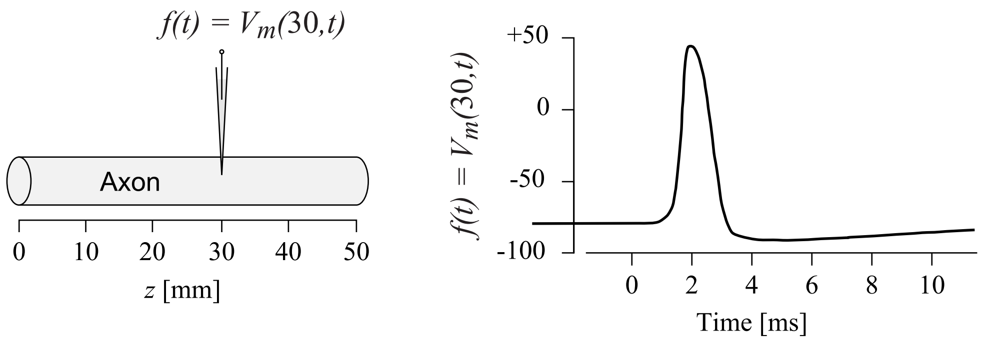
6.003 Homework #14 Solutions

Problems

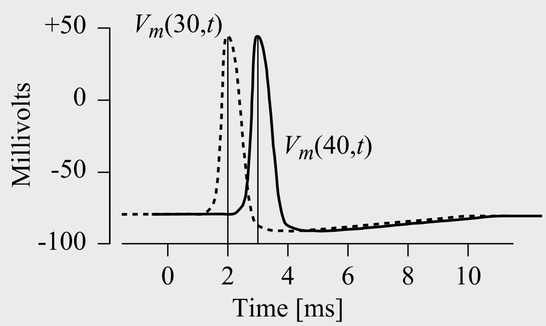
1. Neural signals

The following figure illustrates the measurement of an action potential, which is an electrical pulse that travels along a neuron. Assume that this pulse travels in the positive z direction with constant speed $\nu = 10\,\text{m/s}$ (which is a reasonable assumption for the large unmyelinated fibers found in the squid, where such potentials were first studied). Let $V_m(z,t)$ represent the potential that is measured at position z and time t, where time is measured in milliseconds and distance is measured in millimeters. The right panel illustrates $f(t) = V_m(30,t)$ which is the potential measured as a function of time t at position $z = 30\,\text{mm}$.



Part a. Sketch the dependence of V_m on t at position $z = 40\,\text{mm}$ (i.e., $V_m(40,t)$).

It will take the action potention 1 ms to travel from the reference position at $z = 30\,\text{mm}$ to its new position at $z = 40\,\text{mm}$. Thus, the new waveform $V_m(40,t)$ is a version of $f(t)$ that is shifted by 1 ms to the right.

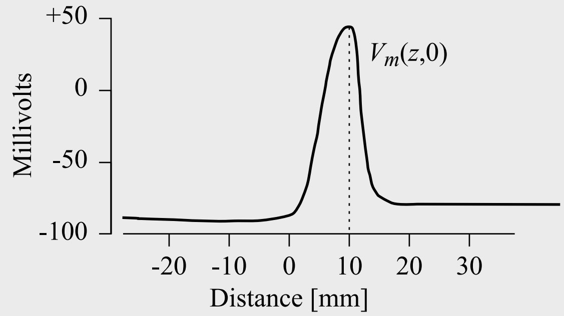


Part b. Sketch the dependence of V_m on z at time $t = 0\,\text{ms}$ (i.e., $V_m(z,0)$).

The action potential peaks at $z = 30\,\text{mm}$ when $t = 2\,\text{ms}$. Since it is traveling to the right at speed $\nu = 10\,\text{mm/ms}$, it must also peak at $z = 10\,\text{mm}$ when $t = 0$. Thus $f(2)$ must map to $z = 10\,\text{mm}$ in the new figure. Similarly, the following function locations map to new positions:

$f(0)$ maps to 30

$f(1)$	maps to	20
$f(2)$	maps to	10
$f(3)$	maps to	0
$f(4)$	maps to	-10



Part c. Determine an expression for $V_m(z,t)$ in terms of $f(\cdot)$ and ν. Explain the relations between this expression and your results from parts a and b.

