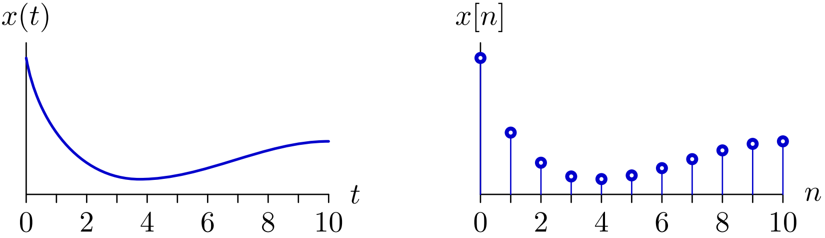
6.003: Signals and Systems

Sampling

November 22, 2011

Sampling

Conversion of a continuous-time signal to discrete time.



We have used sampling a number of times before.

Today: new insights from Fourier representations.

Sampling

Sampling allows the use of modern digital electronics to process, record, transmit, store, and retrieve CT signals.

- audio: MP3, CD, cell phone
- pictures: digital camera, printer
- video: DVD
- everything on the web

Sampling

Sampling is pervasive.

Example: digital cameras record sampled images.

y $I(x, y)$

x

n $I[m, n]$

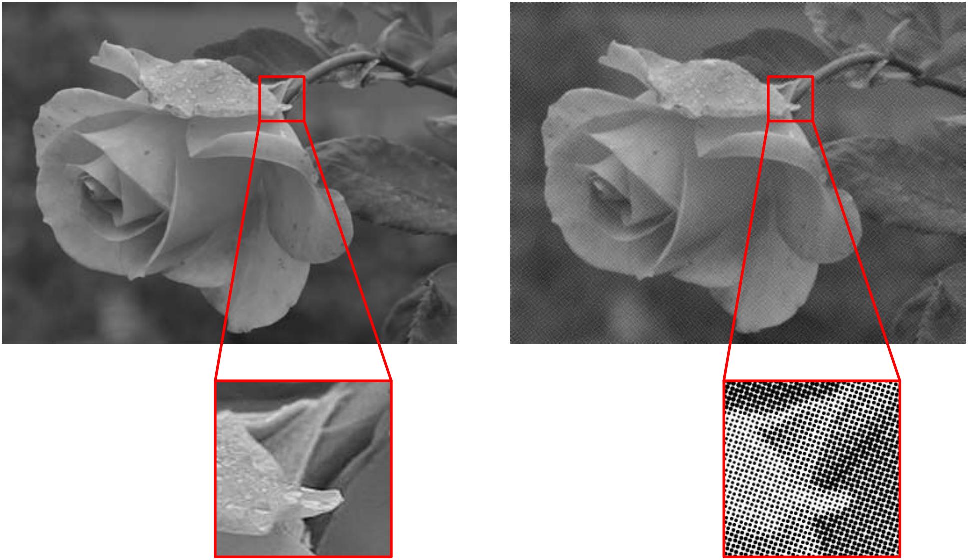
m

Sampling

Photographs in newsprint are “half-tone” images. Each point is black or white and the average conveys brightness.

Sampling

Zoom in to see the binary pattern.



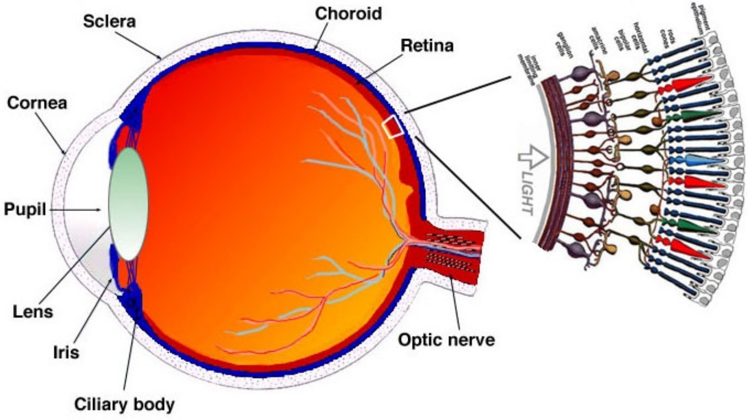
Sampling

Even high-quality photographic paper records discrete images. When AgBr crystals $(0.04 - 1.5\mu\text{m})$ are exposed to light, some of the Ag is reduced to metal. During "development" the exposed grains are completely reduced to metal and unexposed grains are removed.

Images of discrete grains in photographic paper removed due to copyright restrictions.

Sampling

Every image that we see is sampled by the retina, which contains $\approx$ 100 million rods and 6 million cones (average spacing $\approx 3\mu$m) which act as discrete sensors.



Courtesy of Helga Kolb, Eduardo Fernandez, and Ralph Nelson. Used with permission.

http://webvision.med.utah.edu/imageswv/sagschem.jpeg

Check Yourself

Your retina is sampling this slide, which is composed of 1024×768 pixels.

Is the spatial sampling done by your rods and cones adequate to resolve individual pixels in this slide?

