

Massachusetts Institute of Technology  
Department of Electrical Engineering and Computer Science

6.011: INTRODUCTION TO COMMUNICATION,  
CONTROL, AND SIGNAL PROCESSING

**Final Exam — Question Booklet**

Tuesday, May 20, 2003

Your Full Name:	
Recitation Instructor & Time:	

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- Write your name in the space above, because we will be collecting your question booklet in addition to you answer booklet but will return it to you when you collect your answer booklet.
  - This is a closed book exam, but you may use 6 sheets of notes (both sides).
  - Calculators are allowed, but probably will not be useful.
  - There are **five problems** on the exam, with point distributions of **15, 20, 25, 15, 25**, for a total of **100 points**. We have also marked for each problem a rough guide to the number of **minutes** you would spend on the problem if you were dividing your time according to the points on the problem. The time you choose to spend may differ, of course, based on which problems you are more or less comfortable with. However, **budget your time carefully**; if you are stuck on a problem, try and move on to another part or another problem, then return if you have time later. Avoid spending inordinately longer than the prorated time on any particular problem.
  - Be clear and precise in your reasoning and **show all relevant work**. The grade on each problem is based on our best assessment of your level of understanding, as reflected by what you have written. A correct final answer does not guarantee full credit; nor does an incorrect final answer necessarily mean loss of all credit.
  - **If we can't read it, we can't/won't grade it!** So please write neatly and legibly.
  - You are to **hand in only the ANSWER booklet** that is supplied. **No additional pages will be considered in the grading**. You may want to first work things through in the blank areas of this question booklet or on scratch paper, and then neatly transfer to the answer booklet the work you would like us to look at. Let us know if you need additional scratch paper.

**Problem 1 [15 points, 20 minutes]**

Provide brief answers, with appropriate justifications or computations, for each of the following parts.

- (a) Can a causal LTI state-space system that is *not* asymptotically stable become asymptotically stable under appropriate LTI state feedback even if some of its modes are *unreachable*?
- (b) Consider a continuous-time zero-mean wide-sense-stationary (WSS) random process  $x(t)$  whose power spectral density  $S_{xx}(j\omega)$  is *constant* at the value  $N_0 > 0$  for frequencies  $|\omega| < \omega_m$  and 0 outside this band. By considering the autocorrelation function  $R_{xx}(\tau)$  of this process, one can see that for an appropriately chosen  $T$  the samples  $x(nT)$  of the process constitute a discrete-time zero-mean WSS *white* process. Determine an appropriate  $T$ , and specify what the variance of the corresponding  $x(nT)$  is.
- (c) Consider a minimum-error-probability hypothesis test to select between two Gaussians of the same variance  $\sigma^2$  but different means  $\mu_1$  and  $\mu_2 > \mu_1$ , based on a single measurement drawn from one of these densities with equal probability. Use the standard  $Q(x)$  function to express the conditional probability of correctly detecting the Gaussian of mean  $\mu_2$ , and also the conditional probability of falsely detecting this Gaussian. Recall that

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt .$$

- (d) If pulse-amplitude modulation is used to communicate a general discrete-time (DT) signal over a continuous-time lowpass channel whose transmission is bandlimited to  $|\omega| < \omega_m$ , what is the highest symbol repetition rate that can be used before information about the DT signal is lost in the process?

**Problem 2 [20 points, 40 minutes]**

Suppose the zero-mean wide-sense stationary (WSS) process  $x[n]$  is obtained by applying a zero-mean WSS white process  $w[n]$  with power spectral density (PSD)  $S_{ww}(e^{j\Omega}) = \sigma^2$  to the input of a (stable, causal) filter with system function  $M(z) = 1 - 3z^{-1}$ .

- (a) If  $S_{xx}(e^{j\Omega})$  denotes the PSD of  $x[n]$ , find  $S_{xx}(z)$ . Also find the autocovariance function  $C_{xx}[m]$  of the process  $x[n]$ , the variance of the random variable  $x[n+1]$ , and the correlation coefficient  $\rho$  between  $x[n]$  and  $x[n+1]$ .
- (b) Specify the linear minimum mean-square-error (LMMSE) estimator of  $x[n+1]$  based on a measurement of  $x[n]$ , and compute the associated mean-square-error (MSE). Is this MSE less than the variance of  $x[n+1]$  that you computed in (a)?
- (c) Find the system function  $F(z)$  of a stable and causal filter whose inverse  $1/F(z)$  is also stable and causal, and such that

$$S_{xx}(z) = F(z)F(z^{-1}) .$$

Be careful with this part, because errors here will propagate to the next part!

