Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 1, March 16, 2010

QUESTION BOOKLET

Your Full Name:	
Recitation Time :	o'clock

- Check that this **QUESTION BOOKLET** has pages numbered up to 7.
- There are **4 problems, weighted as shown**. (The points indicated on the following pages for the various subparts of the problems are our best guesses for now, but may be modified slightly when we get to grading.)

Problem
1 (5 points)
2 (15 points)
3 (10 points)
4 (20 points)
Total (50 points)

Problem 1 (5 points)

Consider the discrete-time LTI filter specified by its frequency response

$$H(e^{j\Omega}) = \exp\{-j(60\Omega + 25\Omega^3)\}$$

for $|\Omega| < \pi$.

Determine the following properties of the filter frequency response for all $|\Omega| < \pi$:

(a) (1 point) its magnitude, $|H(e^{j\Omega})|$;

(b) (1 point) its phase, $\angle H(e^{j\Omega})$;

(c) (1 point) its group delay, $\tau_g(\Omega)$.

Finally,

(d) (2 points) determine which one of the following plots (labeled A, B, C, D, E) is the impulse response $h[n]$ of this filter; list **two** different features of the chosen response that support your choice. (Adjacent values of $h[n]$ are connected by straight lines in the attached plots, for ease of visualization.)

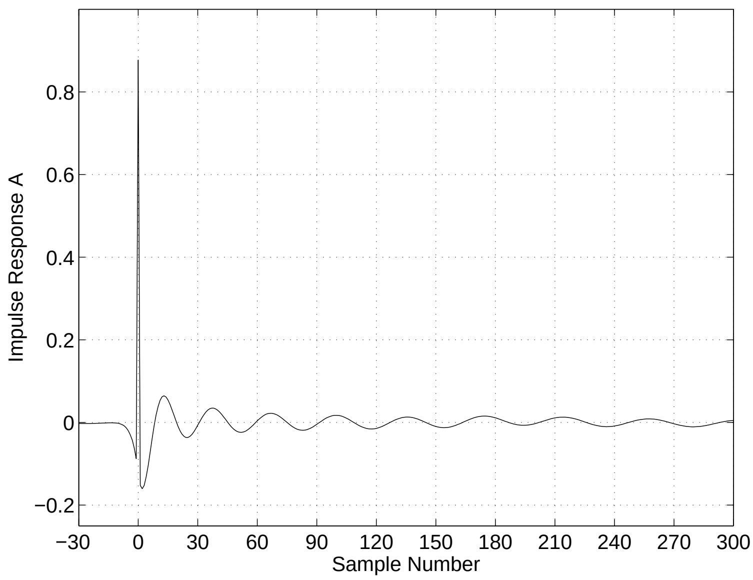


Figure 1: Impulse Response A

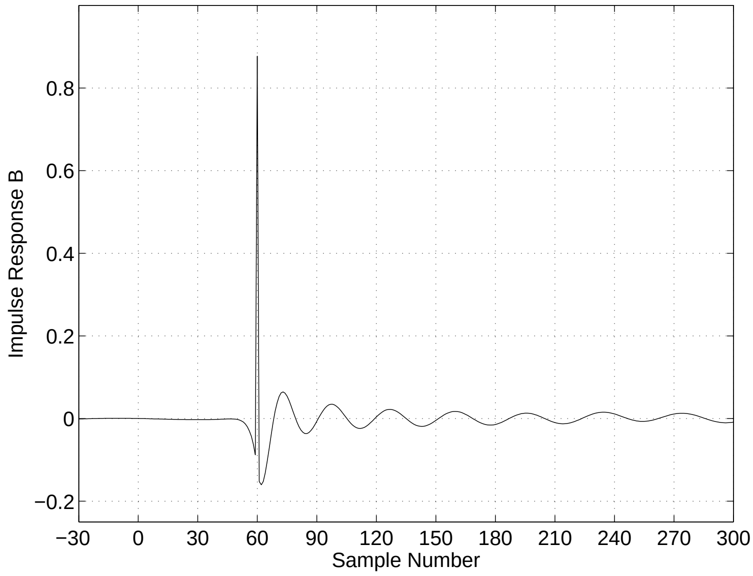