Check Yourself

The spacing of rods and cones limits the angular resolution of your retina to approximately

$$\theta_{\text{eye}} = \frac{\text{rod/cone spacing}}{\text{diameter of eye}} \approx \frac{3 \times 10^{-6}\,\text{m}}{3\,\text{cm}} \approx 10^{-4}\,\text{radians}$$

The angle between pixels viewed from the center of the classroom is approximately

$$\theta_{\text{pixels}} = \frac{\text{screen size / 1024}}{\text{distance to screen}} \approx \frac{3\,\text{m}/1024}{10\,\text{m}} \approx 3 \times 10^{-4}\,\text{radians}$$

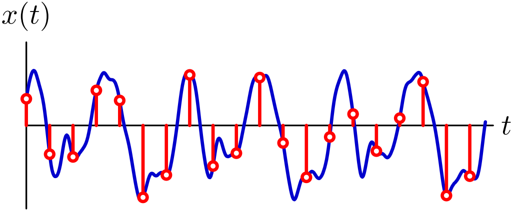
Light from a single pixel falls upon multiple rods and cones.

Sampling

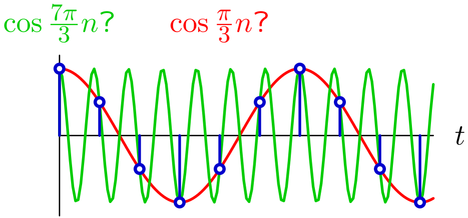
How does sampling affect the information contained in a signal?

Sampling

We would like to sample in a way that preserves information, which may not seem possible.



Information between samples is lost. Therefore, the same samples can represent multiple signals.

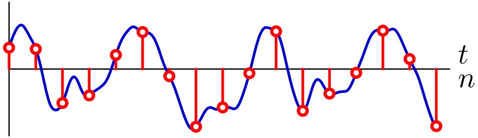


Sampling and Reconstruction

To determine the effect of sampling, compare the original signal $x(t)$ to the signal $x_p(t)$ that is **reconstructed** from the samples $x[n]$.

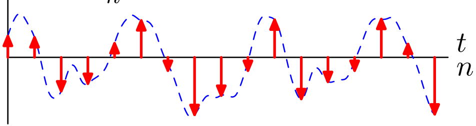
Uniform sampling (sampling interval T).

$$x[n] = x(nT)$$



Impulse reconstruction.

$$x_p(t) = \sum_n x[n]\delta(t - nT)$$



Reconstruction

Impulse reconstruction maps samples $x[n]$ (DT) to $x_p(t)$ (CT).

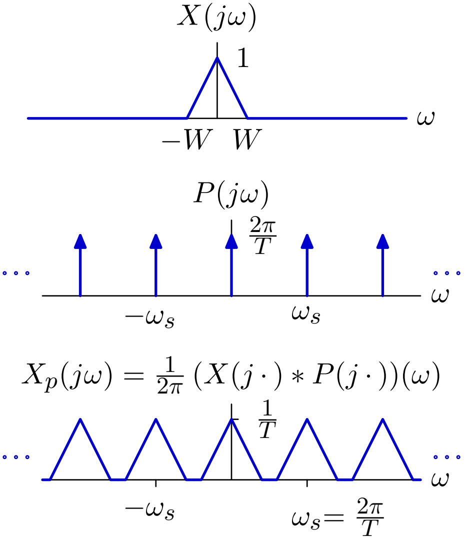
$$\begin{aligned}
x_p(t) &= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) \\
&= x(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\equiv\ p(t)}
\end{aligned}$$

Resulting reconstruction $x_p(t)$ is equivalent to multiplying $x(t)$ by impulse train.

Sampling

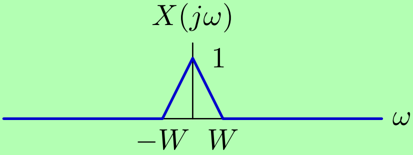
Multiplication by an impulse train in time is equivalent to convolution by an impulse train in frequency.

$\rightarrow$ generates multiple copies of original frequency content.



Check Yourself

What is the relation between the DTFT of $x[n] = x(nT)$ and the CTFT of $x_p(t) = \sum x[n]\delta(t - nT)$ for $X(j\omega)$ below.



1. $X_p(j\omega) = X(e^{j\Omega})|_{\Omega=\omega}$
2. $X_p(j\omega) = X(e^{j\Omega})|_{\Omega=\frac{\omega}{T}}$
3. $X_p(j\omega) = X(e^{j\Omega})|_{\Omega=\omega T}$
4. none of the above

Check Yourself

DTFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

CTFT of $x_p(t)$

$$X_p(j\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)e^{-j\omega t}dt$$

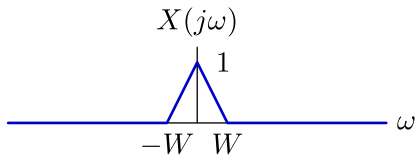
$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT)e^{-j\omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT}$$

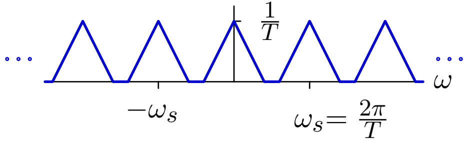
$$= X(e^{j\Omega})\Big|_{\Omega=\omega T}$$

Check Yourself

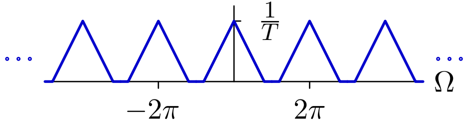
$$X_p(j\omega) = X(e^{j\Omega})\Big|_{\Omega=\omega T}$$



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



$$X(e^{j\Omega}) = X_p(j\omega)\Big|_{\omega=\frac{\Omega}{T}}$$



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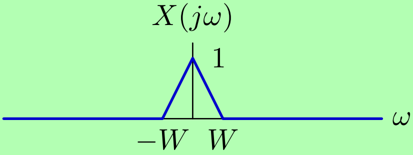

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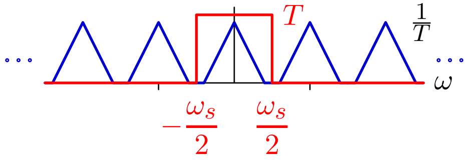
4. none of the above

Sampling

The high frequency copies can be removed with a low-pass filter (also multiply by T to undo the amplitude scaling).

