Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 1, Spring 2003

Question Booklet

- This quiz is **closed book**, but **two** "crib" sheets are allowed.
- Put your name on **each** sheet of the answer booklet, and your recitation instructor's name and time on the cover page of that booklet.
- **Only** the answer booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to the answer booklet the work you would like us to look at. Let us know if you need additional scratch paper.
- The quiz will be graded out of **50 points**. The **three problems** are nominally **weighted as indicated** (but our legal department wishes to let you know that we reserve the right to modify the weighting *slightly* when we grade, if your collective answers and common errors end up suggesting that a modified weighting would be more appropriate).

Problem 1 (20 points)

Parts (a), (b), (c) and (d) here can be done independently of each other.

(a) (**4 points**) The DTFT of a DT signal $x[n]$ is

$$X(e^{j\Omega}) = e^{j2\Omega}\,(1 - e^{-j3\Omega})\ .$$

Completely specify $x[n]$.

(b) (**4 points**) Suppose we know that the DTFT $X(e^{j\Omega})$ of a particular (deterministic) signal $x[n]$ has a *magnitude* of 2 for $|\Omega| < 0.4\pi$, and unknown magnitude for $0.4\pi \leq |\Omega| \leq \pi$. This signal is applied to the input of an ideal lowpass filter whose frequency response $H(e^{j\Omega})$ is 3 in the interval $|\Omega| < 0.25\pi$ and is 0 for $0.25\pi \leq |\Omega| \leq \pi$. What is the *energy* $\sum y^2[n]$ of the output signal $y[n]$? Remember to show your reasoning.

(c) (**6 points**) Suppose X and Y are zero-mean unit-variance random variables. If the linear minimum-mean-squared-error (LMMSE) estimator $\widehat{Y}(X)$ of Y in terms of X is given by

$$\widehat{Y}(X) = \frac{3}{4}X\ ,$$

what is its mean-squared-error (MSE)? Also, suppose the random variable Q is defined by $Q = Y + 3$; what is the LMMSE estimator $\widehat{Q}(X)$ of Q in terms of X, and what is its MSE? Finally, what is the LMMSE estimator $\widehat{X}(Y)$ of X in terms of Y, and what is its MSE?

(d) (**6 points**) Suppose $x(\,\cdot\,)$ is a wide-sense-stationary (WSS) random process with mean μ_x and autocovariance function $C_{xx}(\tau) = 2e^{-|\tau|}$. The process $x(\,\cdot\,)$ turns out to be ergodic in the mean, i.e., the time average equals the ensemble mean:

$$\lim_{T\uparrow\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \mu_x$$

(For **1 point extra credit**: what feature of our characterization of the process guarantees this ergodicity in the mean?)

If now $y(t) = x(t) + Z$, where Z is a zero-mean random variable with variance σ_Z^2, and Z is independent of the process $x(\,\cdot\,)$, determine the mean μ_y and autocovariance function $C_{yy}(\tau)$ of the process $y(\,\cdot\,)$. Also determine what the time-average

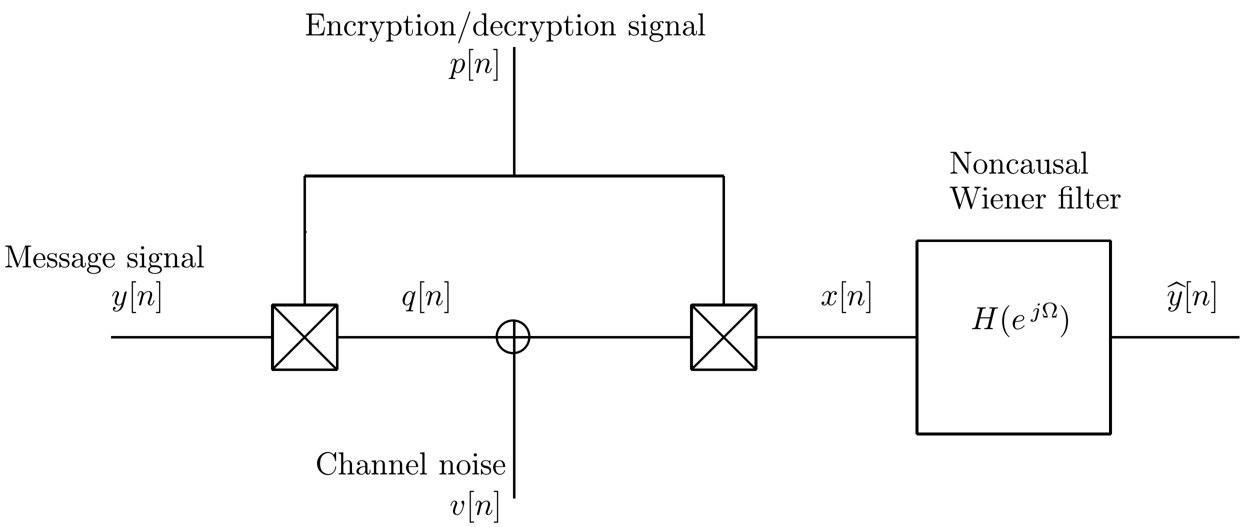
$$\lim_{T\uparrow\infty} \frac{1}{2T}\int_{-T}^{T} y(t)\,dt$$

would be for a general sample function of the process $y(\,\cdot\,)$. Using this result or otherwise, determine if the process $y(\,\cdot\,)$ is ergodic in the mean.

