MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.011 Introduction to Communication, Control and Signal Processing

Spring 2004

FINAL EXAM

Tuesday, May 18th from 9:00 AM – 12:00 NOON

- This is a closed book exam, but four $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides) are allowed. They can be as big as $8\frac{1}{2}'' \times 11''$ or as small as you'd like and you can write on one side or two sides of each, but only four sheets are allowed.
- Everything on the notes must be in your original handwriting (i.e., material cannot be xeroxed from solutions, tables, books, etc).
- You have three hours for this exam.
- Calculators are NOT allowed.
- We will NOT provide a table of transforms.
- There are 7 problems on the exam with the percentage for each part and the total percentage for each problem as indicated. Note that the problems do not all have the same total percentage.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- Please be neat—we cannot grade what we cannot decipher.
- We tried to provide ample space for you to write in. However, the amount of space provided is not an indication of the length of the explanation required. Short, to the point explanations are preferred to long ones that show little understanding.

All work and answers must be in the space provided on the exam booklet. You are welcome to use scratch pages that we provide but when you hand in the exam we will not accept any pages other than the exam booklet. There will be absolutely no exceptions.

Final Exam Grading

As with the other exams, in grading the final exam we will be focusing on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess, from your work, your level of understanding. On each part of an exam question we will also indicate the percentage of the total exam grade represented by that part, and your numerical score on the exam will then be calculated accordingly.

Our assessment of your level of understanding will be based upon what is given in your solution. A correct final answer with no explanation will not receive full credit, and may even receive no credit at all. An incorrect final answer having a solution and explanation that shows excellent understanding quite likely will receive full (or close to full) credit.

Graded Exams and Final Course Grade:
Graded exams and the final course grade can be picked up after Monday, May 24th, between noon and 3pm. If you would like your graded exam mailed to you, please leave an addressed, stamped envelope with us at the end of the exam. We'll use the envelope as is, so please be sure to address it properly and with enough postage. Also, we'll guarantee that we'll put it into the proper mailbox but won't guarantee anything beyond that. Please look over the grading of the exam before leaving. We will not consider any regrading of the exam once you take it away.

OUT OF CONSIDERATION FOR THE 6.011 STAFF, UNDER NO CIRCUMSTANCES WILL THE GRADE BE AVAILABLE BY PHONE OR EMAIL. PLEASE DON'T EVEN ASK.

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Full Name:____

	Points	Grader
1		
2(a)		
2(b)		
3(a)		
3(b)		
3(c)		
4(a)		
4(b)		
5(a)		
5(b)		
5(c)		
6(a)		
6(b)		
6(c)		
6(d)		
7(a)		
7(b)		
Total		

FOR THE EXAM, YOU MAY FIND SOME, NONE, OR ALL OF THE FOLLOWING USEFUL:

- Parseval's identity:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{<2\pi>} |X(e^{j\Omega})|^2 d\Omega.$$

- Univariate Gaussian PDF: $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2}(\frac{x-m}{\sigma})^2)$

- Two random variables X and Y are said to have a bivariate Gaussian joint PDF if the joint density of the *centered* (i.e. zero-mean) and *normalized* (i.e. unit-variance) random variables

$$V = \frac{X - \mu_X}{\sigma_X}$$

$$W = \frac{Y - \mu_Y}{\sigma_Y}$$

  is given by

$$f_{V,W}(v, w) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp(-\frac{(v^2 - 2\rho vw + w^2)}{2(1-\rho^2)})$$

  Here μ_X and μ_Y are the means of X and Y respectively, and σ_X, σ_Y are the respective standard deviations of X and Y. The number ρ is called the correlation coefficient of X and Y, and is defined by

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \text{ with } \sigma_{XY} = E[XY] - \mu_X \mu_Y$$

  where σ_{XY} is the covariance of X and Y.

- The Cauchy-Schwarz inequality, tells us that the following inequality holds for any two square-integrable functions $a(t)$ and $b(t)$:

$$\left(\int_{-\infty}^{\infty} a(t)b(t)dt\right)^2 \leq \int_{-\infty}^{\infty} a^2(t)dt \int_{-\infty}^{\infty} b^2(t)dt$$

  with equality if and only if $a(t) = kb(t)$ where k is a constant.