$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Impulse reconstruction followed by ideal low-pass filtering is called **bandlimited reconstruction.**

The Sampling Theorem

If signal is bandlimited $\rightarrow$ sample without loosing information.

If $x(t)$ is bandlimited so that

$$X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_m$$

then $x(t)$ is uniquely determined by its samples $x(nT)$ if

$$\omega_s = \frac{2\pi}{T} > 2\omega_m.$$

The minimum sampling frequency, $2\omega_m$, is called the "Nyquist rate."

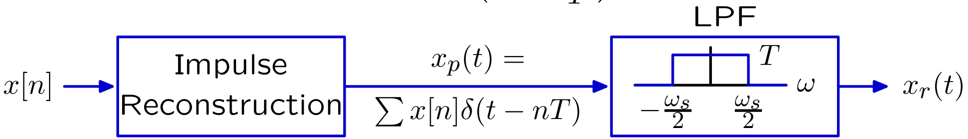
Summary

Three important ideas.

Sampling

$$x(t) \rightarrow x[n] = x(nT)$$

Bandlimited Reconstruction $\left(\omega_s = \dfrac{2\pi}{T}\right)$



Sampling Theorem: If $X(j\omega) = 0 \ \forall \ |\omega| > \dfrac{\omega_s}{2}$ then $x_r(t) = x(t)$.

Check Yourself

We can hear sounds with frequency components between 20 Hz and 20 kHz.

What is the maximum sampling interval T that can be used to sample a signal without loss of audible information?

1. $100\,\mu s$
2. $50\,\mu s$
3. $25\,\mu s$
4. $100\pi\,\mu s$
5. $50\pi\,\mu s$
6. $25\pi\,\mu s$

Check Yourself

$$2\pi f_m = \omega_m < \frac{\omega_s}{2} = \frac{2\pi}{2T}$$

$$T < \frac{1}{2f_m} = \frac{1}{2 \times 20 \text{ kHz}} = 25\,\mu s$$

Check Yourself

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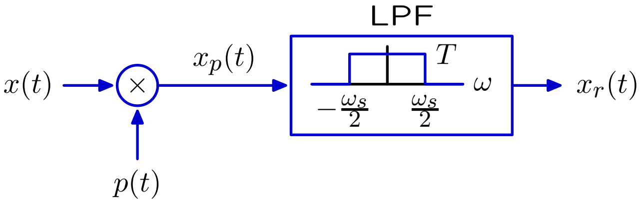
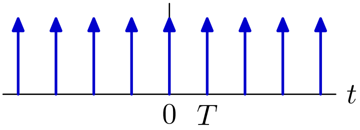
CT Model of Sampling and Reconstruction

Sampling followed by bandlimited reconstruction is equivalent to multiplying by an impulse train and then low-pass filtering.

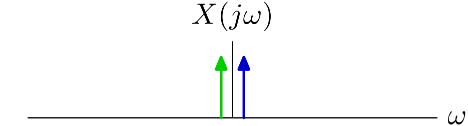


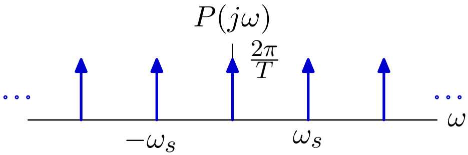
$p(t) =$ "sampling function"



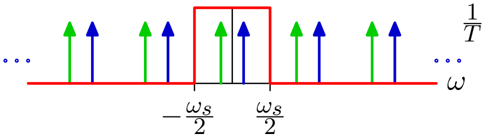
Aliasing

What happens if X contains frequencies $|\omega| > \dfrac{\pi}{T}$?





$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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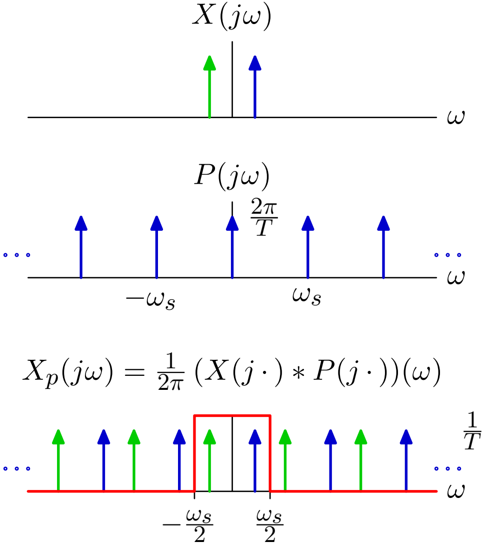

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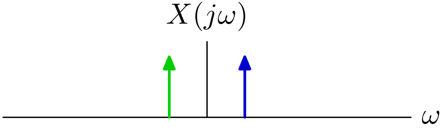

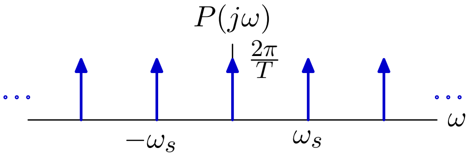

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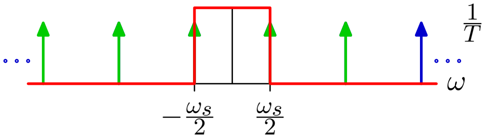