Figure 2: Impulse Response B

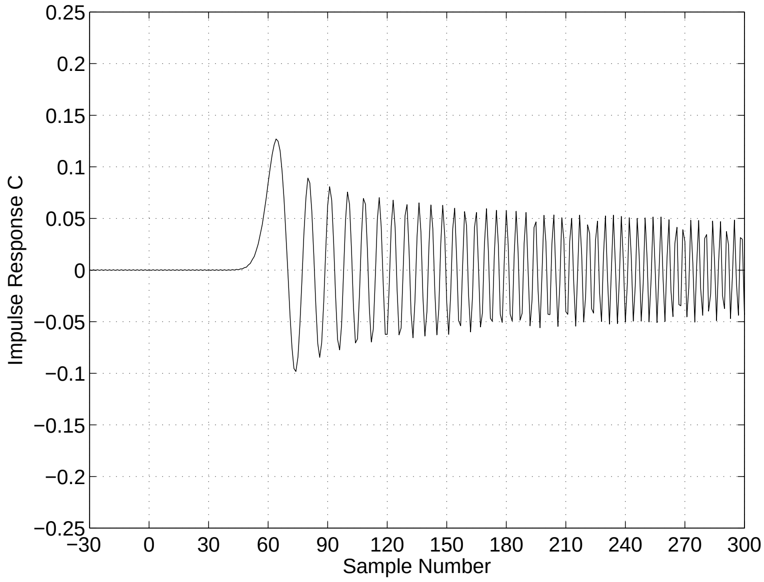


Figure 3: Impulse Response C

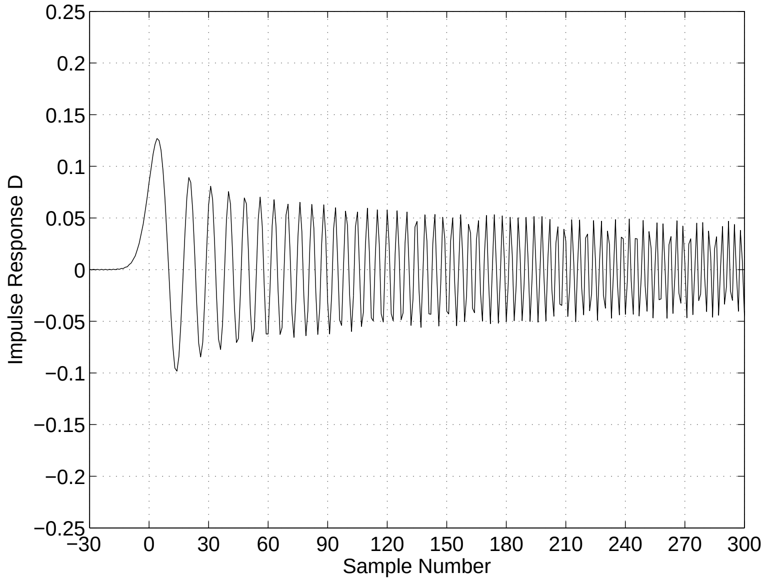


Figure 4: Impulse Response D

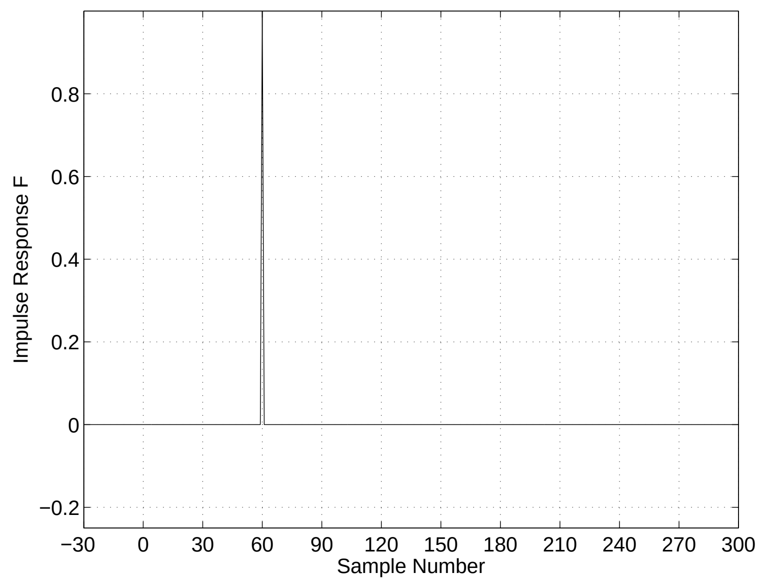


Figure 5: Impulse Response E

Problem 2 (15 points)

Suppose we know that a specific real signal $x(t)$ has deterministic autocorrelation function

$$\overline{R}_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)\,dt = 9\frac{\sin(2\tau)}{\pi\tau} \ .$$

(a) (3 points) What is the energy spectral density $\overline{S}_{xx}(j\omega)$ of this signal? (A careful and fully labeled sketch of this energy spectral density as a function of ω will suffice as an answer.) *Check your answer carefully, as errors here will propagate!* As one check on your answer, compute the energy of the signal, namely

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x^2(t)\,dt \ ,$$

in both the time domain and the frequency domain — i.e., respectively using $\overline{R}_{xx}(\tau)$ and $\overline{S}_{xx}(j\omega)$ — to be sure you get the same answer both ways.

(b) (2 points) What is the magnitude of the Fourier transform of $x(t)$, i.e., what is $|X(j\omega)|$? (Again, a careful and fully labeled sketch of $|X(j\omega)|$ as a function of ω will suffice as an answer.)