- MMSE for two bivariate Gaussian random variables, X and Y:

$$\widehat{Y}_{MMSE}(x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$

- The trigonometric identity:

$$cos(x + y) = cos(x)cos(y) - sin(x)sin(y)$$

Problem 1 (6%)

$x(t)$ and $y(t)$ are two real-valued jointly WSS random processes. The autocorrelation function of $x(t)$ is $R_{xx}(\tau) = e^{-|\tau|}$. State whether or not it is possible to specify a choice for $y(t)$ so that the cross-power spectral density $S_{xy}(j\omega)$ is as shown in Figure 1-1. (Note that the amplitude at $\omega = 1$ is $j = \sqrt{-1}$)

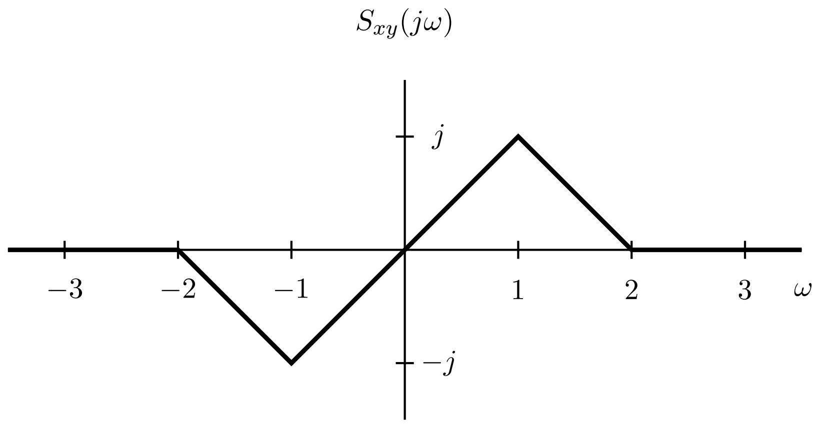


Figure 1-1:

If your answer is no explain why not. If your answer is yes, explain how you might specify or construct $y(t)$.

YES **NO**

Explanation and Reasoning:

Problem 2 (12%)

The signal $x(t)$ is a zero-mean WSS random process with autocorrelation function $R_{xx}(\tau)$. Consider the random process $y(t)$ defined in terms of $x(t)$ as:

$$y(t) = x(t) \cdot \cos(2\pi t + \phi).$$

Specify for the following cases whether or not $y(t)$ is WSS. Clearly justify your answers in a few lines. You may find the following trigonometric identity useful:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

(a) (6%) If $\phi = 0$.

It is WSS **It is NOT WSS**

Explanation and Reasoning:

(b) (6%) If ϕ is equally distributed in the interval $[0, 2\pi]$ and is independent of $x(t)$.

It is WSS **It is NOT WSS**

Explanation and Reasoning:

Problem 3 (15%)

In Figure 3-1 we show a PAM system in which the transmitted sequence $a[n]$ is a zero mean WSS Gaussian random sequence with autocorrelation function $R_{aa}[m] = \left(\frac{1}{2}\right)^{|m|}$.

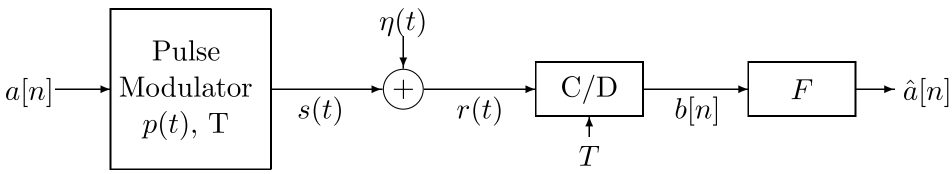


Figure 3-1: PAM System

The channel introduces additive noise $\eta(t)$. The received signal $r(t)$ is sampled to obtain $b[n]$. $b[n]$ is then processed with a memoryless affine system F whose output $\hat{a}[n]$ is an estimate of $a[n]$.