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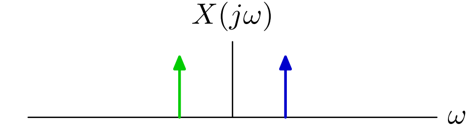

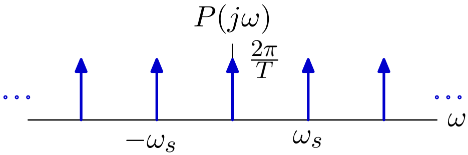

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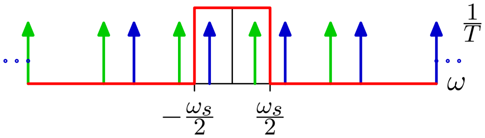
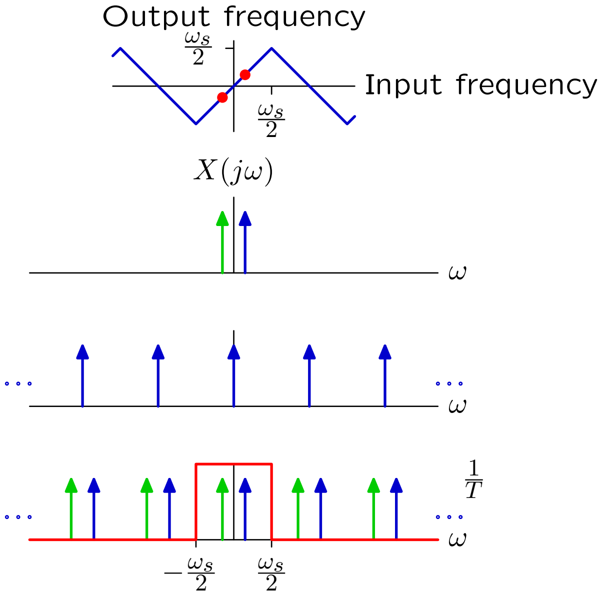


Aliasing

The effect of aliasing is to wrap frequencies.



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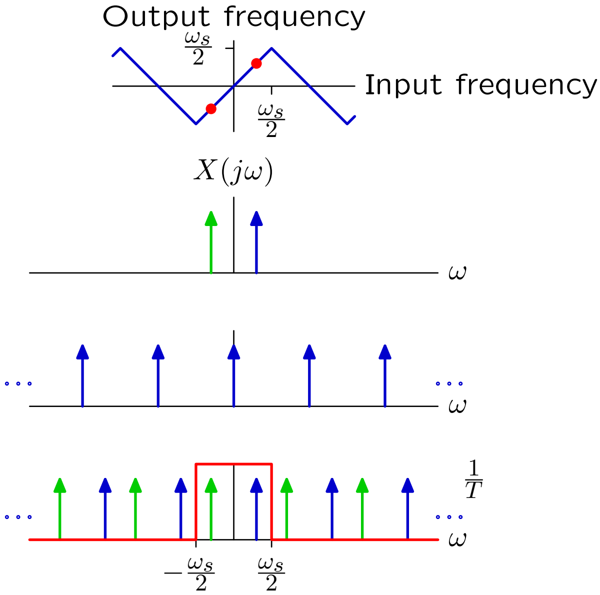

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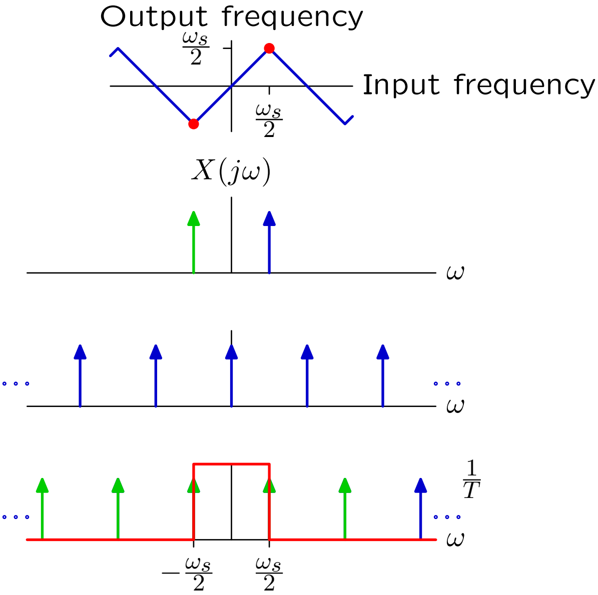