(c) (2 points) Keeping in mind your answer to (b), explicitly write down an expression for one possible signal $x(t)$ that has the deterministic autocorrelation function $\overline{R}_{xx}(\tau)$ specified above. Be sure to explain your reasoning!

(d) (2 points) Your answer in (c) is not unique; for example, the signal $-x(t)$ would have the same deterministic autocorrelation function. Describe precisely the relation – in either the time domain or the frequency domain – between any other (correct but otherwise arbitrary) answer to (c) and the specific one you have written down in (c).

(e) (3 points) Suppose the $x(t)$ above is the input to an ideal lowpass filter that has gain 1 for frequencies ω satisfying $|\omega| < 1$, and gain 0 elsewhere. Denoting the corresponding output of the filter by $y(t)$, determine its energy spectral density $\overline{S}_{yy}(j\omega)$ — draw a careful and fully labeled sketch! — and also compute the energy $\mathcal{E}_y$ of $y(t)$.

(f) (3 points) Suppose another signal $f(t)$ has deterministic autocorrelation function

$$\overline{R}_{ff}(\tau) = \int_{-\infty}^{\infty} f(t)f(t-\tau)\,dt = \cos(10\tau)\,\frac{\sin(2\tau)}{\tau} \ .$$

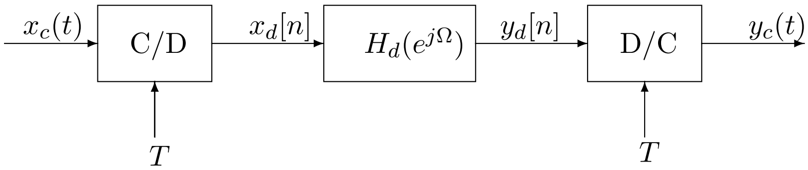
Determine the magnitude of the Fourier transform of the signal, i.e., determine $|F(j\omega)|$. (As before, a careful and fully labeled sketch will suffice as the answer.) Now determine the value of