The associated relationships are:

- $s(t) = \sum_{n=-\infty}^{\infty} a[n]p(t - nT)$
- $r(t) = \sum_{n=-\infty}^{\infty} a[n]p(t - nT) + \eta(t)$
- $b[n] = r(nT)$
- $\eta(t)$ is zero-mean WSS noise with autocorrelation function $R_{\eta\eta}(\tau) = Ne^{-|\tau|}$ and is independent of $a[n]$.
- $\hat{a}[n] = k_0 + k_1 b[n]$

(a) (5%) If $p(t)$ is as shown in Figure 3-2 determine whether or not there is ISI present in $r(t)$.

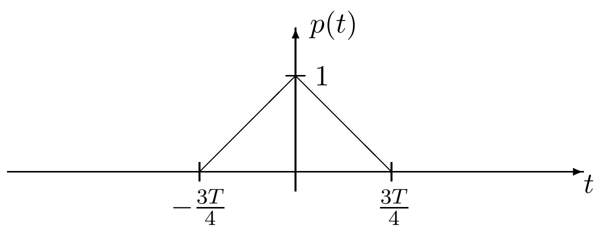


Figure 3-2:

Is there ISI present in $r(t)$?

YES ISI **NO ISI**

Explanation:

(b) (5%) For this part assume that $p(t)$ is chosen so that there is no ISI in $r(t)$, and that $p(0) = 1$. $\hat{a}[n]$, the output of the system F has the form, $\hat{a}[n] = k_0 + k_1 b[n]$. Determine k_0 and k_1 to minimize the mean square error ε, given as:

$$\varepsilon = E\left[(a[n] - \hat{a}[n])^2\right].$$

$k_0 =$

$k_1 =$

Work to be looked at:

(c) (5%) For this part assume that $p(t)$ is chosen so that there is no ISI in $r(t)$, and that $p(0) = 1$.You are at the transmitter, therefore know what $a[n]$ is and you are trying to estimate what $b[n]$ will be at the receiver. With $\hat{b}[n]$ denoting the estimate at the transmitter of $b[n]$, determine the estimate $\hat{b}[n]$ that will minimize the mean square error ε_T defined as:

$$\varepsilon_T = E\left[(b[n] - \hat{b}[n])^2\right]$$

$\hat{b}[n] =$

Work to be looked at:

Problem 4 (12%)

Consider the following signal

$$s(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT) \cos \omega_1 t + \sum_{n=-\infty}^{\infty} b_n p(t - nT) \cos \omega_2 t,$$

where $f_1 = \omega_1/2\pi = 1$ kHz and $f_2 = \omega_2/2\pi = 3$ kHz, and

$$p(t) = \mathrm{sinc}(0.5 \cdot 10^3 t),$$

where t is in seconds.
Suppose this signal is transmitted over the channel whose frequency response, $H(j\omega)$ is characterized as below:

