Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control, and Signal Processing

Final Exam — Question Booklet

Tuesday, May 20, 2003

Your Full Name:	
Recitation Instructor & Time:	

- Write your name in the space above, because we will be collecting your question booklet in addition to you answer booklet but will return it to you when you collect your answer booklet.
- This is a closed book exam, but you may use 6 sheets of notes (both sides).
- Calculators are allowed, but probably will not be useful.
- There are **five problems** on the exam, with point distributions of **15, 20, 25, 15, 25**, for a total of **100 points**. We have also marked for each problem a rough guide to the number of **minutes** you would spend on the problem if you were dividing your time according to the points on the problem. The time you choose to spend may differ, of course, based on which problems you are more or less comfortable with. However, **budget your time carefully**; if you are stuck on a problem, try and move on to another part or another problem, then return if you have time later. Avoid spending inordinately longer than the prorated time on any particular problem.
- Be clear and precise in your reasoning and **show all relevant work.** The grade on each problem is based on our best assessment of your level of understanding, as reflected by what you have written. A correct final answer does not guarantee full credit; nor does an incorrect final answer necessarily mean loss of all credit.
- **If we can't read it, we can't/won't grade it!** So please write neatly and legibly.
- You are to **hand in only the ANSWER booklet** that is supplied. **No additional pages will be considered in the grading.** You may want to first work things through in the blank areas of this question booklet or on scratch paper, and then neatly transfer to the answer booklet the work you would like us to look at. Let us know if you need additional scratch paper.

Problem 1 [15 points, 20 minutes]

Provide brief answers, with appropriate justifications or computations, for each of the following parts.

(a) Can a causal LTI state-space system that is *not* asymptotically stable become asymptotically stable under appropriate LTI state feedback even if some of its modes are *unreachable*?

Answer: Yes, as long as unreachable modes are asymptotically stable. All reachable modes can be shifted by state feedback.

(b) Consider a continuous-time zero-mean wide-sense-stationary (WSS) random process $x(t)$ whose power spectral density $S_{xx}(j\omega)$ is *constant* at the value $N_0 > 0$ for frequencies $|\omega| < \omega_m$ and 0 outside this band. By considering the autocorrelation function $R_{xx}(\tau)$ of this process, one can see that for an appropriately chosen T the samples $x(nT)$ of the process constitute a discrete-time zero-mean WSS *white* process. Determine an appropriate T, and specify what the variance of the corresponding $x(nT)$ is.

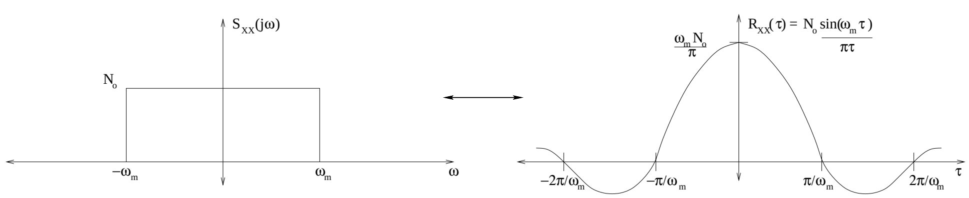

Figure 1: The power spectral density and autocorrelation function of $x(t)$.

Answer: Figure 1 shows the PSD and autocorrelation function of $x(t)$. Samples taken $T = \frac{\pi}{\omega_m}$ apart, or at any integer multiple of this, have zero mean and are uncorrelated. Thus, the variance of the corresponding samples $x(nT)$ is $var[x(nT)] = R_{xx}(0) = \frac{w_n N_o}{\pi}$.

(c) Consider a minimum-error-probability hypothesis test to select between two Gaussians of the same variance σ^2 but different means μ_1 and $\mu_2 > \mu_1$, based on a single measurement drawn from one of these densities with equal probability. Use the standard $Q(x)$ function to express the conditional probability of correctly detecting the Gaussian of mean μ_2, and also the conditional probability of falsely detecting this Gaussian. Recall that

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \ .$$

Answer: The optimum test is a threshold test on the measurement for this case of 2 equal-variance Gaussians. Since the a priori probabilites are equal, the threshold is just where the conditional densities cross, which is the halfway point between their means: $\frac{\mu_2+\mu_1}{2}$.

The conditional probability of correctly detecting the Gaussian of mean $\mu 2$ can be expressed as:

$$P_D = Q(\frac{\frac{\mu_1+\mu_2}{2} - \mu_2}{\sigma}) = Q(\frac{\mu_1 - \mu_2}{2\sigma})$$

The conditional density of falsely detecting this Gaussian is:

$$P_{FA} = Q(\frac{\frac{\mu_1+\mu_2}{2} - \mu_1}{\sigma}) = Q(\frac{\mu_2 - \mu_1}{2\sigma})$$

(d) If pulse-amplitude modulation is used to communicate a general discrete-time (DT) signal over a continuous-time lowpass channel whose transmission is bandlimited to $|\omega| < \omega_m$, what is the highest symbol repetition rate that can be used before information about the DT signal is lost in the process?