$$V_m(z,t) = \boxed{f\left(t - \frac{z-30}{\nu}\right)}$$

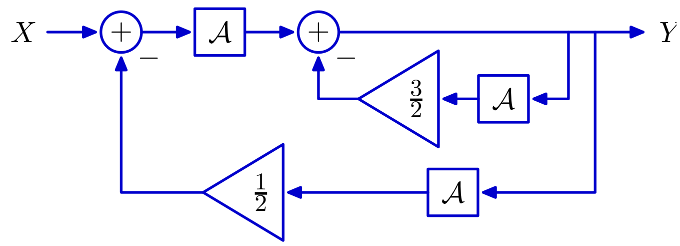
The definition of $f(t)$ provides a starting point: $V_m(30,t) = f(t)$. In part a, we found that $V_m(40,t) = f(t-1)$. This result generalizes: shifting to a more positive location (i.e., adding z_0 to z) adds a time delay of z_0/ν. Expressed as an equation, $V_m(30+z_0,t) = f(t-\frac{z_0}{\nu})$. Substituting $z = 30 + z_0$, we get the general relation

$$V_m(z,t) = f\left(t - \frac{z-30}{\nu}\right).$$

To understand our result from part b, substitute $t = 0$ to obtain $V_m(z,0) = f(0 - \frac{z-30}{\nu})$. Thus we must scale the x-axis by ν (to convert the time axis to a space axis) then shift the space axis by 30 mm (so that the peak is now at $z = -10$ mm) and finally, flip the plot about the x-axis (bringing the peak to $z = 10$ mm).

2. Characterizing block diagrams

Consider the system defined by the following block diagram:



a. Determine the system functional $H = \frac{Y}{X}$.

Let W represent the output of the topmost integrator. Then

$$W = \mathcal{A}(X - \frac{1}{2}\mathcal{A}Y) = \mathcal{A}X - \frac{1}{2}\mathcal{A}^2Y$$

and

$$Y = W - \frac{3}{2}\mathcal{A}Y\,.$$

Substituting the former into the latter we find that

$$Y = \mathcal{A}X - \frac{1}{2}\mathcal{A}^2Y - \frac{3}{2}\mathcal{A}Y\,.$$

Solving for $\dfrac{Y}{X}$ yields the answer,

$$\frac{Y}{X} = \frac{\mathcal{A}}{1 + \frac{3}{2}\mathcal{A} + \frac{1}{2}\mathcal{A}^2}.$$

b. Determine the poles of the system.

Substituting $\mathcal{A} \to \dfrac{1}{s}$ in the system functional yields

$$\frac{Y}{X} = \frac{\frac{1}{s}}{1 + \frac{3}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2}} = \frac{s}{s^2 + \frac{3}{2}s + \frac{1}{2}} = \frac{s}{(s + \frac{1}{2})(s+1)}\,.$$

The poles are then the roots of the denominator: $-\frac{1}{2}$, and -1.

c. Determine the impulse response of the system.

Expand the system functional using partial fractions:

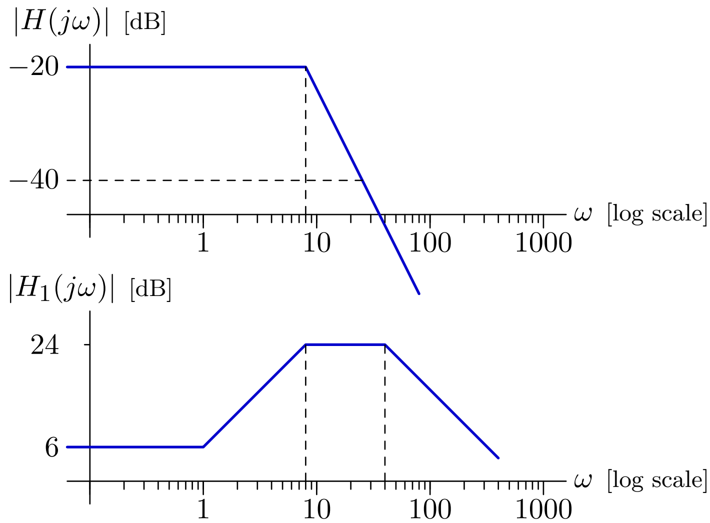
$$\frac{Y}{X} = \frac{\mathcal{A}}{1 + \frac{3}{2}\mathcal{A} + \frac{1}{2}\mathcal{A}^2} = \frac{\alpha\mathcal{A}}{1+\mathcal{A}} + \frac{\beta\mathcal{A}}{1 + \frac{1}{2}\mathcal{A}} = \frac{2\mathcal{A}}{1+\mathcal{A}} - \frac{\mathcal{A}}{1+\frac{1}{2}\mathcal{A}}$$

