Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

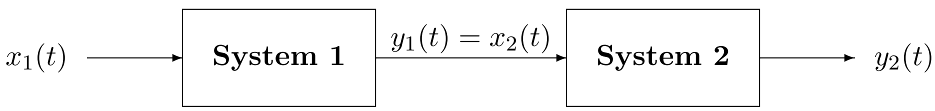
6.011: Introduction to Communication, Control and Signal Processing

QUIZ 2, Spring 2003

Question Booklet

- This quiz is **closed book**, but **two** "crib" sheets are allowed.
- Put your name on **each** sheet of the answer booklet, and your recitation instructor's name and time on the cover page of that booklet.
- **Only** the answer booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to the answer booklet the work you would like us to look at. Let us know if you need additional scratch paper.
- The quiz will be graded out of **50 points**. The **three problems** are nominally **weighted as indicated** (but our legal department wishes to let you know that we reserve the right to modify the weighting *slightly* when we grade, if your collective answers and common errors end up suggesting that a modified weighting would be more appropriate).

Problem 1 (36 points)



(a) Suppose the transfer function of System 1 in the block diagram above is

$$H_1(s) = \frac{s}{s-1} = \frac{1}{s-1} + 1 \ .$$

(i) Find a first-order state-space model for System 1, using $q_1(t)$ to denote its state variable, and arranging things such that

$$y_1(t) = q_1(t) + x_1(t) \ .$$

(ii) Is your state-space model for System 1:

- reachable?
- observable?
- asymptotically stable?

(b) Suppose System 2 in the block diagram above is described by the first-order state-space model

$$\begin{aligned} \dot{q}_2(t) &= \mu q_2(t) + x_2(t) \\ y_2(t) &= 2q_2(t) \end{aligned}$$

where μ is a parameter, and we are given that $\mu \neq 1$.

(i) What is the transfer function $H_2(s)$ of System 2?

(ii) For what values of μ, if any, is the state-space model of System 2:

- unreachable?
- unobservable?
- asymptotically stable?

(c) (i) Combine the state-space models in (a) and (b) to obtain a second-order state-space model of the form

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x_1(t) \ , \quad y_2(t) = \mathbf{c}^T\mathbf{q}(t) + \mathtt{d}x_1(t)$$

for the overall system in the above block diagram, using $\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$ as the overall state vector $\mathbf{q}(t)$, and of course $x_1(t)$ as the overall input, and $y_2(t)$ as the overall output.

(ii) Compute the transfer function $H(s)$ from $x_1(t)$ to $y_2(t)$ using the model in (c)(i), and verify that it equals $H_1(s)H_2(s)$. This is a little grungy, but important as a check on your answer in (c)(i).

(iii) What are the eigenvalues of $\mathbf{A}$ in (c)(i)? Check that the eigenvalues you obtain are consistent with what you expect from your results in (c)(ii). What are the eigen*vectors* associated with these eigenvalues?

There are values of μ for which one can find nonzero initial conditions $\mathbf{q}(0)$ such that the resulting zero-input solution $\mathbf{q}(t)$ (i.e., the solution with $x_1(t) \equiv 0$) decays to 0 as $t \to \infty$. Find *all* such values of μ, and for each such μ specify *all* initial conditions that lead to such decaying zero-input solutions.

(iv) For what values of μ, if any, is the overall system in (c)(ii):

- unreachable? — which natural frequencies are unreachable?
- unobservable? — which natural frequencies are unobservable?

Interpret your results in terms of pole-zero cancellations in the block diagram on the previous page.

(d) Suppose you can measure both state variables $q_1(t)$ and $q_2(t)$, so that you can choose

$$x_1(t) = g_1 q_1(t) + g_2 q_2(t) \ .$$

The resulting closed-loop system is evidently still described by a second-order LTI state-space model. What choice of g_1 and g_2 will result in the closed-loop natural (or characteristic) frequencies being at $-1 \pm j1$? You can express your answer in term of μ. (If you've done things correctly, you will find that when $\mu = 2$, for example, you get $g_1 = 0$ and $g_2 = -5$; and when $\mu = 1$, you get $g_1 = 1$ and $g_2 = -5$.) Now determine for what values of μ, if any, your expressions for g_1 and/or g_2 have infinite magnitude, and reconcile your answer with what you found in (c)(iv).

(e) Suppose you can only measure the input $x_1(t)$ and the output $y_2(t)$. Fully specify a scheme for estimating the state variables $q_1(t)$ and $q_2(t)$, in such a way that the error between each of the actual and estimated state variables can be expressed as a linear combination of two *decaying* exponential terms with *time constants* of 0.5 and 0.25 respectively. Will your estimation scheme work for all values of μ? Again, reconcile your answer with what you found in (c)(iv).

Problem 2 (7 points)

Consider a nonlinear time-invariant state-space model described in the form

$$\begin{aligned} \dot{q}_1(t) &= q_2(t) \\ \dot{q}_2(t) &= -\beta q_1^3(t) + x(t) \ , \end{aligned}$$

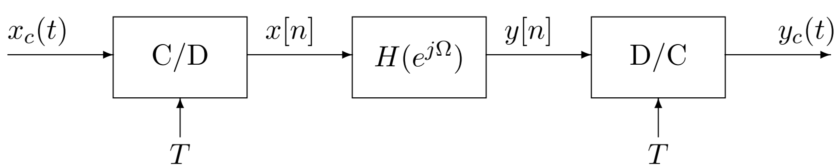
where β is some positive constant.

(a) If the input $x(t)$ is fixed at a constant positive value $\overline{x} > 0$, determine the possible equilibrium values $\overline{q}_1$ and $\overline{q}_2$ of $q_1(t)$ and $q_2(t)$ respectively.

(b) If the input actually deviates by a small amount $\check{x}(t) = x(t) - \overline{x}$ from its equilibrium value, and if the state variables correspondingly deviate by small amounts $\check{q}_1(t) = q_1(t) - \overline{q}_1$ and $\check{q}_2(t) = q_2(t) - \overline{q}_2$ respectively from their equilibrium values, find a *linearized* LTI state-space model that approximately describes how these small deviations are related to each other. In other words, find an LTI state-space model that has $\check{q}_1(t)$ and $\check{q}_2(t)$ as state variables, and $\check{x}(t)$ as input. Is your linearized model asymptotically stable? [In deriving the linearized model, it may help you to recall that $(1+\epsilon)^3 \approx 1 + 3\epsilon$ for $|\epsilon| \ll 1$.]

Problem 3 (7 points)

Consider our standard system for DT processing of CT signals, where the C/D converter samples the continuous-time signal $x_c(t)$ with a sampling interval of T seconds, while the ideal D/C converter at the output produces a bandlimited interpolation of the samples $y[n]$ using a reconstruction interval of T seconds.



Suppose the LTI discrete-time system between these two converters is a *notch* filter, i.e. has a frequency response $H(e^{j\Omega})$ whose value is 0 at $\Omega = \pm\Omega_o$ (where $\Omega_o > 0$ is termed the "notch frequency") and whose value is nonzero everywhere else in the interval $|\Omega| < \pi$. Suppose the input signal is of the form

$$x_c(t) = \cos(\omega_{\text{in}} t + \theta) \ .$$

Determine **ALL** — and we mean **all**! — values of ω_{in} for which the output $y_c(t)$ will be identically 0.