Answer: If $a[n]$ is the sequence to be transmitted, $p(t)$ is the pulse shape used, $h(t)$ is the impulse response of the channel, and $r(t)$ is the received signal, we can express the transform of $r(t)$ as:

$$R(j\omega) = A(e^{j\Omega})\Big|_{\Omega=\omega T} P(j\omega)H(j\omega)$$

Since $A(e^{j\Omega})$ has information for $|\Omega| < \pi$, we need to pass frequencies in the band $|\omega| < \frac{\pi}{T}$, so $\omega_m = \frac{\pi}{T}$. Thus the highest symbol repetition rate that can be used without any loss of information of $a[n]$ is $\frac{1}{T} = \frac{\omega_m}{\pi}$.

Problem 2 [20 points, 40 minutes]

Suppose the zero-mean wide-sense stationary (WSS) process $x[n]$ is obtained by applying a zero-mean WSS white process $w[n]$ with power spectral density (PSD) $S_{ww}(e^{j\Omega}) = \sigma^2$ to the input of a (stable, causal) filter with system function $M(z) = 1 - 3z^{-1}$.

(a) If $S_{xx}(e^{j\Omega})$ denotes the PSD of $x[n]$, find $S_{xx}(z)$. Also find the autocovariance function $C_{xx}[m]$ of the process $x[n]$, the variance of the random variable $x[n+1]$, and the correlation coefficient ρ between $x[n]$ and $x[n+1]$.

Answer: $S_{xx}(z)$ can be expressed as:

$$\begin{aligned} S_{xx}(z) &= M(z)M(z^{-1})S_{ww}(z) \\ &= \sigma^2(1-3z^{-1})(1-3z) \\ &= \sigma^2(-3z+10-3z^{-1}) \end{aligned}$$

Taking the inverse transform of $S_{xx}(z)$ produces the autocorrelation function $R_{xx}[m]$. We know that $E(x[n]) = 0$ since $E(w[n]) = 0$; this gives $C_{xx}[m] = R_{xx}[m]$. Thus, $C_{xx}[m] = \sigma^2(-3\delta[m+1] + 10\delta[m] - 3\delta[m-1])$.

The variance of $x[n+1]$ is the value of the autocovariance function evaluated at a lag of zero. Thus, $var(x[n+1]) = 10\sigma^2$.

The correlation coefficient ρ between $x[n]$ and $x[n+1]$ can be expressed as $\rho = \frac{C_{xx}[1]}{C_{xx}[0]} = \frac{-3\sigma^2}{10\sigma^2} = -0.3$.

(b) Specify the linear minimum mean-square-error (LMMSE) estimator of $x[n+1]$ based on a measurement of $x[n]$, and compute the associated mean-square-error (MSE). Is this MSE less than the variance of $x[n+1]$ that you computed in (a)?

Answer: Since $x[n+1]$ and $x[n]$ are both zero-mean and have the same variance, The LMMSE estimate can be expressed as:

$$\begin{aligned} \hat{x}[n+1] &= \rho x[n] \\ &= -0.3x[n] \end{aligned}$$

Thus, the associated MSE is just:

$$\begin{aligned} MSE &= E((x[n+1]-\hat{x}[n+1])^2) \\ &= 10\sigma^2(1-\rho^2) \\ &= 10\sigma^2(1-\frac{9}{100}) \\ &= 9.1\sigma^2 \end{aligned}$$

The MSE is less than the variance computed in 2(a) since we have taken advantage of the knowledge of the correlated variable x[n].

(c) Find the system function $F(z)$ of a stable and causal filter whose inverse $1/F(z)$ is also stable and causal, and such that

$$S_{xx}(z) = F(z)F(z^{-1}) \; .$$

Be careful with this part, because errors here will propagate to the next part!

Answer: From the result in part (a), we can factor $S_{xx}(z)$ as:

$$\begin{aligned} S_{xx}(z) &= \sigma^2(1-3z^{-1})(1-3z) \\ &= \sigma^2 z(1-3z^{-1})z^{-1}(1-3z) \\ &= \sigma^2(z-3)(z^{-1}-3) \end{aligned}$$

Having written it in this form, we notice that it has poles at 0 and ∞, and zeros at 3 and $\frac{1}{3}$. Picking $F(z)$ to be stable and causal (and have a stable and causal inverse), we need its pole at 0 and its zero at $\frac{1}{3}$. Thus, $F(z) = \pm\sigma(z^{-1}-3)$.

(d) Find the system function of the causal Wiener filter that generates an estimate of $x[n+1]$ based on *all past* $x[k]$, $k \le n$, i.e., find the system function of the one-step predictor. Do you expect that the MSE for this case will be less than or equal to or greater than what you computed in (b)? (You don't need to actually determine the MSE; we are only interested in what your intuition is for the situation.)

Answer: We thus choose $y[n] = x[n+1]$, so that the causal Wiener filter will produce an estimate $\hat{y}[n]$ based on all past values of $x[n]$. We note that:

$$\begin{aligned} R_{yx}[m] &= E(y[n+m]x[n]) \\ &= E(x[n+m+1]x[n]) \\ &= R_{xx}[m+1] \end{aligned}$$

Taking the z-transform of both sides results in $S_{yx}(z) = zS_{xx}(z)$. The transfer function $H(z)$ of the causal Wiener filter is given by:

$$H(z) = \frac{1}{F(z)}\left[\frac{S_{yx}(z)}{F(z^{-1})}\right]_+$$

Substituting the corresponding values results in:

$$\begin{aligned} H(z) &= \frac{1}{F(z)}\left[\frac{zS_{xx}(z)}{F(z^{-1})}\right]_+ \\ &= \frac{1}{\pm\sigma(z^{-1}-3)}\left[\pm z\sigma(z^{-1}-3)\right]_+ \\ &= \frac{1}{z^{-1}-3} \\ H(z) &= -\frac{1}{3}\frac{z}{z-\frac{1}{3}} \end{aligned}$$