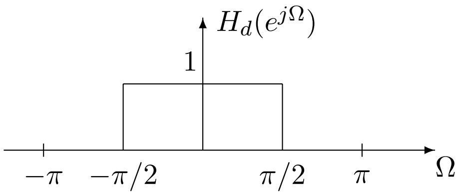
$$\int_{-\infty}^{\infty} x(t)f(t-\tau)\,dt \ ,$$

where $x(t)$ is the signal above. Explain your reasoning carefully.

Problem 3 (10 points)

The first figure below shows our standard configuration for DT processing of a CT signal $x_c(t)$ to produce a CT signal $y_c(t)$. Here T as usual denotes the sampling and reconstruction interval, and the D/C converter implements an ideal bandlimited interpolation. The second figure displays the frequency response of the DT filter, which is an ideal lowpass filter.





Suppose the CT input $x_c(t)$ is chosen to be

$$x_c(t) = \sin(2\pi f_1 t) \ ,$$

where $f_1 = 1300$ Hz.

(a) (1 point) What minimum value does the sampling frequency $1/T$ have to exceed in order to avoid aliasing at the C/D converter, for this signal?

For each of the following parts, fully specify what the output $y_c(t)$ is for the indicated choice of the sampling/reconstruction frequency $1/T$.

(b) (3 points) $1/T = 8000$ Hz.

(c) (3 points) $1/T = 4000$ Hz.

(d) (3 points) $1/T = 1600$ Hz.

Problem 4 (20 points)

Consider the following CT LTI state-space model:

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = [\ 1 \quad 1\ ]\, \mathbf{q}(t)\ .$$

(a) (3 points) Determine the natural frequencies (or eigenvalues) λ_1 and λ_2 of the system, and the associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. (Compute these *carefully*, because errors will propagate, and you will lose significant points! You should check that what you claim is an eigenvalue/eigenvector pair does indeed satisfy the defining matrix equation for such a pair.)

(b) (1 point) Is the system asymptotically stable?

(c) (3 points) Determine the transfer function $H(s)$ of the system. What feature of this $H(s)$ tells you the system is reachable and observable?

(d) (4 points) When the initial state of the system is at zero, i.e., when $\mathbf{q}(0) = \mathbf{0}$, and when the input for $t \geq 0$ is $x(t) = 5e^{-t}$, the output $y(t)$ of the system for $t \geq 0$ is

$$y(t) = e^{3t} - e^{-2t}\ .$$

(You can use this fact as a check on some of your preceding answers!) What initial condition $\mathbf{q}(0)$ would we need in order to have $y(t) \equiv 0$ for $t \geq 0$, with this same input $x(t) = 5e^{-t}$ for $t \geq 0$.

(e) (3 points) Suppose we implement a state feedback control of the form

$$x(t) = \mathbf{g}^T \mathbf{q}(t) + p(t)$$

on the given system, where the feedback gain vector is $\mathbf{g}^T = [g_1 \quad g_2]$, and $p(t)$ is some new external input. Write down the resulting state-space model, with input $p(t)$, state vector $\mathbf{q}(t)$, and output $y(t)$. You need to show the model in detail, making explicit the entries of all matrices and vectors involved in the state evolution equation and the output equation.

(f) (4 points) For your closed-loop system in (e), determine what choice of g_1 and g_2 will result in a closed-loop characteristic polynomial of

$$\nu(s) = s^2 + 3s + 2\ .$$

Is the resulting system observable? Be sure to show your reasoning.

(g) (2 points) With the state feedback gains picked as in (f), suppose $p(t)$ in the closed-loop system is kept constant at the value $\overline{p} = 6$ for all time. What is the corresponding equilibrium value $\overline{\mathbf{q}}$ of the state vector $\mathbf{q}(t)$?