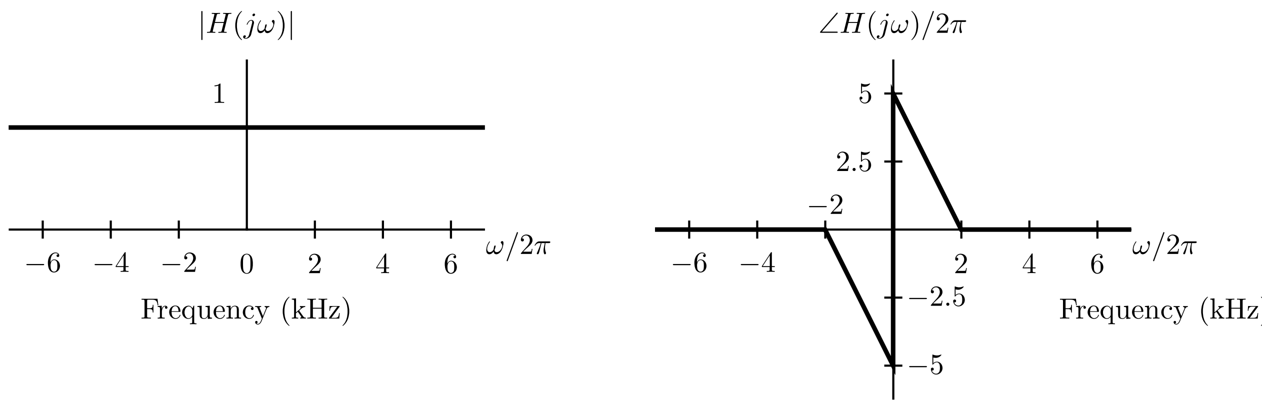
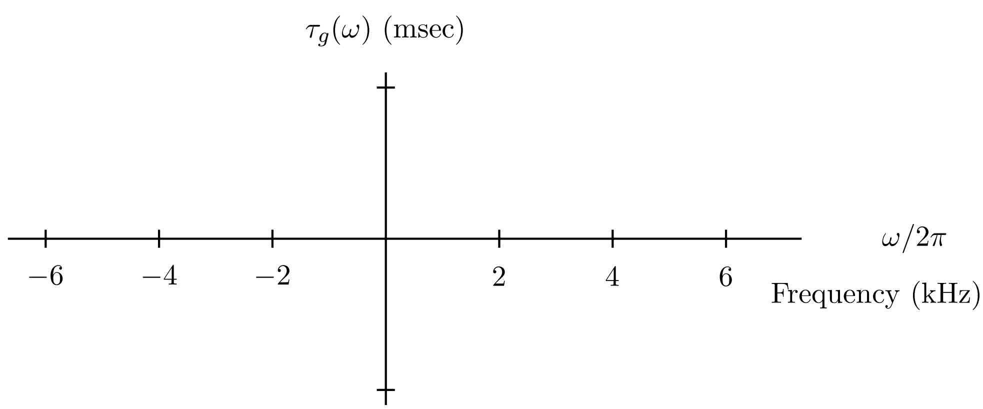


Figure 4-1: Frequency response of $H(j\omega)$

(a) (6%) Sketch the group delay as a function of ω for this channel.



(b) (6%) Give an expression for the channel output, $y(t)$.

$y(t) =$

Work to be looked at:

Problem 5 (16%)

Consider a digital communication system in which an i.i.d. bit stream $s[n]$ of ones and zeros is transmitted over a faulty, memoryless channel. P_0 denotes the probability that a zero is sent and P_1 denotes the probability that a one is sent. Of course $P_1 = 1 - P_0$. The probability of a one being received as a zero is $\frac{1}{4}$ and the probability of a zero being received as a one is $\frac{1}{4}$. We then process the received signal $r[n]$ through a memoryless, possibly nonlinear system H to obtain an estimate $\hat{s}[n]$ of $s[n]$ from $r[n]$. The overall system is depicted in Figure 5-1.

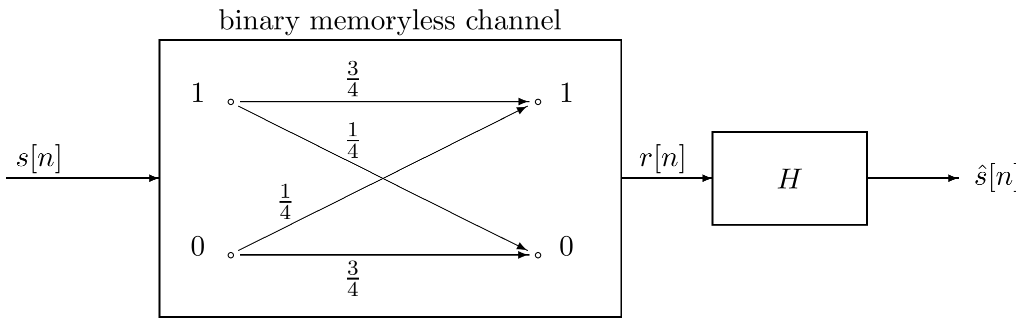


Figure 5-1:

(a) (6%) Determine H in terms of P_0 so that the error probability P_e is minimized, where P_e is defined as the probability that $\hat{s}[n]$ is **not** equal to $s[n]$ at a given time index n.