Since more information is being used in this case than in 2(b), we can expected the MSE to be less. We can recall from Equation (L10-8) from Lecture 10 that the MSE increases from its value in the noncausal case (which, for a prediction problem is 0!) by the following amount:

$$\Delta MMSE = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left[\frac{S_{yx}(e^{j\Omega})}{F^*(e^{j\Omega})}\right]_-\right|^2 d\Omega$$

which is the energy of the strictly anticausal part. In this case, we have $\left[\frac{S_{yx}(e^{j\Omega})}{F^*(e^{j\Omega})}\right]_- = 3\sigma z$, or $3\sigma e^{j\Omega}$ for z on the unit circle (which is what the integral above considers). Hence, we get $MMSE = 9\sigma^2$, which is not a big improvement over $9.1\sigma^2$ obtained in 2(b).

Problem 3 [25 points, 45 minutes]

A particular causal first-order discrete-time LTI system is governed by a model in state-space form:

$$q[n+1] = 3q[n] + x[n] + d[n]$$

where $x[n]$ is a *known* control input and $d[n]$ is an *unknown* wide-sense-stationary (WSS), zero-mean, white-noise disturbance input with $E(\, d^2[n]\,) = \sigma_d^2$. We would like to use an *observer* to construct an estimate $\widehat{q}[n]$ of $q[n]$, using the noisy output measurements

$$y[n] = 2q[n] + v[n] \ ,$$

where the measurement noise $v[n]$ is also an *unknown* WSS, zero-mean, white-noise process with $E(\, v^2[n]\,) = \sigma_v^2$. Assume the measurement noise is uncorrelated with the system disturbance: $E(\, v[n]d[k]\,) = 0$ for all n, k.

(a) Specify which of the following equations you would implement as your (causal) observer, explaining your reasoning. In each case, ℓ denotes the observer gain.

(i) $\widehat{q}[n+1] = 3\widehat{q}[n] + x[n] + d[n] - \ell\,(\, y[n] - 2\widehat{q}[n] - v[n]\,)$.
(ii) $\widehat{q}[n+1] = 3\widehat{q}[n] + x[n] - \ell\,(\, y[n] - 2\widehat{q}[n] - v[n]\,)$.
(iii) $\widehat{q}[n+1] = 3\widehat{q}[n] + x[n] - \ell\,(\, y[n] - 2\widehat{q}[n]\,)$.
(iv) $\widehat{q}[n+1] = 3\widehat{q}[n] - \ell\,(\, y[n] - 2\widehat{q}[n]\,)$.
(v) $\widehat{q}[n+1] = 3\widehat{q}[n] - \ell\,(\, y[n] - 2\widehat{q}[n] - v[n]\,)$.
(vi) Something other than the above (specify).

Answer: We pick choice (iii), since we do not know $d[n]$ nor $v[n]$, but know $x[n]$.

(b) Obtain a state-space model for the observer error, $\widetilde{q}[n] = q[n] - \widehat{q}[n]$, writing it in the form

$$\widetilde{q}[n+1] = \alpha\,\widetilde{q}[n] + p[n] \ ,$$

with α and $p[n]$ expressed in terms of the parameters and signals specified in the problem statement (but with $p[n]$ not involving $\widetilde{q}[n]$, of course). Check: If you have done things correctly, you should find that $\alpha = 0$ when $\ell = -\frac{3}{2}$.

Answer: Substituting the expression for $q[n]$ above and our choice for the observer in 3(a), we can express the state-space model for the observer error as:

$$\begin{aligned}\widetilde{q}[n+1] &= 3\widetilde{q}[n] + d[n] + \ell(2\widetilde{q}[n] + v[n]) \\ &= (3+2\ell)\widetilde{q}[n] + d[n] + \ell v[n]\end{aligned}$$

We can easily see that $\alpha = 3 + 2\ell$ and $p[n] = d[n] + \ell v[n]$. Substituting $\ell = -\frac{3}{2}$ in the former gives $\alpha = 0$.

(c) Determine the system function of the error system in (b) and the corresponding impulse response, i.e., find the system function and corresponding impulse response that relate $\widetilde{q}[n]$ to the input $p[n]$.

Answer: Taking the transform of both sides of the error system, we get:

$$\begin{aligned} zQ(z) &= \alpha Q(z) + P(z) \\ Q(z)(z-\alpha) &= P(z) \\ H(z) &= \frac{Q(z)}{P(z)} = \frac{1}{z-\alpha} \\ &= \frac{z^{-1}}{1-\alpha z^{-1}} \end{aligned}$$

We can see that the system is causal. Taking the inverse Z-transform of the system function, we get the impulse response as $h[n] = \alpha^{n-1}u[n-1]$.

(d) Note that the input process $p[n]$ in (b) is WSS and zero-mean. Determine its autocovariance function $C_{pp}[m]$ in terms of parameters specified in the problem statement.

