Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 2, April 14, 2005

Answer Booklet

Your Full Name:	
Recitation Instructor & Time :	at o'clock

- This quiz is **closed book**, but **three** sheets of notes are allowed. Calculators will not be necessary and are not allowed.
- This answer booklet has space for **all** answers, and for relevant reasoning. Check that the answer booklet has pages numbered up to 20.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this answer booklet the work you would like us to look at. Let us know if you need additional scratch paper. **Only** this answer booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **three problems**, **weighted as indicated on the quiz**. The quiz will be graded out of **50 points**.
- **DO NOT DISCUSS THIS QUIZ WITH 6.011 STUDENTS WHO HAVE NOT YET TAKEN IT TODAY!**

Problem	Your Score
1 (16 points)	
2 (16 points)	
3 (18 points)	
Total (50 points)	

Problem 1 (16 points)

A particular object of unit mass, constrained to move in a straight line, is acted on by an external force $x(t)$ and restrained by a cubic spring. The system can be described by the equation

$$\frac{d^2p(t)}{dt^2} + kp(t) - \epsilon p^3(t) = x(t) \ ,$$

where $p(t)$ denotes the position of the mass and $p^3(t)$ is the cube of the position (*not* its third derivative!); the quantities k and ϵ are known positive constants.

1(a) (4 points) Obtain a state-space model for the above system, using physically meaningful state variables; take $x(t)$ to be the input and let the output $y(t)$ be the position of the mass.

$$q_1(t) = p(t)$$
$$q_2(t) = \dot{p}(t)$$
$$\dot{q_2}(t) = -kp(t) + \epsilon p^3(t) + x(t)$$

State-space model:

$$\dot{q_1}(t) = q_2(t)$$
$$\dot{q_2}(t) = \{-k + \epsilon q_1^2(t)\}q_1(t) + x(t)$$
$$y(t) = q_1(t)$$

1(b) (5 points) Suppose $x(t) \equiv 0$ and the system is in equilibrium. You will find that there are *three* possible equilibrium conditions of the system. Determine the values of your state variables in each of these three equilibrium conditions, expressing your results in terms of the parameters k and ϵ.

$$\dot{q}_1(t) = \dot{\bar{q}}_1 = 0 \Rightarrow q_2(t) = \bar{q}_2 = 0$$

$$\dot{q}_2(t) = \dot{\bar{q}}_2 = 0 \Rightarrow \{-k + \epsilon q_1^2(t)\}q_1(t) = \{-k + \epsilon \bar{q}_1^2(t)\}\bar{q}_1 = 0$$

so equilibria are:

$$\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{k}{\epsilon}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{\frac{k}{\epsilon}} \\ 0 \end{bmatrix}$$

The (constant) values of the state variables corresponding to each of the three equilibria are:

First equilibrium:

$$\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second equilibrium:

$$\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{k}{\epsilon}} \\ 0 \end{bmatrix}$$

Third equilibrium:

$$\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{k}{\epsilon}} \\ 0 \end{bmatrix}$$

1(c) (7 points) For *each* of the three equilibrium positions you identified in 1(b), obtain a linearized state-space model that approximately describes small deviations away from the equilibrium. We are looking for the standard "$\mathbf{A}, \mathbf{b}, \mathbf{c}^T, \mathtt{d}$" description for each linearized model. Which of these three linearized models, if any, is asymptotically stable? Explain your answer.

$$\begin{aligned}\begin{bmatrix} \dot{\tilde{q}}_1(t) \\ \dot{\tilde{q}}_2(t) \end{bmatrix} &= \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix}_{\bar{\mathbf{q}},\bar{x}} \begin{bmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \end{bmatrix}_{\bar{\mathbf{q}},\bar{x}} \tilde{x}(t) \\ &= \begin{bmatrix} 0 & 1 \\ -k + 3\epsilon\bar{q}_1^2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{x}(t)\end{aligned}$$

$$\tilde{y}(t) = \tilde{q}_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{bmatrix}$$

The standard "$\mathbf{A}, \mathbf{b}, \mathbf{c}^T, \mathtt{d}$ for each of the three linearized models are as follows (**in the same order as the equilibria listed in the previous part**):

First equilibrium:

$$\begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$0$$

Second equilibrium:

$$\begin{bmatrix} 0 & 1 \\ -k + 3\frac{\epsilon k}{\epsilon} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2k & 0 \end{bmatrix}$$

same $\mathbf{b}$,$\mathbf{c}^T$, $\mathtt{d}$.

Third equilibrium:
same as the second

Indicate which of the above, if any, are asymptotically stable. Reasoning:
None. Their eigenvalues are $j\sqrt{k}$,$-j\sqrt{k}$; $\sqrt{2k}$, $-\sqrt{2k}$; $\sqrt{2k}$, $-\sqrt{2k}$. All have real part non-negative.

Problem 2 (16 points)

A particular second-order continuous-time causal LTI system has natural frequencies $\lambda_1 = -3$ and $\lambda_2 = -4$ (these are the eigenvalues of the matrix that governs state evolution), with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Its input-output transfer function is

$$H(s) = \frac{s+1}{(s+3)(s+4)} \ .$$

2(a) (2 points) Is the system reachable? Is it observable? Explain.

Yes, reachable and observable, since both natural frequencies are evident in the transfer function, with no pole/zero cancellation possible.