H:

Work to be looked at:

(b) (6%) Determine all the possible values of P_0 for which P_e will be minimized using the system in part (a).

$P_0 =$

Work to be looked at:

(c) (4%) In this part assume that the system H has been designed for us and according to the manufacturer it has $P_M = \frac{1}{10}$ and the receiver operating characteristic (ROC) specified by:

$$\text{ROC:} \quad P_D = (P_{FA})^{\frac{1}{10}}$$

where

$$\begin{aligned} P_D &= Prob\,(\text{declare that a "one" was sent} \mid \text{a "one" was sent}) \\ P_{FA} &= Prob(\text{declare that a "one" was sent} \mid \text{a "zero" was sent}) \\ P_M &= Prob(\text{declare that a "zero" was sent} \mid \text{a "one" was sent}). \end{aligned}$$

The overall system in Figure 5-1 can be represented as a new binary memoryless channel as depicted in figure 5-2. Determine the new probabilities P_a, P_b, P_c, and P_d.

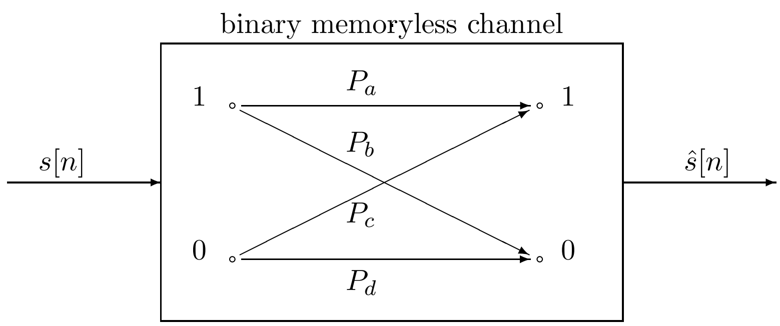


Figure 5-2:

$P_a =$ $P_b =$ $P_c =$ $P_d =$

Work to be looked at:

Work to be looked at:

Problem 6 (24%)

Consider the system L specified by

$$\frac{d\mathbf{q}(t)}{dt} = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$
$$y(t) = \mathbf{c}^T\mathbf{q}(t),$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{c}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}.$$

It is driven by a WSS process $x(t)$ with $\mu_x = 0$ and $R_{xx}(\tau) = \sigma_x^2\delta(\tau)$. We would like to estimate the states of the system $\mathbf{q}(t)$ from the corrupted output $z(t)$. A WSS sensor noise $v(t)$ has $\mu_v = 0$ and $R_{vv}(\tau) = \sigma_v^2\delta(\tau)$ and is independent of $x(t)$.

Consider the case when an observer defined below is used to estimate the states:

$$\frac{d\widehat{\mathbf{q}}_{\text{obs}}(t)}{dt} = \mathbf{A}\widehat{\mathbf{q}}_{\text{obs}}(t) + \mathbf{b}x(t) - \mathbf{l}(y(t) + v(t) - \widehat{y}(t))$$
$$\hat{y}(t) = \mathbf{c}^T\widehat{\mathbf{q}}_{\text{obs}}(t),$$

where $\mathbf{l}^T = [l_1 \; l_2]$, and the state reconstruction error, $\widetilde{\mathbf{q}}_{\text{obs}}(t)$ is defined as follows:

$$\widetilde{\mathbf{q}}_{\text{obs}}(t) = \mathbf{q}(t) - \widehat{\mathbf{q}}_{\text{obs}}(t).$$