Answer: We recall that $p[n] = d[n] + \ell v[n]$, and that $d[n]$ and $v[n]$ are uncorrelated. Thus, it directly follows from evaluating $C_{pp}[m] = E(p[n+m]p[n])$ that:

$$\begin{aligned} C_{pp}[m] &= C_{dd}[m] + \ell^2 C_{vv}[m] \\ &= \delta[m](\sigma_d^2 + \ell^2\sigma_v^2) \end{aligned}$$

(e) For $\ell = -\frac{3}{2}$, determine the mean $E(\,\widetilde{q}[n]\,)$ of the observer error, its second moment $E(\,\widetilde{q}^2[n]\,)$, and its variance.

Answer: For $\ell = -\frac{3}{2}$, $\alpha = 0$. Evaluating the error model at this value of α reduces it to $\widetilde{q}[n+1] = p[n]$. Solving for the moments of $\widetilde{q}[n]$:

$$\begin{aligned} E(\widetilde{q}[n]) &= E(p[n-1]) = E(d[n]) + \ell E(v[n]) = 0 + 0 = 0 \\ E(\widetilde{q}^2[n]) &= E(p^2[n-1]) = C_{pp}[0] = \sigma_d^2 + \ell^2\sigma_v^2 = \sigma_d^2 + \frac{9}{4}\sigma_v^2 \\ var(\widetilde{q}[n]) &= \sigma_d^2 + \frac{9}{4}\sigma_v^2 \end{aligned}$$

We note that the variance is the same as the second moment in this case since $\widetilde{q}[n]$ is zero-mean.

(f) If we no longer fix ℓ to have the value specified in (e), what constraints must ℓ satisfy if the observer error $\widetilde{q}[n]$ is to be a zero-mean WSS process (assuming the observer has been running since the beginning of time, i.e., starting infinitely far in the past)? Verify that the choice of ℓ in (e) satisfies the constraints that you specify here.

Answer: We need to choose ℓ such that the error system is stable. That is, we want the (single) eigenvalue λ to satisfy $|\lambda| < 1$, where $\lambda = \alpha$ for this error system. This translates to:

$$-1 < 3 + 2\ell < 1$$

$$-2 < \ell < -1$$

We note that $\ell = -\frac{3}{2}$ satisfies this constraint.

(g) Assume the constraints on ℓ that you specified in (f) are satisfied and that the observer has been running since the beginning of time. Find a general expression for the mean $E(\,\widetilde{q}[n]\,)$ of the observer error, its second moment $E(\,\widetilde{q}^2[n]\,)$, and its variance. You might find it helpful to recall that

$$\sum_{k=0}^{\infty} \alpha^{2k} = \frac{1}{1-\alpha^2}\ .$$

Answer: With ℓ satisfying the preceding constraints, what we have is a zero-mean WSS process $p[n]$ going through a stable LTI filter with unit sample response $h[n]$ and output $\widetilde{q}[n]$. Hence, it follows that the output is also zero-mean, that is, $E(\widetilde{q}[n]) = 0$. Since it is zero-mean, its second moment and variance are equal. Solving for the two:

$$\begin{aligned}
E(\widetilde{q}^2[n]) &= var(\widetilde{q}[n]) = R_{\widetilde{q}\widetilde{q}}[0] = h * \bar{h} * R_{pp}[0] \\
&= \left(\Sigma_{k=1}^{\infty}\alpha^{2(k-1)}\right)(\sigma_d^2 + \ell^2\sigma_v^2) \\
&= \left(\Sigma_{k=0}^{\infty}\alpha^{2(k)}\right)(\sigma_d^2 + \ell^2\sigma_v^2) \\
&= \frac{(\sigma_d^2 + \ell^2\sigma_v^2)}{1-\alpha^2} \\
&= \frac{(\sigma_d^2 + \ell^2\sigma_v^2)}{1-(3+2\ell)^2}
\end{aligned}$$

We can check that for $\ell = -\frac{3}{2}$, we get $var(\widetilde{q}[n]) = \sigma_d^2 + \ell^2\sigma_v^2$, as in 3(e).

(h) Given your error variance expression in (g), you could in principle try and choose the value of ℓ that *minimizes* this error variance, but the calculations are messy in the general case. Instead, to simply get an idea of the possibilities, evaluate your expression here for the case $\sigma_d^2 = 0$ and $\ell = -\frac{4}{3}$, and show that the error variance in this case is smaller that what you get (still for $\sigma_d^2 = 0$) with the earlier choice in (e) of $\ell = -\frac{3}{2}$.

Answer: Substituting $\sigma_d^2 = 0$ and $\ell = -\frac{4}{3}$:

$$\begin{aligned} var(\widetilde{q}[n]) &= \frac{(\sigma_d^2 + \ell^2\sigma_v^2)}{1-(3+2\ell)^2} \\ &= \frac{\ell^2\sigma_v^2}{1-(3+2\ell)^2} \\ &= \frac{\frac{16}{9}\sigma_v^2}{1-(3-\frac{8}{3})^2} \\ &= 2\sigma_v^2 \end{aligned}$$

For the case where $\sigma_d^2 = 0$ and $\ell = -\frac{3}{2}$:

$$\begin{aligned} var(\widetilde{q}[n]) &= \frac{\frac{9}{4}\sigma_v^2}{1-(3-3)^2} \\ &= 2.25\sigma_v^2 \end{aligned}$$

Comparing the two, we can see that the variance is less for $\ell = -\frac{4}{3}$:

$$2\sigma_v^2 < 2.25\sigma_v^2$$

Problem 4 [15 points, 25 minutes]

A discrete-time LTI system has frequency response

$$H(e^{j\Omega}) = e^{-j\Omega/4} \ , \quad |\Omega| < \pi \ .$$

(a) Determine the phase angle $\angle H(e^{j\Omega})$ of the system.