Each term in the partial fraction expansion contributes one fundamental mode to h,

$$h(t) = (2e^{-t} - e^{-t/2})\,u(t)$$

3. Bode Plots

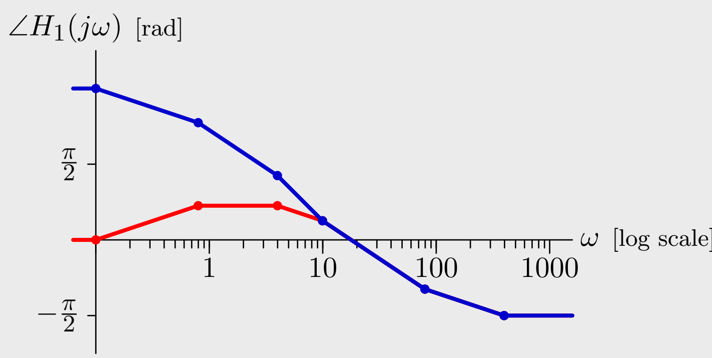
Our goal is to design a stable CT LTI system H by cascading two causal CT LTI systems: H_1 and H_2. The magnitudes of $H(j\omega)$ and $H_1(j\omega)$ are specified by the following straight-line approximations. We are free to choose other aspects of the systems.



H_1 and H_2 have to be stable as well as causal because we're talking about their frequency responses, and H has to be causal because H_1 and H_2 are. This implies that all poles must be in the left half-plane.

a. Determine all system functions $H_1(s)$ that are consistent with these design specifications, and plot the straight-line approximation to the phase angle of each (as a function of ω).

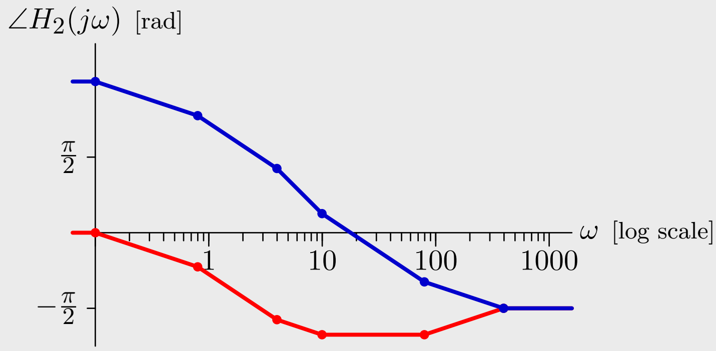
The frequency response of H_1 breaks up at $\omega = 1$ and then down at $\omega = 8$ and 40. The two breaks downward require poles at $s = -8$ and $s = -40$ respectively, The break upward can be achieved with a zero at $s = 1$ (blue) or at $s = -1$ (red).



H_1 could also be multiplied by -1 without having any effect on the magnitude function. Multiplying by -1 would shift the phase curves up or down by π.

b. Determine all system functions $H_2(s)$ that are consistent with these design specifications, and plot the straight-line approximation to the phase angle of each (as a function of ω).

To compensate for H_1, the frequency response of H_2 must break downward at $\omega = 1$ and upward at $\omega = 40$. In addition, H_2 must break downward at $\omega = 8$ so that the slope of H changes from 0 to $-40\,$dB/decade at $\omega = 10$. H_2 can be achieved with poles at $s = -1$ and -8 and a zero at $s = 40$ (blue) or at $s = -40$ (red).



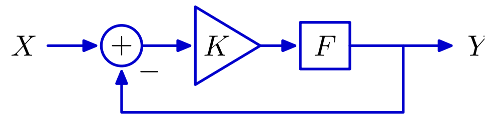
H_2 could also be multiplied by -1 without having any effect on the magnitude function. Multiplying by -1 would shift the phase curves up or down by π.

4. Controlling Systems

Use a proportional controller (gain K) to control a plant whose input and output are related by

$$F = \frac{R^2}{1+\mathcal{R}-2\mathcal{R}^2}$$

as shown below.



a. Determine the range of K for which the unit-sample response of the closed-loop system converges to zero.