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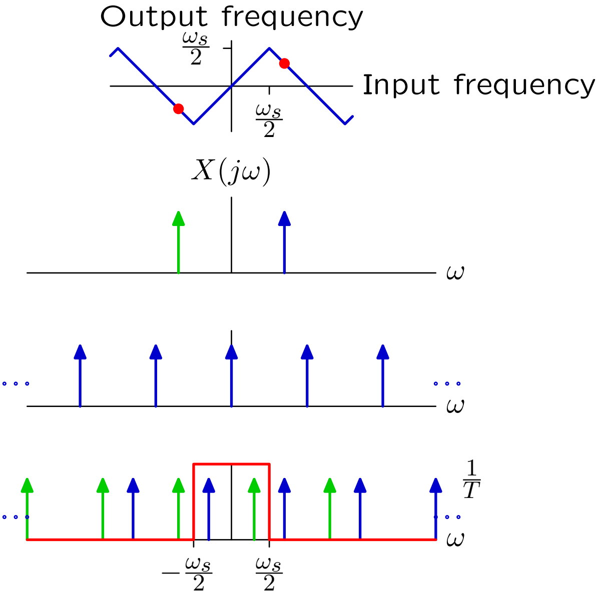
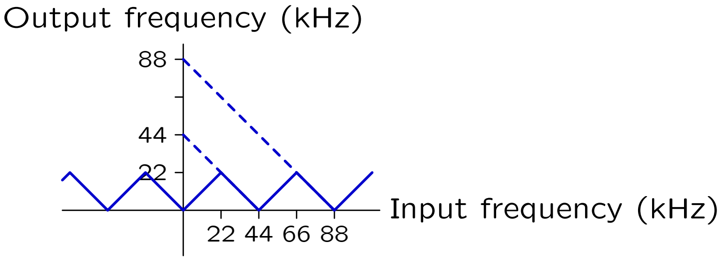


Check Yourself

A periodic signal, period of 0.1 ms, is sampled at 44 kHz. To what frequency does the third harmonic alias?

1. 18 kHz
2. 16 kHz
3. 14 kHz
4. 8 kHz
5. 6 kHz
0. none of the above

Check Yourself



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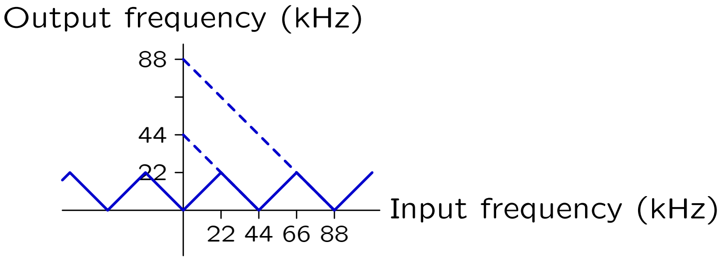

Harmonic	Alias
10 kHz	10 kHz
20 kHz	20 kHz
30 kHz	44 kHz-30 kHz =14 kHz

Check Yourself

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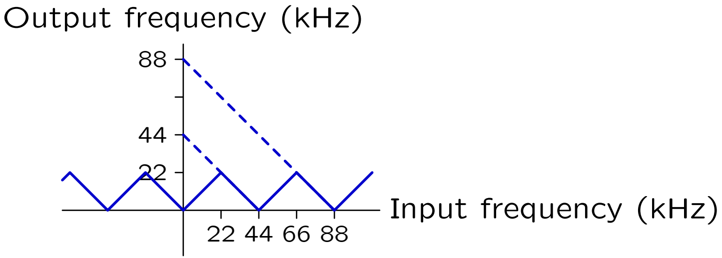
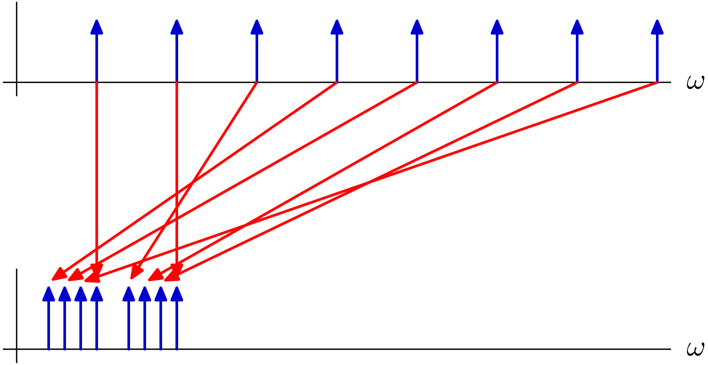


Harmonic	Alias
10 kHz	10 kHz
20 kHz	20 kHz
30 kHz	44 kHz-30 kHz =14 kHz
40 kHz	44 kHz-40 kHz = 4 kHz
50 kHz	50 kHz-44 kHz = 6 kHz
60 kHz	60 kHz-44 kHz =16 kHz
70 kHz	88 kHz-70 kHz =18 kHz
80 kHz	88 kHz-80 kHz = 8 kHz

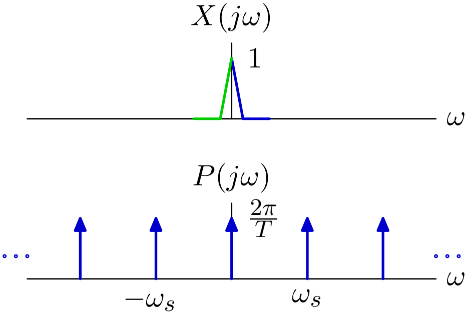
Check Yourself

Scrambled harmonics.

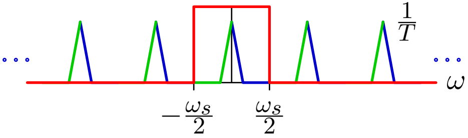


Aliasing

High frequency components of complex signals also wrap.