Answer: The phase angle is $\angle H(e^{j\Omega}) = -\frac{\Omega}{4}$ for $|\Omega| < \pi$, and repeats periodically with period 2π outside this interval.

(b) If $h[n]$ denotes the unit-sample response of the system, evaluate the following three expressions (you should be able to do this without first explicitly computing $h[n]$):

$$h[0] \ ; \qquad \sum_{n=-\infty}^{\infty} h[n] \ ; \qquad \sum_{n=-\infty}^{\infty} (h[n])^2 \ .$$

Answer: Given the system function, we can determine the value for $h[0]$ by evaluating the synthesis equation for $h[n]$ at $n = 0$:

$$\begin{aligned} h[0] &= \frac{1}{2\pi} \int_{-pi}^{\pi} e^{-j\frac{\Omega}{4}} d\Omega \\ &= \frac{1}{2\pi} \frac{e^{-j\frac{\Omega}{4}}}{-\frac{j}{4}} \Bigg|_{-\pi}^{\pi} \\ &= \frac{4}{\pi} \frac{e^{+j\frac{\pi}{4}} + e^{-j\frac{\pi}{4}}}{2j} \\ &= \frac{4}{\pi} sin(\frac{\pi}{4}) \end{aligned}$$

Similarly, we can find the value for $\sum_{n=-\infty}^{\infty} h[n]$ by evaluating the analysis equation at $\Omega = 0$:

$$\sum_{n=-\infty}^{\infty} h[n] = H(e^{j0}) = e^{-j\frac{0}{4}} = 1$$

Finally, we can use Parseval's theorem to compute for the energy of the impulse response of the filter, $\sum_{n=-\infty}^{\infty} (h[n])^2$:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (h[n])^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| e^{-j\frac{\Omega}{4}} \right|^2 d\Omega \\ &= \frac{1}{2\pi} (2\pi) \\ &= 1 \end{aligned}$$

(c) Determine $h[n]$, and check that your answer yields the value of $h[0]$ you computed in (b). Is the system causal or not?

Answer: The system is a quarter-sample delay, so the impulse response is:

$$h[n] = \frac{sin(\pi(n - \frac{1}{4}))}{\pi(n - \frac{1}{4})}$$

The system is not causal since $h[n] \neq 0$ for $n < 0$. We can check that at $n = 0$, $h[0] = \frac{sin(-\frac{\pi}{4})}{-\frac{\pi}{4}} = \frac{4}{\pi} sin(\frac{\pi}{4})$, which is the same as in 4(b).

(d) If the input sequence $x[n]$ to the above system is given by

$$x[n] = 2\frac{\sin(\pi n/2)}{\pi n} \quad \text{for } n \neq 0 , \qquad x[0] = 1 , \tag{1}$$

find a simple and explicit expression for the output $y[n]$ of the system. (Just expressing $y[n]$ via a convolution sum will *not* do!)

Answer: Again, since the system is a quarter-sample delay, the output $y[n]$ will just be:

$$y[n] = 2\frac{\sin(\pi(n - \frac{1}{4})/2)}{\pi(n - \frac{1}{4})}$$

Problem 5 [25 points, 45 minutes]

A signal $X[n]$ that we will be measuring for $n = 1, 2$ is known to be generated according to one of the following two hypotheses:

$$\begin{aligned} H_{no}: \quad X[n] &= W[n] \\ H_{yes}: \quad X[n] &= s[n] + W[n] \end{aligned}$$

where $s[1]$, $s[2]$ are specified (deterministic) numbers, with $0 < s[i] \leq 1$ for $i = 1, 2$, and where $W[1]$, $W[2]$ are i.i.d. random variables *uniformly distributed* in the interval $[-1, 1]$ (and hence with mean 0 and variance $\frac{1}{3}$). Given measurements $x[1]$ and $x[2]$ of $X[1]$ and $X[2]$ respectively, we would like to decide between the hypotheses H_{no} and H_{yes}.

Most of part (c) below can be done independently of parts (a) and (b).

(a) One strategy for processing the measurements is to only look at a linear combination of the measurements, of the form

$$r = g[1]x[1] + g[2]x[2] \ .$$

To analyze decision schemes that are based on consideration of the number r, consider the random variable

$$R = g[1]X[1] + g[2]X[2] \ .$$

Determine the mean and variance of R under each of the hypotheses, and note that the variance does not depend on which hypothesis applies. (Note: you do *not* need to find the densities of R under the two hypotheses in order to find these conditional means and variances!)

