

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.011: Introduction to Communication, Control and Signal Processing
QUIZ 2, April 14, 2005
Questions

- This quiz is **closed book**, but **three** sheets of notes are allowed. Calculators will not be necessary and are not allowed.
- The accompanying answer booklet has space for **all** answers, and for relevant reasoning. Check that the answer booklet has pages numbered up to ******.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to the answer booklet the work you would like us to look at. Let us know if you need additional scratch paper. **Only** the answer booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered**. Absolutely no exceptions!
- There are **three problems, weighted as indicated on the quiz**. The quiz will be graded out of **50 points**.
- **DO NOT DISCUSS THIS QUIZ WITH 6.011 STUDENTS WHO HAVE NOT YET TAKEN IT TODAY!**

Problem 1 (16 points)

A particular object of unit mass, constrained to move in a straight line, is acted on by an external force $x(t)$ and restrained by a cubic spring. The system can be described by the equation

$$\frac{d^2p(t)}{dt^2} + kp(t) - \epsilon p^3(t) = x(t) ,$$

where $p(t)$ denotes the position of the mass and $p^3(t)$ is the cube of the position (*not* its third derivative!); the quantities k and ϵ are known positive constants.

- (a) (4 points) Obtain a state-space model for the above system, using physically meaningful state variables; take $x(t)$ to be the input and let the output $y(t)$ be the position of the mass.
- (b) (5 points) Suppose $x(t) \equiv 0$ and the system is in equilibrium. You will find that there are *three* possible equilibrium conditions of the system. Determine the values of your state variables in each of these three equilibrium conditions, expressing your results in terms of the parameters k and ϵ .
- (c) (7 points) For *each* of the three equilibrium positions you identified in (b), obtain a linearized state-space model that approximately describes small deviations away from the equilibrium. We are looking for the standard “ $\mathbf{A}, \mathbf{b}, \mathbf{c}^T, \mathbf{d}$ ” description for each linearized model. Which of these three linearized models, if any, is asymptotically stable? Explain your answer.

Problem 2 (16 points)

[Part (c) below can be done even if you don't get part (b).]

A particular second-order continuous-time causal LTI system has natural frequencies $\lambda_1 = -3$ and $\lambda_2 = -4$ (these are the eigenvalues of the matrix that governs state evolution), with associated eigenvectors \mathbf{v}_1 and \mathbf{v}_2 respectively. Its input-output transfer function is

$$H(s) = \frac{s + 1}{(s + 3)(s + 4)}.$$

- (a) (2 points) Is the system reachable? Is it observable? Explain.
- (b) (4 points) Suppose the system is initially at rest, i.e., its initial state is zero. Is it now possible to choose the input in such a way that the state moves out along the eigenvector \mathbf{v}_1 , with *no component* along \mathbf{v}_2 during the entire motion? Explain your answer carefully.
- (c) (10 points) Suppose the output of the above system is applied to the input of another causal second-order LTI system with transfer function

$$G(s) = \frac{s + 3}{s(s + 5)}.$$

The input to the combined system is then just the original input to the first system, while the output of the combined system is the output of the second system:

$$\text{---} > \frac{s + 1}{(s + 3)(s + 4)} \text{---} > \frac{s + 3}{s(s + 5)} \text{---} >$$

- (i) (2 points) How many state variables are there in the state-space description of the combined system, and what are the natural frequencies of this combined system?
- (ii) (2 points) Is the combined system asymptotically stable? Explain.
- (iii) (3 points) Is the combined system reachable from the input of the first system? Is it observable from the output of the second system? Explain.
- (iv) (3 points) If you were to build an observer for the combined system (using measurements of the input to the first system and the output of the second system), could you get the estimation error of the observer to decay? If not, why not; and if so, could you get the error to decay arbitrarily fast?

Remember to **explain** all your answers, to receive the indicated credit!

Problem 3 (18 points)

Consider the causal discrete-time LTI system

$$\mathbf{q}[n+1] = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \mathbf{q}[n] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x[n] + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w[n]$$

where $x[n]$ is a control input and $w[n]$ is a disturbance input.

- (a) (3 points) What are the natural frequencies of the system (i.e., the eigenvalues of the state evolution matrix)? Is the system asymptotically stable?
- (b) (6 points) Suppose you use the LTI state feedback

$$x[n] = g_1 q_1[n] + g_2 q_2[n].$$

What choice of the gains g_1 and g_2 will yield the closed-loop characteristic polynomial $z(z + 0.5)$? For this choice, write down the eigenvalues of the matrix that describes the state evolution of the closed-loop system, and compute the associated eigenvectors.

- (c) (2 points) Suppose the system output is $y[n] = q_1[n]$. With $x[n]$ chosen as in (b), is the closed-loop system observable? Show reasoning.
- (d) (3 points) With $x[n]$ as in (b) and $y[n]$ as in (c), what is the transfer function from $w[n]$ to $y[n]$ for the closed-loop system?
- (e) (4 points) Determine in *two* distinct ways, using the results in (b), (c), (d), whether or not the closed-loop system is reachable from the disturbance input $w[n]$. Explain your two approaches clearly.