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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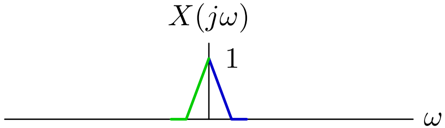

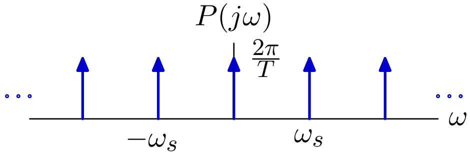

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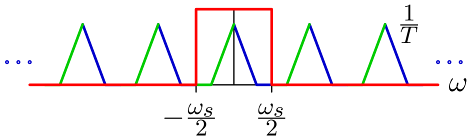

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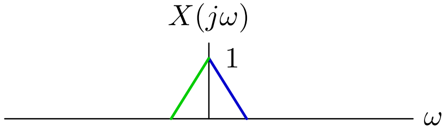

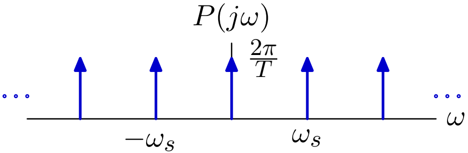

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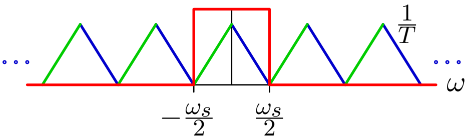

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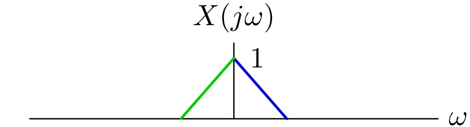

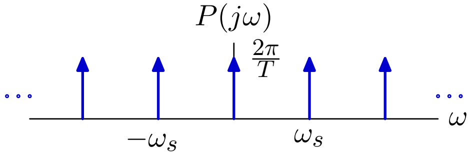

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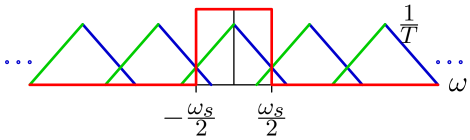
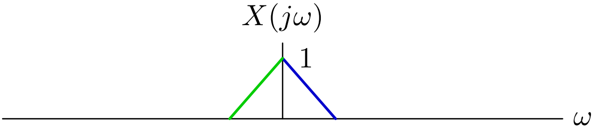
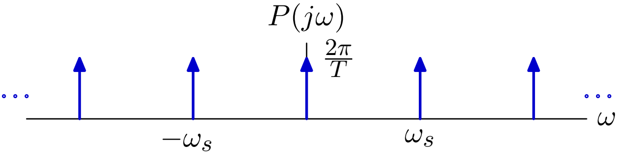


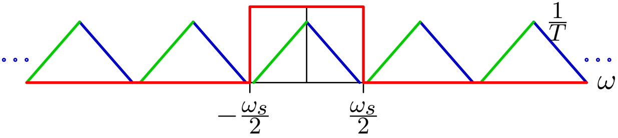
Aliasing

Aliasing increases as the sampling rate decreases.



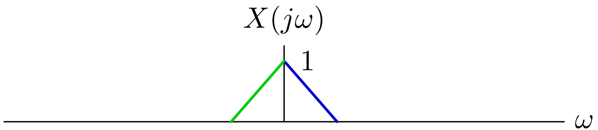


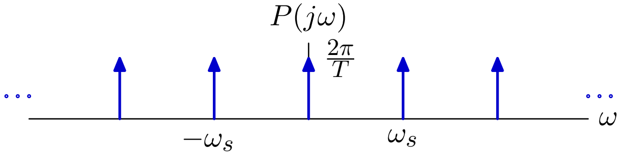
$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



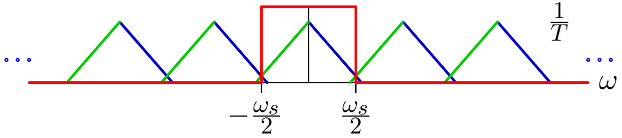
Aliasing

Aliasing increases as the sampling rate decreases.





$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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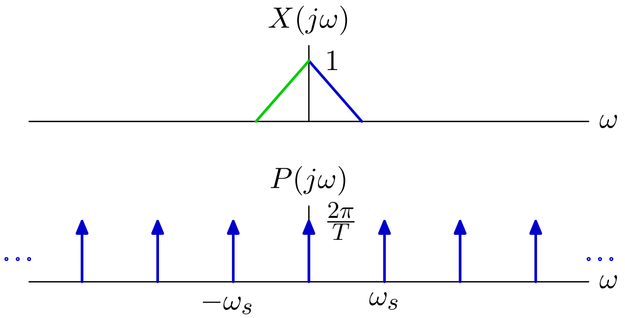
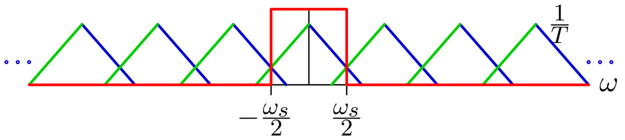


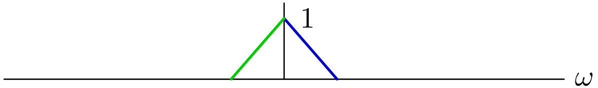
$$X_p(j\omega) = \frac{1}{2\pi}\,(X(j\,\cdot\,) * P(j\,\cdot\,))(\omega)$$



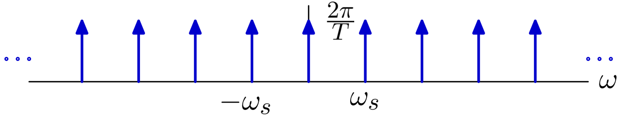
Aliasing

Aliasing increases as the sampling rate decreases.