Now choose $g[1]$ and $g[2]$ to maximize the *relative* distance between these means, where "relative" signifies that the distance is to be measured relative to the standard deviation of R under hypothesis H_{no} (or, equivalently, under H_{yes}). Equivalently, maximize the following "signal-to-noise" ratio (SNR):

$$\frac{\Big(E[R|H_{yes}] - E[R|H_{no}]\Big)^2}{\text{variance}(R|H_{no})} \ .$$

Answer: Under H_{no}, we get $R = g[1]W[1] + g[2]W[2]$. Thus, it follows that:

$$\begin{aligned} E(R|H_{no}) &= g[1]E(W[1]) + g[2]E(W[2]) = 0 \\ var(R|H_{no}) &= \frac{1}{3}(g^2[1] + g^2[2]) \end{aligned}$$

Under H_{yes}, we have $R = g[1](s[1] + W[1]) + g[2](s[2] + W[2])$. Evaluating its mean, we get:

$$\begin{aligned} E(R|H_{yes}) &= g[1](s[1] + E(W[1])) + g[2](s[2] + E(W[2])) \\ &= g[1]s[1] + g[2]s[2] \end{aligned}$$

The variance in this case is the same as in H_{no} since the noise components are the same. Substituting the values we found above in the SNR expression results in:

$$\begin{aligned} SNR &= \frac{\left(g[1]s[1] + g[2]s[2] - 0\right)^2}{\frac{1}{3}(g^2[1] + g^2[2])} \\ &= \frac{<\mathbf{g}, \mathbf{s}>^2}{\frac{1}{3} <\mathbf{g}, \mathbf{g}>} \end{aligned}$$

where $\mathbf{g} = \begin{bmatrix} g[1] \\ g[2] \end{bmatrix}$, $\mathbf{s} = \begin{bmatrix} s[1] \\ s[2] \end{bmatrix}$ and $< \cdot, \cdot >$ is the inner product operation.
From the Cauchy-Schwartz inequality, it can be shown that:

$$<\mathbf{g}, \mathbf{s}>^2 \quad \leq \quad <\mathbf{g}, \mathbf{g}><\mathbf{s}, \mathbf{s}>$$

where equality holds if and only if $\mathbf{g} = c\mathbf{s}$ for some constant c (and we need $c \neq 0$ to have nontrivial processing). Essentially, as the result shows, the vector $\mathbf{g}$ is a matched filter for $\mathbf{s}$.

(b) In the particular case where $s[1] = s[2] = 1$, which we shall focus on for the rest of this problem, it turns out that the choice $g[1] = g[2] = c$ will serve, for any nonzero constant c, to maximize the SNR in (a). Taking $c = 3$, draw fully labeled sketches of the conditional densities of R under each of the hypotheses, i.e., sketches of $f_{R|H}(r|H_{no})$ and $f_{R|H}(r|H_{yes})$.

Suppose now that the prior probabilities on the two hypotheses are $p(H_{no}) = \frac{2}{3}$ and hence $p(H_{yes}) = \frac{1}{3}$. Specify a decision rule that, on the basis of knowledge that $R = r$, decides between H_{no} and H_{yes} with minimum probability of error. Also compute the probability of error associated with this decision rule. (It will probably help you to shade on the appropriate sketch the regions corresponding to the conditional probability of a false "yes" and to the conditional probability of a false "no".)

Answer: Under H_{no}, we have $R = 3W[1] + 3W[2]$. Thus, the (conditional) density of R is derived by convolving the densities of the scaled random variables $3W[1]$ and $3W[2]$.

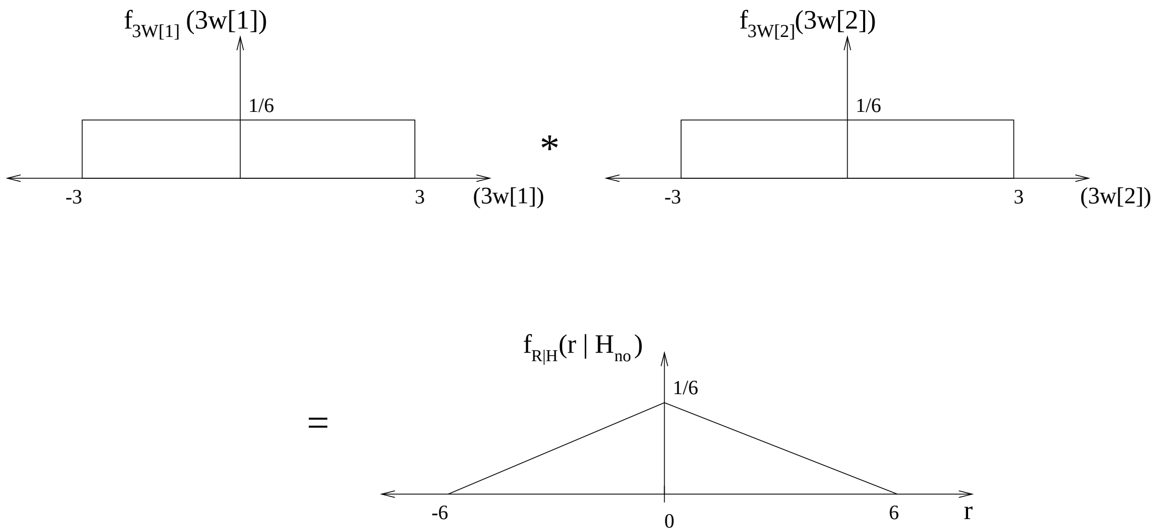

Figure 2: The probability density function of R under H_{no}.

Under hypothesis H_{yes}, we have $R = 3(1+W[1])+3(1+W[2]) = 6+(3W[1]+3W[2])$. Thus, the pdf of R will be the same as in H_{no} except that its mean is shifted by 6.