Problem 2 (20 points)

The message signal $y[n]$ in the figure below is to be encrypted and transmitted across a noisy channel, then decrypted and filtered at the receiver. We model $y[n]$ as a wide-sense stationary (WSS) zero-mean random process with autocorrelation function $R_{yy}[m]$ and corresponding power spectral density (PSD) $S_{yy}(e^{j\Omega})$.

The signal $p[n]$ is used for both the encryption at the transmitter and the decryption at the receiver, and is an i.i.d. process that takes the values $+1$ or -1 with equal probability at each time; it is independent of the process $y[\cdot]$. Note that $p^2[n] = 1$ for all n. The transmitted signal $q[n]$ is the product $p[n]y[n]$.



(a) (**8 points**) Determine the respective means μ_p and μ_q of the processes $p[n]$ and $q[n]$, their respective autocorrelations $R_{pp}[m]$ and $R_{qq}[m]$ (expressed in terms of $R_{yy}[\cdot]$), and also the cross-correlation $R_{yq}[m]$ between the message signal and the transmitted signal. Would an intruder who was able to intercept the transmitted process $q[\cdot]$ have any use for a (possibly noncausal) linear estimator of $y[n]$ based on measurements of $q[\cdot]$? Explain your answer.

The channel adds a noise signal $v[n]$ to the transmitted signal, so that the received signal is

$$q[n] + v[n] = p[n]y[n] + v[n] \ .$$

Assume $v[n]$ is a (zero-mean and) white WSS process, with $R_{vv}[m] = \sigma_v^2 \delta[m]$, and suppose it is independent of the processes $y[\cdot]$ and $p[\cdot]$.

We assume, as indicated in the block diagram above, that the intended receiver knows the specific encryption signal $p[n]$ (i.e., the specific sample function from the ensemble that was used for encryption). If there was no channel noise (i.e., if we had $v[n] = 0$), the decryption would then simply involve multiplying the received signal by $p[n]$, because

$$p[n]q[n] = p[n]\left(p[n]y[n]\right) = p^2[n]y[n] = y[n] \ ,$$

where the last equality is a consequence of having $p^2[n] = 1$.

In the presence of noise, we can still attempt to decrypt in the same manner, but will follow it up by a further stage of filtering. The signal to be filtered is thus

$$x[n] = p[n]\Big(p[n]y[n] + v[n]\Big) = y[n] + p[n]v[n] \ .$$

(b) (**5 points**) Determine μ_x, $R_{xx}[m]$, and $R_{yx}[m]$.

(c) (**7 points**) Suppose the filter at the receiver is to be a (stable) *noncausal Wiener filter*, constructed so as to produce the linear minimum-mean-squared-error (LMMSE) estimate $\widehat{y}[n]$ of $y[n]$. Determine the frequency response $H(e^{j\Omega})$ of this filter, and explicitly check that it is what you would expect it to be in the two limiting cases of $\sigma_v^2 = 0$ and $\sigma_v^2 \uparrow \infty$. Also write an expression, in terms of $S_{yy}(e^{j\Omega})$ and σ_v^2, for the mean-squared-error (MSE) obtained with this filter, and explicitly check that it is what you would expect it to be in the preceding two limiting cases.

Problem 3 (10 points)

Suppose the autocorrelation function $R_{xx}[m]$ of a zero-mean wide-sense stationary (WSS) process $x[n]$ has the following Z-transform:

$$S_{xx}(z) = \frac{1}{a(z)a(z^{-1})}$$

where

$$a(z) = z^L + a_1 z^{L-1} + a_2 z^{L-2} + \cdots + a_L$$

is a polynomial of degree L whose roots are all inside the unit circle. We can evidently also write $S_{xx}(z)$ as

$$S_{xx}(z) = \frac{z^L}{a(z)} \frac{z^{-L}}{a(z^{-1})} = \frac{1}{1 + a_1 z^{-1} + \cdots + a_L z^{-L}} \; \frac{1}{1 + a_1 z + \cdots + a_L z^L} \; .$$

(a) (**4 points**) Find the system function $M(z)$ of a stable and causal filter with a stable and causal inverse such that $M(z)M(z^{-1}) = S_{xx}(z)$.

(b) (**6 points**) Find the system function $H_1(z)$ and the corresponding unit-sample response $h_1[n]$ of a (stable) *causal* Wiener filter that uses measurements of $x[\cdot]$ up to and including time n in order to produce the linear minimum-mean-squared-error (LMMSE) estimate of $x[n+1]$ (so the filter is the *one-step Wiener predictor*).

Hint: You may or may not (depending on how you tackle the problem) find it convenient to use the relation

$$\frac{z^{L+1}}{a(z)} = z - \frac{a_1 z^L + a_2 z^{L-1} + \cdots + a_L z}{a(z)} \; ,$$

along with the observation that

$$\frac{a_1 z^L + a_2 z^{L-1} + \cdots + a_L z}{a(z)}$$

has an inverse transform that is a causal and stable (i.e., absolutely summable) signal.

Extra credit: For **2 points** extra credit, find the system function $H_2(z)$ and unit-sample response $h_2[n]$ of the causal *two-step* Wiener predictor for LMMSE estimation of $x[n+2]$ from measurements of $x[\cdot]$ up to and including time n. You may leave your answer in terms of the coefficients $p_1, \cdots, p_L$ defined through the identity below:

$$\frac{z^{L+2}}{a(z)} = z^2 - a_1 z - \frac{p_1 z^L + p_2 z^{L-1} + \cdots + p_L z}{a(z)} \; .$$

(These coefficients $p_1, \cdots, p_L$ can easily be written explicitly in terms of $a_1, \cdots, a_L$, but that's not important to do here.)