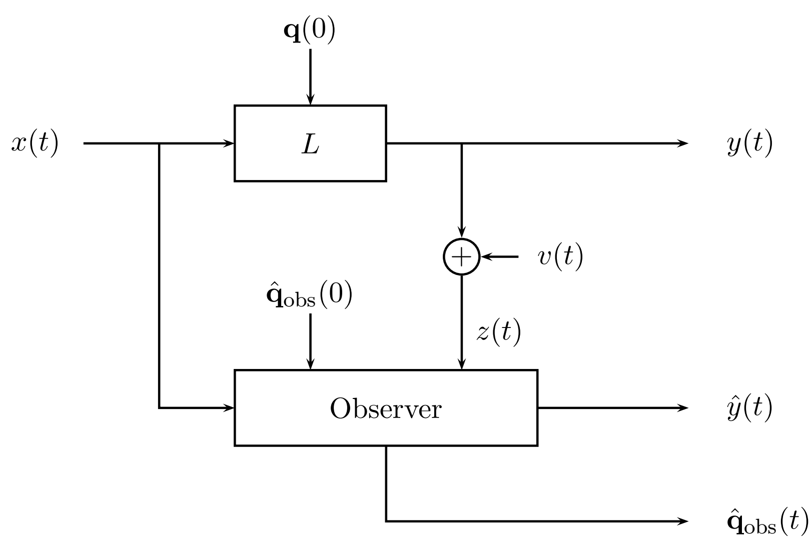


Figure 6-1: Observer based system

(a) (5%) Suppose that $\sigma_x^2 \neq 0$, $\sigma_v^2 = 0$ and $\widehat{\mathbf{q}}_{\text{obs}}(0) = \mathbf{q}(0)$. For $\mathbf{l}^T = [-2\ 0]$, determine $\widetilde{\mathbf{q}}_{\text{obs}}(t)$.

$\widetilde{\mathbf{q}}_{\text{obs}}(t) =$

Work to be looked at:

(b) (5%) Suppose that $\sigma_x^2 \neq 0$, $\sigma_v^2 \neq 0$ and $\hat{\mathbf{q}}_{\text{obs}}(0) \neq \mathbf{q}(0)$. The observer state reconstruction error dynamics can be written in the following form:

$$\frac{d\widetilde{\mathbf{q}}_{\text{obs}}(t)}{dt} = \widetilde{\mathbf{A}}\widetilde{\mathbf{q}}_{\text{obs}}(t) + \mathbf{p}(t).$$

With $\mathbf{l}^T = [-2\ 0]$, determine $\widetilde{\mathbf{A}}$ and $\mathbf{p}(t)$.

$\widetilde{\mathbf{A}} =$ $\mathbf{p}(t) =$

Work to be looked at:

Now consider the case when a noncausal Wiener filter is used to estimate the state $q_2(t)$ from the signal $z(t)$. $H(s)$ is the ZSR of the system L, i.e., the transfer function from $x(t)$ to $y(t)$. We assume that the system L is in steady state, i.e. that it is characterized by its ZSR system function $H(s)$. The estimated state is denoted by $\widehat{q}_{2,\text{wiener}}(t)$ and the estimation error is defined as follows:

$$\widetilde{q}_{2,\text{wiener}}(t) = q_2(t) - \widehat{q}_{2,\text{wiener}}(t).$$

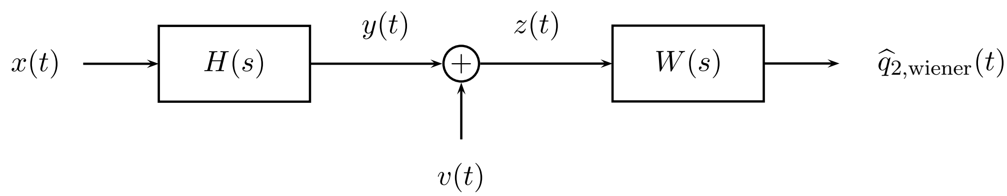


Figure 6-2: Noncausal Wiener filter based system

(c) (7%) Determine $S_{q_1 q_2}(j\omega)$, the cross power spectrum between $q_1(t)$ and $q_2(t)$ in terms of σ_x^2 or σ_v^2.