- (d) Find the system function of the causal Wiener filter that generates an estimate of  $x[n+1]$  based on *all past*  $x[k]$ ,  $k \leq n$ , i.e., find the system function of the one-step predictor. Do you expect that the MSE for this case will be less than or equal to or greater than what you computed in (b)? (You don't need to actually determine the MSE; we are only interested in what your intuition is for the situation.)

**Problem 3 [25 points, 45 minutes]**

A particular causal first-order discrete-time LTI system is governed by a model in state-space form:

$$q[n + 1] = 3q[n] + x[n] + d[n]$$

where  $x[n]$  is a *known* control input and  $d[n]$  is an *unknown* wide-sense-stationary (WSS), zero-mean, white-noise disturbance input with  $E(d^2[n]) = \sigma_d^2$ . We would like to use an *observer* to construct an estimate  $\hat{q}[n]$  of  $q[n]$ , using the noisy output measurements

$$y[n] = 2q[n] + v[n],$$

where the measurement noise  $v[n]$  is also an *unknown* WSS, zero-mean, white-noise process with  $E(v^2[n]) = \sigma_v^2$ . Assume the measurement noise is uncorrelated with the system disturbance:  $E(v[n]d[k]) = 0$  for all  $n, k$ .

- (a) Specify which of the following equations you would implement as your (causal) observer, explaining your reasoning. In each case,  $\ell$  denotes the observer gain.

(i)  $\hat{q}[n + 1] = 3\hat{q}[n] + x[n] + d[n] - \ell (y[n] - 2\hat{q}[n] - v[n])$  .

(ii)  $\hat{q}[n + 1] = 3\hat{q}[n] + x[n] - \ell (y[n] - 2\hat{q}[n] - v[n])$  .

(iii)  $\hat{q}[n + 1] = 3\hat{q}[n] + x[n] - \ell (y[n] - 2\hat{q}[n])$  .

(iv)  $\hat{q}[n + 1] = 3\hat{q}[n] - \ell (y[n] - 2\hat{q}[n])$  .

(v)  $\hat{q}[n + 1] = 3\hat{q}[n] - \ell (y[n] - 2\hat{q}[n] - v[n])$  .

(vi) Something other than the above (specify).

- (b) Obtain a state-space model for the observer error,  $\tilde{q}[n] = q[n] - \hat{q}[n]$ , writing it in the form

$$\tilde{q}[n + 1] = \alpha \tilde{q}[n] + p[n],$$

with  $\alpha$  and  $p[n]$  expressed in terms of the parameters and signals specified in the problem statement (but with  $p[n]$  not involving  $\tilde{q}[n]$ , of course). Check: If you have done things correctly, you should find that  $\alpha = 0$  when  $\ell = -\frac{3}{2}$ .

- (c) Determine the system function of the error system in (b) and the corresponding impulse response, i.e., find the system function and corresponding impulse response that relate  $\tilde{q}[n]$  to the input  $p[n]$ .
- (d) Note that the input process  $p[n]$  in (b) is WSS and zero-mean. Determine its autocovariance function  $C_{pp}[m]$  in terms of parameters specified in the problem statement.
- (e) For  $\ell = -\frac{3}{2}$ , determine the mean  $E(\tilde{q}[n])$  of the observer error, its second moment  $E(\tilde{q}^2[n])$ , and its variance.

- (f) If we no longer fix  $\ell$  to have the value specified in (e), what constraints must  $\ell$  satisfy if the observer error  $\tilde{q}[n]$  is to be a zero-mean WSS process (assuming the observer has been running since the beginning of time, i.e., starting infinitely far in the past)? Verify that the choice of  $\ell$  in (e) satisfies the constraints that you specify here.
- (g) Assume the constraints on  $\ell$  that you specified in (f) are satisfied and that the observer has been running since the beginning of time. Find a general expression for the mean  $E(\tilde{q}[n])$  of the observer error, its second moment  $E(\tilde{q}^2[n])$ , and its variance. You might find it helpful to recall that

$$\sum_{k=0}^{\infty} \alpha^{2k} = \frac{1}{1 - \alpha^2}.$$

- (h) Given your error variance expression in (g), you could in principle try and choose the value of  $\ell$  that *minimizes* this error variance, but the calculations are messy in the general case. Instead, to simply get an idea of the possibilities, evaluate your expression here for the case  $\sigma_d^2 = 0$  and  $\ell = -\frac{4}{3}$ , and show that the error variance in this case is smaller than what you get (still for  $\sigma_d^2 = 0$ ) with the earlier choice in (e) of  $\ell = -\frac{3}{2}$ .

