*) and its mirror image $(-q^*, -q)$. We then need to assign one of each mirrored real pole and zero and one of each mirrored conjugate pair of poles and zeros to $H(s)$, and the mirror image to $H(-s)$.

If we assume (or know) that $H(s)$ is causal, in addition to being stable, then we would assign the left-half plane poles of each pair to $H(s)$. With no further knowledge or assumption we have no guidance on the assignment of the zeros other than the requirement of assigning one of each mirror image pair to $H(s)$ and the other to $H(-s)$. If we further know or assume that the system is minimum-phase, then the left-half-plane zeros from each mirrored pair are assigned to $H(s)$, and the right-half-plane zeros to $H(-s)$. This process of factoring $H(s)H(-s)$ to obtain $H(s)$ is referred to as spectral factorization.

EXAMPLE 3.4 Spectral factorization

Consider a frequency response magnitude that has been measured or approximated as

$$|H(j\omega)|^2 = \frac{\omega^2 + 1}{\omega^4 + 13\omega^2 + 36} = \frac{\omega^2 + 1}{(\omega^2 + 4)(\omega^2 + 9)} \quad (3.37)$$

Making the substitution $\omega^2 = -s^2$, we obtain

$$H(s)H(-s) = \frac{-s^2 + 1}{(-s^2 + 4)(-s^2 + 9)} \quad (3.38)$$

which we further factor as

$$H(s)H(-s) = \frac{(s + 1)(-s + 1)}{(s + 2)(-s + 2)(s + 3)(-s + 3)} \quad (3.39)$$

It now remains to associate appropriate factors with $H(s)$ and $H(-s)$. Assuming the system is causal in addition to being stable, the two left-half plane poles at $s = -2$ and $s = -3$ must be associated with $H(s)$. With no further assumptions, either one of the numerator factors can be associated with $H(s)$ and the other with $H(-s)$. However, if we know or assume that $H(s)$ is minimum phase, then we would assign the left-half plane zero to $H(s)$, resulting in the choice

$$H(s) = \frac{(s + 1)}{(s + 2)(s + 3)} \quad (3.40)$$

In the discrete-time case, a similar development leads to an expression for $H(z)H(1/z)$ from knowledge of $|H(e^{j\Omega})|^2$. The zeros of $H(z)H(1/z)$ occur in conjugate reciprocal pairs, and similarly for the poles. We again have to split such conjugate reciprocal pairs, assigning one of each to $H(z)$, the other to $H(1/z)$, based on whatever additional knowledge we have. For instance, if $H(z)$ is known to be causal in addition to being stable, then all the poles of $H(z)H(1/z)$ that are in the unit circle are assigned to $H(z)$; and if $H(z)$ is known to be minimum phase as well, then all the zeros of $H(z)H(1/z)$ that are in the unit circle are assigned to $H(z)$.

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