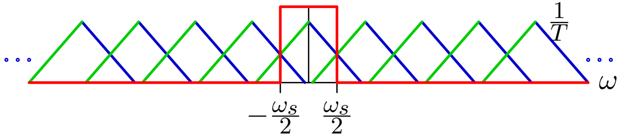
$X(j\omega)$



$P(j\omega)$



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Aliasing Demonstration

Sampling Music

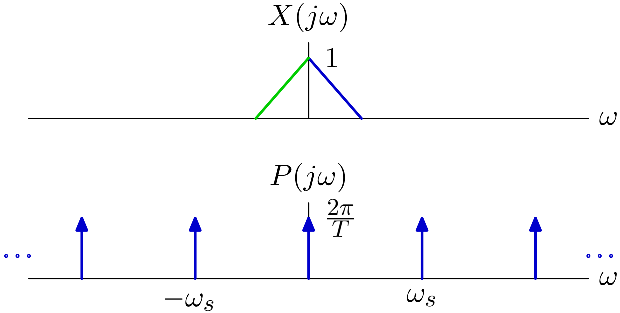
$$\omega_s = \frac{2\pi}{T} = 2\pi f_s$$

- $f_s = 44.1$ kHz
- $f_s = 22$ kHz
- $f_s = 11$ kHz
- $f_s = 5.5$ kHz
- $f_s = 2.8$ kHz

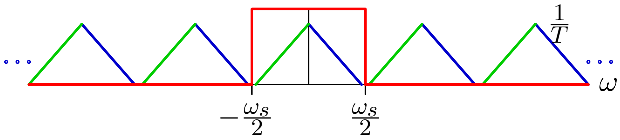
J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

Aliasing

Aliasing increases as the sampling rate decreases.

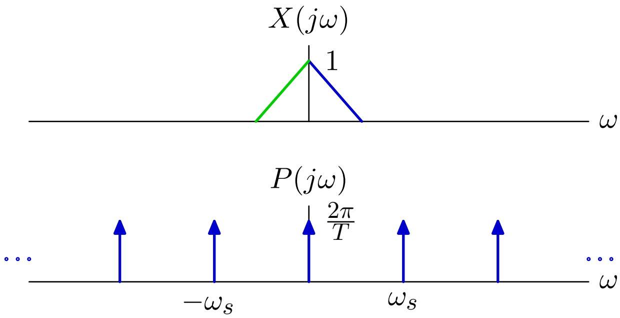


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$

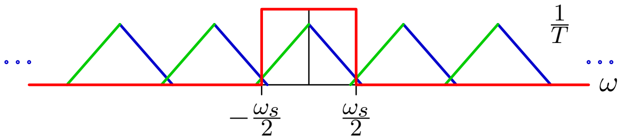


Aliasing

Aliasing increases as the sampling rate decreases.

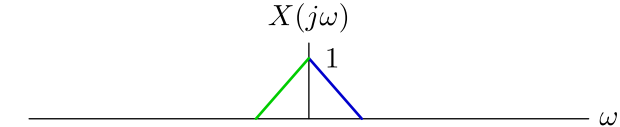


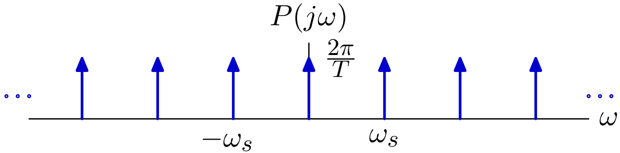
$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



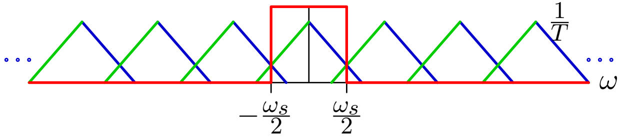
Aliasing

Aliasing increases as the sampling rate decreases.



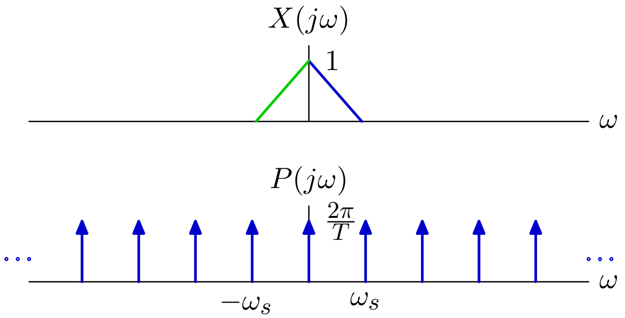


$$X_p(j\omega) = \frac{1}{2\pi}\,(X(j\,\cdot\,) * P(j\,\cdot\,))(\omega)$$

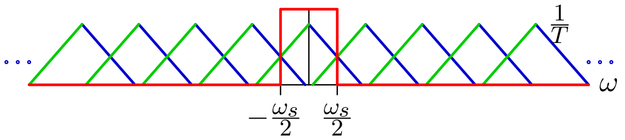


Aliasing

Aliasing increases as the sampling rate decreases.