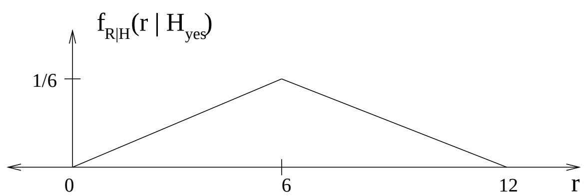

Figure 3: The probability density function of R under H_{yes}.

With the a priori probabilities specified as $p(H_{no}) = \frac{2}{3}$ and $p(H_{yes}) = \frac{1}{3}$, we can use the likelihood ratio test and compare $p(H_{no})f_{R|H}(r|H_{no})$ with $p(H_{yes})f_{R|H}(r|H_{yes})$. The superimposed densities (normalized by the a priori probabilities) are plotted in Figure 3.

In order to find the decision rule, we need to find the r where the lines $\frac{\frac{1}{18}r}{6}$ and $-\frac{\frac{1}{9}(r-6)}{6}$, which represent $p(H_{yes})f_{R|H}(r|H_{yes})$ with $p(H_{no})f_{R|H}(r|H_{no})$, respectively, intersect. Solving

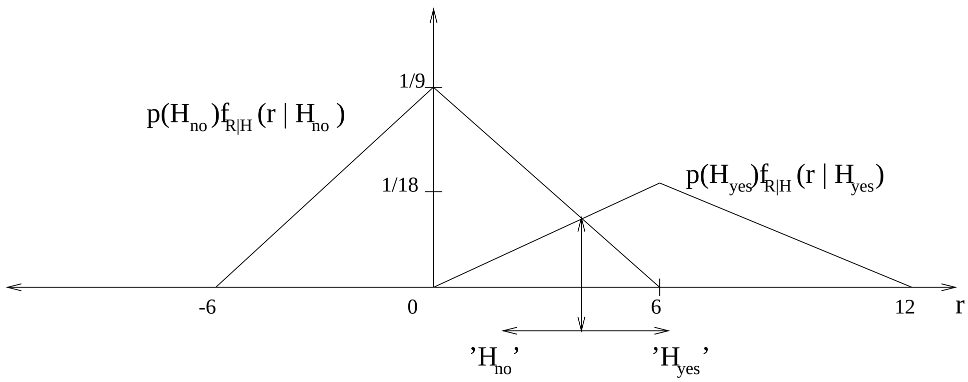

Figure 4: The superimposed (scaled) densities for R under H_{yes} and H_{no}.

for this r gives us:

$$\begin{aligned} \frac{\frac{1}{18}r}{6} &= -\frac{\frac{1}{9}(r-6)}{6} \\ \frac{1}{2}r &= 6-r \\ r &= 4 \end{aligned}$$

Formalizing this result, the decision rule is as follows:

$$r \gtrless_{'H'_{no}}^{'H'_{yes}} 4$$

The associated probability of error with this decision rule can be expressed as:

$$\begin{aligned} P_e &= p(H_{no})p(false\ 'yes') + p(H_{yes})p(false\ 'no') \\ &= \frac{2}{3}\frac{1}{18} + \frac{1}{3}\frac{2}{9} \\ &= \frac{1}{9} \end{aligned}$$

Graphically, the probability of error can be shown as the sum of the shaded areas under the conditional densities in Figure 4.

(c) If we did *not* hastily commit ourselves to working with a scalar measurement obtained by taking a linear combination of the measurements $x[1]$ and $x[2]$, we might perhaps have done better. Accordingly, first sketch or fully describe the conditional densities

$$f_{X[1],X[2]|H}\Big(x[1], x[2] \mid H_{no}\Big) \qquad \text{and} \qquad f_{X[1],X[2]|H}\Big(x[1], x[2] \mid H_{yes}\Big)$$

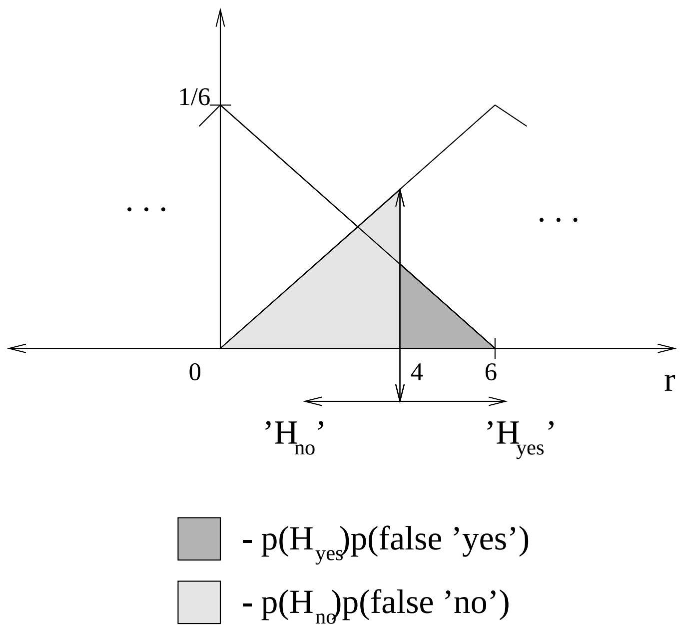

Figure 5: The probability of error under the decision rule is the sum of the shaded areas.

for the case where $s[1] = s[2] = 1$. Then specify a decision rule that will pick between the two hypotheses with minimum probability of error, on the basis of knowledge that $X[1] = x[1]$ and $X[2] = x[2]$, and still with the prior probabilities specified in (b), namely $p(H_{no}) = \frac{2}{3}$ and $p(H_{yes}) = \frac{1}{3}$. Determine the probability of error associated with this decision rule, and compare with your result in (b).