2(b) (4 points) Suppose the system is initially at rest, i.e., its initial state is zero. Is it now possible to choose the input in such as way that the state moves out along the eigenvector $\mathbf{v}_1$, with *no component* along $\mathbf{v}_2$ during the entire motion? Explain your answer carefully.

$$\mathbf{q}(t) = \mathbf{v}_1 r_1(t) + \mathbf{v}_2 r_2(t)$$

where:

$$\dot{r}_1(t) = \lambda_1 r_1(t) + \beta_1 x(t),\ r_1(0) = 0$$

$$\dot{r}_2(t) = \lambda_2 r_2(t) + \beta_2 x(t),\ r_2(0) = 0$$

Since $\beta_2 \neq 0$ (because of reachability of the second mode), any nonzero $x(t)$ - as would be needed to make $r_1(t) \neq 0$ and drive along $\mathbf{v}_1$ - would also force $r_2(t) \neq 0$ and result in a component along $\mathbf{v}_2$. So, not possible.

2(c) Suppose the output of the above system is applied to the input of another causal second-order LTI system with transfer function

$$G(s) = \frac{s+3}{s(s+5)} \ .$$

The input to the combined system is then just the original input to the first system, while the output of the combined system is the output of the second system:

$$---> \quad \frac{s+1}{(s+3)(s+4)} \quad ---> \quad \frac{s+3}{s(s+5)} \quad --->$$

2(c)(i) (2 points) How many state variables are there in the state-space description of the combined system, and what are the natural frequencies of this combined system?

The interconnection requires 2 state variables to describe the first system and 2 for the second, so 4 state variables. (Because of the pole-zero cancellation, there are 3rd-order state-space models that have the same transfer function, but that's not what's asked.)

The natural frequencies of a cascade of two systems comprises the natural frequencies of the individual systems, hence, -3, -4, 0, and -5 in this case.

2(c)(ii) (2 points) Is the combined system asymptotically stable? Explain.

No, because there is an eigenvalue at 0.

2(c)(iii) (3 points) Is the combined system reachable from the input of the first system? Is it observable from the output of the second system? Explain.

Observability is lost because a zero of the second system cancels (hides) a pole of the first. However, reachability is maintained because the zero of the first subsystem does not cancel (shield) any poles of the second system.

2(c)(iv) (3 points) If you were to build an observer for the combined system (using measurements of the input to the first system and the output of the second system), could you get the estimation error of the observer to decay? If not, why not; and if so, could you get the error to decay arbitrarily fast? Explain.

The error dynamics of the observer would retain the unobservable mode at -3, but all the other modes could be shifted to arbitrary self-conjugate positions. Hence, since the unobservable mode is stable, we can always get all modes stable, hence a decaying estimation error. However, arbitrarily fast decay won't be possible.

Problem 3 (18 points)

Consider the causal discrete-time LTI system

$$\mathbf{q}[n+1] = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \mathbf{q}[n] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x[n] + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w[n]$$

where $x[n]$ is a control input and $w[n]$ is a disturbance input.

3(a) (3 points) What are the natural frequencies of the system (i.e., the eigenvalues of the state evolution matrix)? Is the system asymptotically stable?

Characteristic polynomial: $z(z+5)+6 = (z+2)(z+3)$.

So natural frequencies are -2,-3. These are not strictly in the unit circle, so the system is not asymptotically stable.

Natural frequencies: -2, -3.

Is the system asymptotically stable? No.

3(b) (6 points) Suppose you use the LTI state feedback

$$x[n] = g_1 q_1[n] + g_2 q_2[n] \ .$$

What choice of the gains g_1 and g_2 will yield the closed-loop characteristic polynomial $z(z+0.5)$? For this choice, write down the eigenvalues of the matrix that describes the state evolution of the closed-loop system, and compute the associated eigenvectors.

$\begin{bmatrix} 0 & 1 \\ -6+g_1 & -5+g_2 \end{bmatrix}$ has characteristic polynomial $z(z+5-g_2)+6-g_1$. To make this equal to $z(z+0.5)$, pick $g_2 = 4.5$, $g_1 = 6$.

Roots of $z(z+0.5)$ are 0, -0.5, so these are the eigenvalues.

The closed-loop matrix is:

$$\begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}$$

and its eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$.

$g_1 = 6$ $g_2 = 4.5$

Eigenvalues: 0, -0.5.

Their respective eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

3(c) (2 points) Suppose the system output is $y[n] = q_1[n]$. With $x[n]$ chosen as in (b), is the closed-loop system observable? Show reasoning.

$$\mathbf{c}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$\mathbf{c}^T\mathbf{v}_1 = 1 \neq 0$, and $\mathbf{c}^T\mathbf{v}_2 = 1 \neq 0$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of closed-loop system. So the system is observable.

3(d) (3 points) With $x[n]$ as in (b) and $y[n]$ as in (c), what is the transfer function from $w[n]$ to $y[n]$ for the closed-loop system?

$$\begin{aligned} H(z) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & -1 \\ 0 & z+0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{z(z+0.5)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z+0.5 & 1 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{z+0.5}{z(z+0.5)} \\ &= \frac{1}{z} \end{aligned}$$

Transfer function: $\frac{1}{z}$.

3(e) (4 points) Determine in *two* distinct ways, using the results in (b), (c), (d), whether or not the closed-loop system is reachable from the disturbance input $w[n]$. Explain your two approaches clearly.

The system is observable, but there is a hidden mode, so the system must be unreachable (and the hidden mode at -0.5 is the unreachable one).

The disturbance acts through the vector:

$$\mathbf{b}_{\text{dist}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1.\mathbf{v}_1 + 0.\mathbf{v}_2$$

Hence, $\beta_1 = 1$ and $\beta_2 = 0$, so the mode at -0.5 is unreachable.