**Problem 4 [15 points, 25 minutes]**

A discrete-time LTI system has frequency response

$$H(e^{j\Omega}) = e^{-j\Omega/4}, \quad |\Omega| < \pi.$$

- (a) Determine the phase angle  $\angle H(e^{j\Omega})$  of the system.
- (b) If  $h[n]$  denotes the unit-sample response of the system, evaluate the following three expressions (you should be able to do this without first explicitly computing  $h[n]$ ):

$$h[0]; \quad \sum_{n=-\infty}^{\infty} h[n]; \quad \sum_{n=-\infty}^{\infty} (h[n])^2.$$

- (c) Determine  $h[n]$ , and check that your answer yields the value of  $h[0]$  you computed in (b). Is the system causal or not?
- (d) If the input sequence  $x[n]$  to the above system is given by

$$x[n] = 2 \frac{\sin(\pi n/2)}{\pi n} \quad \text{for } n \neq 0, \quad x[0] = 1, \quad (1)$$

find a simple and explicit expression for the output  $y[n]$  of the system. (Just expressing  $y[n]$  via a convolution sum will *not* do!)

**Problem 5 [25 points, 45 minutes]**

A signal  $X[n]$  that we will be measuring for  $n = 1, 2$  is known to be generated according to one of the following two hypotheses:

$$\begin{aligned} H_{no} : \quad X[n] &= W[n] \\ H_{yes} : \quad X[n] &= s[n] + W[n] \end{aligned}$$

where  $s[1], s[2]$  are specified (deterministic) numbers, with  $0 < s[i] \leq 1$  for  $i = 1, 2$ , and where  $W[1], W[2]$  are i.i.d. random variables *uniformly distributed* in the interval  $[-1, 1]$  (and hence with mean 0 and variance  $\frac{1}{3}$ ). Given measurements  $x[1]$  and  $x[2]$  of  $X[1]$  and  $X[2]$  respectively, we would like to decide between the hypotheses  $H_{no}$  and  $H_{yes}$ .

Most of part (c) below can be done independently of parts (a) and (b).

- (a) One strategy for processing the measurements is to only look at a linear combination of the measurements, of the form

$$r = g[1]x[1] + g[2]x[2].$$

To analyze decision schemes that are based on consideration of the number  $r$ , consider the random variable

$$R = g[1]X[1] + g[2]X[2].$$

Determine the mean and variance of  $R$  under each of the hypotheses, and note that the variance does not depend on which hypothesis applies. (Note: you do *not* need to find the densities of  $R$  under the two hypotheses in order to find these conditional means and variances!)

Now choose  $g[1]$  and  $g[2]$  to maximize the *relative* distance between these means, where “relative” signifies that the distance is to be measured relative to the standard deviation of  $R$  under hypothesis  $H_{no}$  (or, equivalently, under  $H_{yes}$ ). Equivalently, maximize the following “signal-to-noise” ratio (SNR):

$$\frac{\left(E[R|H_{yes}] - E[R|H_{no}]\right)^2}{\text{variance}(R|H_{no})}.$$

- (b) In the particular case where  $s[1] = s[2] = 1$ , which we shall focus on for the rest of this problem, it turns out that the choice  $g[1] = g[2] = c$  will serve, for any nonzero constant  $c$ , to maximize the SNR in (a). Taking  $c = 3$ , draw fully labeled sketches of the conditional densities of  $R$  under each of the hypotheses, i.e., sketches of  $f_{R|H}(r|H_{no})$  and  $f_{R|H}(r|H_{yes})$ .

Suppose now that the prior probabilities on the two hypotheses are  $p(H_{no}) = \frac{2}{3}$  and hence  $p(H_{yes}) = \frac{1}{3}$ . Specify a decision rule that, on the basis of knowledge that  $R = r$ , decides between  $H_{no}$  and  $H_{yes}$  with minimum probability of error. Also compute the probability of error associated with this decision rule. (It will probably help you to shade on the appropriate sketch the regions corresponding to the conditional probability of a false “yes” and to the conditional probability of a false “no”.)

- (c) If we did *not* hastily commit ourselves to working with a scalar measurement obtained by taking a linear combination of the measurements  $x[1]$  and  $x[2]$ , we might perhaps have done better. Accordingly, first sketch or fully describe the conditional densities

$$f_{X[1],X[2]|H}(x[1],x[2]|H_{no}) \quad \text{and} \quad f_{X[1],X[2]|H}(x[1],x[2]|H_{yes})$$

for the case where  $s[1] = s[2] = 1$ . Then specify a decision rule that will pick between the two hypotheses with minimum probability of error, on the basis of knowledge that  $X[1] = x[1]$  and  $X[2] = x[2]$ , and still with the prior probabilities specified in (b), namely  $p(H_{no}) = \frac{2}{3}$  and hence  $p(H_{yes}) = \frac{1}{3}$ . Determine the probability of error associated with this decision rule, and compare with your result in (b).