Answer: Since $X[1]$ and $X[2]$ under each hypothesis are independent and have the same uniform distribution, their (conditional) joint densities are also uniform with a height of $\frac{1}{4}$, as shown in Figure 5.

With the a priori probabilities as $p(H_{no}) = \frac{2}{3}$ and $p(H_{yes}) = \frac{1}{3}$ as in 5(b), we now compare the scaled densities $\frac{2}{3} f_{X[1],X[2] \big| H}(x[1], x[2] \big| H_{no})$ and $\frac{1}{3} f_{X[1],X[2] \big| H}(x[1], x[2] \big| H_{yes})$. Proceeding in the same manner as in (b), the decision rule with the minimum probability of error can be derived as:

$$max(x[1], x[2]) \gtrless^{'H'_{yes}}_{'H'_{no}} 1$$

With this decision rule, we get $P(false\,'yes') = 0$, while $P(false\,'no') = \frac{1}{4}$. Thus, the associated probability of error is $P_e = \frac{1}{3}\frac{1}{4} = \frac{1}{12}$.

Comparing this result with the one in 5(b), we can see that the probability of error has dropped significantly, as expected, since we have not carried out the preliminary (and in this case, suboptimal) matched filtering.

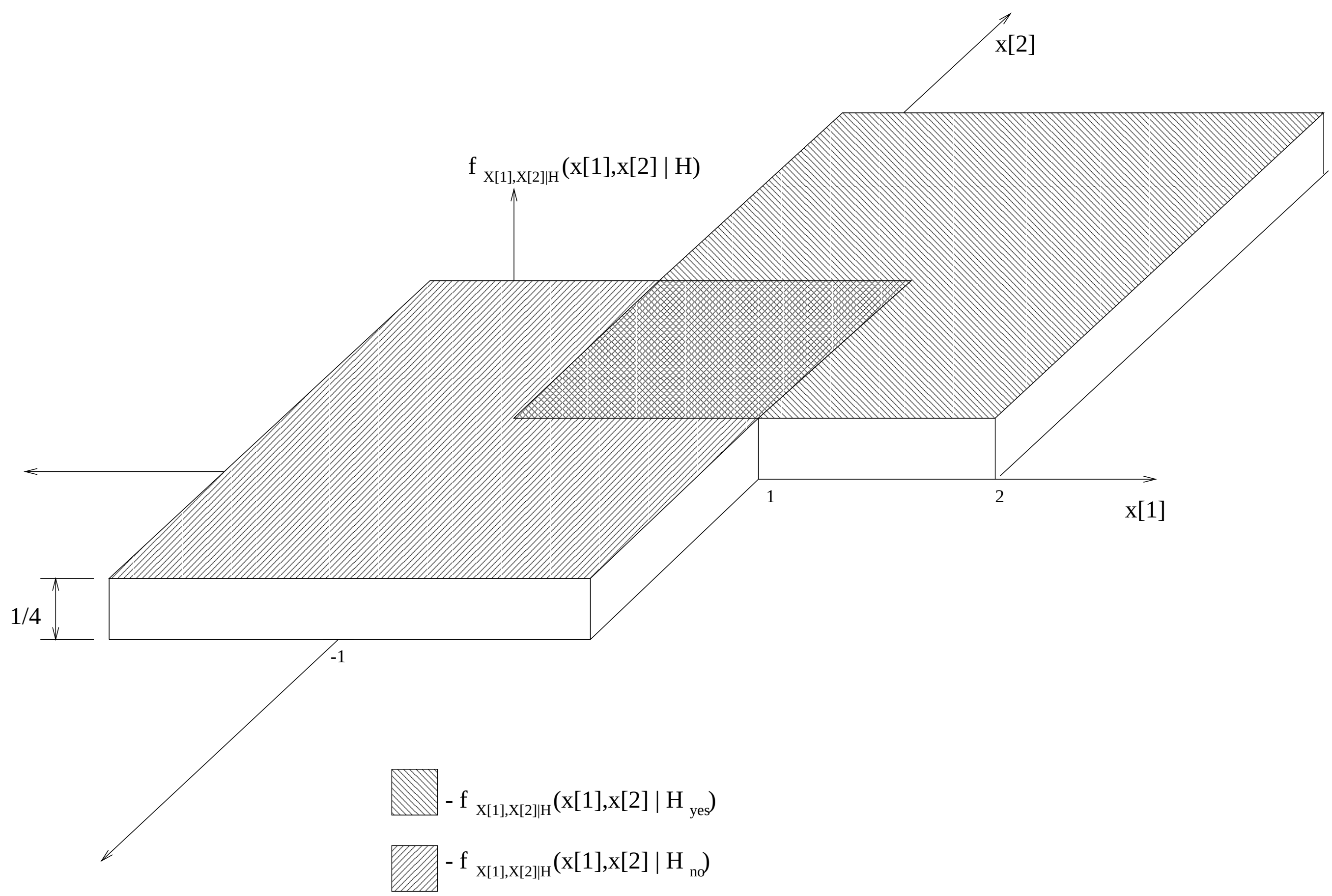

Figure 6: The joint distributions of $X[1]$ and $X[2]$ udner each hypothesis.