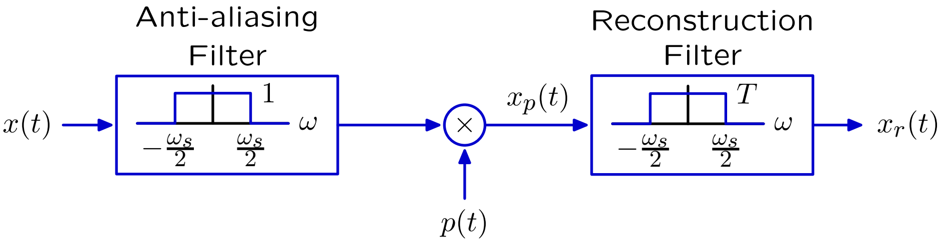


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



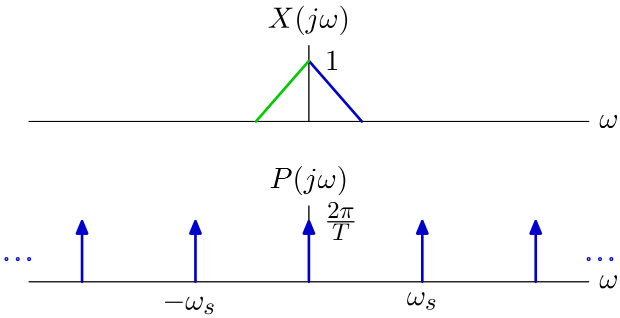
Anti-Aliasing Filter

To avoid aliasing, remove frequency components that alias before sampling.

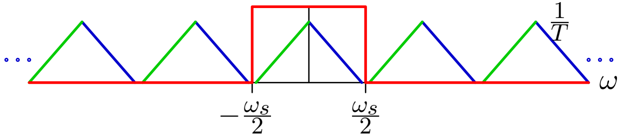


Aliasing

Aliasing increases as the sampling rate decreases.



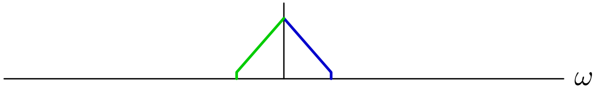
$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



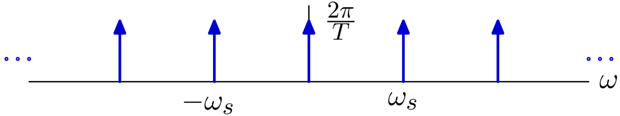
Aliasing

Aliasing increases as the sampling rate decreases.

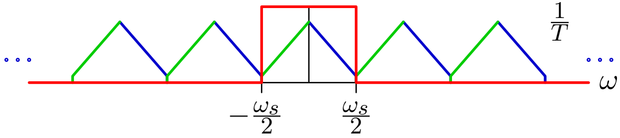
Anti-aliased $X(j\omega)$



$P(j\omega)$

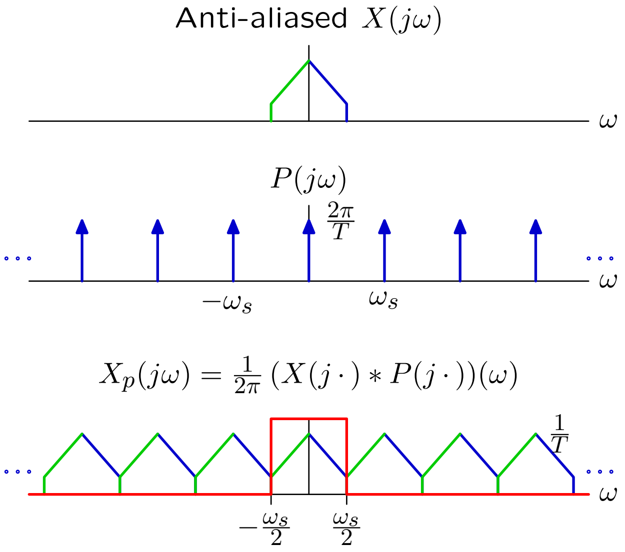


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Aliasing

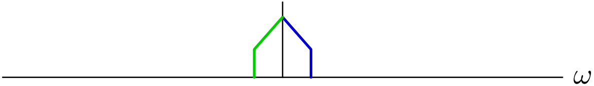
Aliasing increases as the sampling rate decreases.



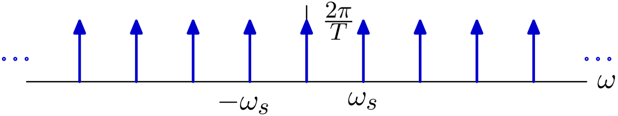
Aliasing

Aliasing increases as the sampling rate decreases.

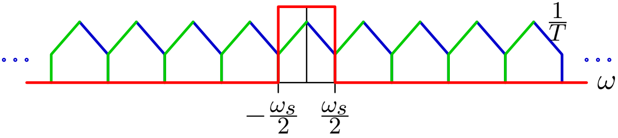
Anti-aliased $X(j\omega)$



$P(j\omega)$



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Anti-Aliasing Demonstration

Sampling Music

$$\omega_s = \frac{2\pi}{T} = 2\pi f_s$$

- $f_s = 11$ kHz without anti-aliasing
- $f_s = 11$ kHz with anti-aliasing
- $f_s = 5.5$ kHz without anti-aliasing
- $f_s = 5.5$ kHz with anti-aliasing
- $f_s = 2.8$ kHz without anti-aliasing
- $f_s = 2.8$ kHz with anti-aliasing

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

Sampling: Summary

Effects of sampling are easy to visualize with Fourier representations.

Signals that are bandlimited in frequency (e.g., $-W < \omega < W$) can be sampled without loss of information.

The minimum sampling frequency for sampling without loss of information is called the Nyquist rate. The Nyquist rate is twice the highest frequency contained in a bandlimited signal.

Sampling at frequencies below the Nyquist rate causes aliasing.

Aliasing can be eliminated by pre-filtering to remove frequency components that would otherwise alias.