$S_{q_1 q_2}(j\omega) =$

Work to be looked at:

(d) (7%) Determine the frequency response $W(j\omega)$ of the noncausal Wiener filter $W(s)$ that minimizes $E\{\widetilde{q}^2_{2,\text{wiener}}(t)\}$. Give your answer in terms of σ_x^2 or σ_v^2.

$W(j\omega) =$

Work to be looked at:

Problem 7 (15%)

In Figure 7-1 we show a PAM system in which the transmitted sequence $a[n] = A\delta[n]$, where A is some non-zero constant.

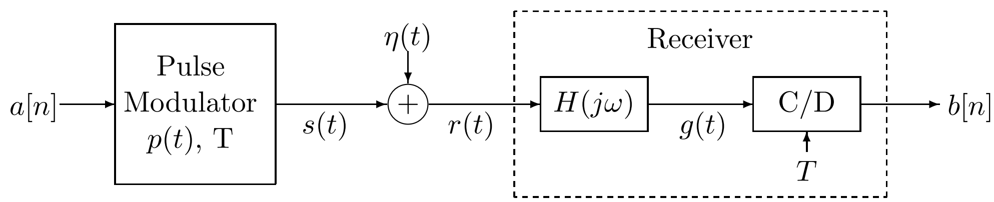


Figure 7-1: PAM system with receiver filter $h(t)$

$P(j\omega)$, the Fourier transform of $p(t)$, is shown in Figure 7-2.

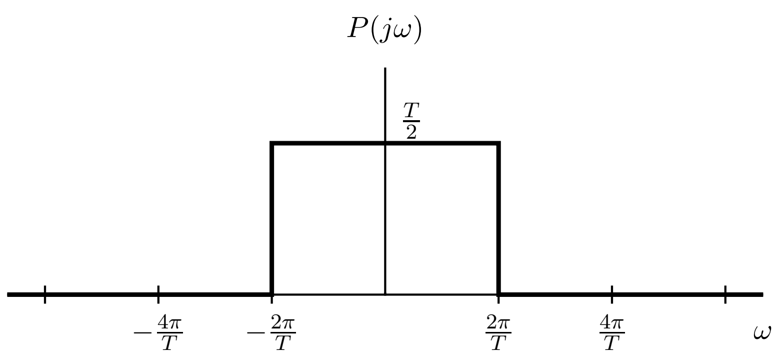


Figure 7-2:

(a) (7%) $\eta(t)$ is a WSS Gaussian random process with autocorrelation function $R_{\eta\eta} = \sigma^2\delta(\tau)$. We can express $b[0]$ as

$$b[0] = b_a[0] + b_\eta[0]$$

where $b_a[0]$ is the value of $b[0]$ if the noise $\eta(t)$ were 0.

Determine a choice for $H(j\omega)$ so that $b_a[0] = a[0]$ and $E\left\{(b_\eta[0])^2\right\}$ is minimized.

$H(j\omega) =$

Work to be looked at:

(b) (8%) If $\eta(t)$ is a zero-mean WSS random process with power spectral density $S_{\eta\eta}(j\omega)$ given in Figure 7-3.

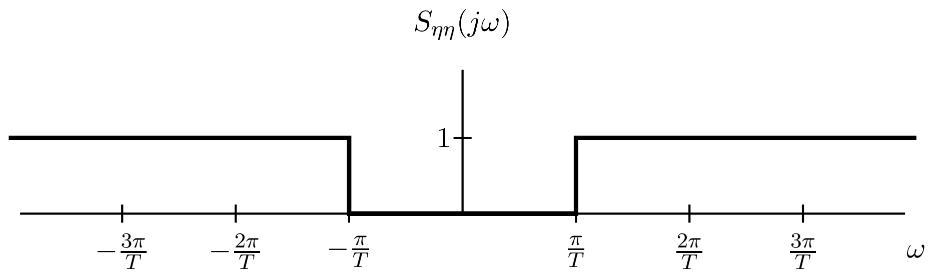


Figure 7-3:

Determine a choice for $H(j\omega)$ so that $b_a[0] = a[0]$ and $E\left\{(b_\eta[0])^2\right\}$ is minimized.

$H(j\omega) =$

Work to be looked at:

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