Using Black's equation, we can write

$$\frac{Y}{X} = \frac{\frac{K\mathcal{R}^2}{1+\mathcal{R}-2\mathcal{R}^2}}{1+\frac{K\mathcal{R}^2}{1+\mathcal{R}-2\mathcal{R}^2}} = \frac{K\mathcal{R}^2}{1+\mathcal{R}-(2-K)\mathcal{R}^2}$$

The closed-loop poles can be found by substituting $\mathcal{R} \to \frac{1}{z}$:

$$\frac{Y}{X} = \frac{K}{z^2+z-(2-K)}$$

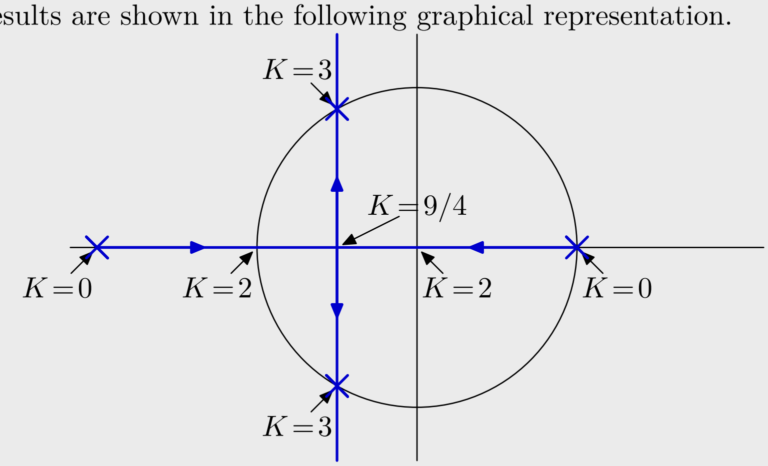
and solving for the roots of the denominator:

$$z = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2-K}$$

The unit-sample response will converge to zero iff the poles are inside the unit circle.

When $K = 0$, the poles are at $z = -2$ and $z = 1$ (not convergent). As K increases, the poles move toward each other, creating a double pole at $z = -\frac{1}{2}$ when $K = \frac{9}{4}$. The response will converge when the pole that started at $z = -2$ reaches $z = -1$, i.e., at $K = 2$. The poles will split away from $z = -\frac{1}{2}$ for $K > \frac{9}{4}$ and will stay inside the unit circle if $\frac{1}{4}+2-K > -\frac{3}{4}$, i.e., if $K < 3$.

These results are shown in the following graphical representation.



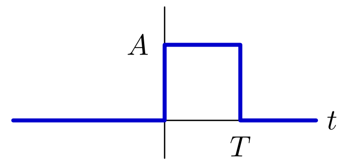
Thus, the unit-sample response will converge if $2 < K < 3$.

b. Determine the range of K for which the closed-loop poles are real-valued numbers with magnitudes less than 1.

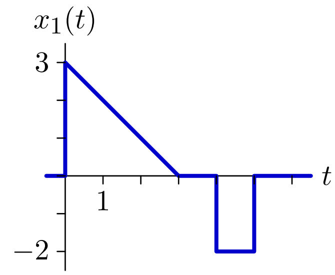
From the plot in the previous part, it follows that the closed-loop poles are on the real axis and have magnitudes less than one when $2 < K < \frac{9}{4}$.

5. CT responses

We are given that the impulse response of a CT LTI system is of the form



where A and T are unknown. When the system is subjected to the input



the output $y_1(t)$ is zero at $t = 5$. When the input is

$$x_2(t) = \sin\left(\frac{\pi t}{3}\right) u(t),$$

the output $y_2(t)$ is equal to 9 at $t = 9$. Determine A and T. Also determine $y_2(t)$ for all t.

The first fact implies that

$$y_1(5) = \int_{-\infty}^{\infty} x_1(\tau)h(5-\tau)d\tau = A\int_{5-T}^{5} x_1(\tau)d\tau = 0.$$

If the lower limit is 1, the area of the triangle between $\tau = 1$ and $\tau = 3$ is 2 and cancels the area of the rectangle between $\tau = 4$ and $\tau = 5$. Therefore $T = 4$. From the second fact, we have

$$\begin{aligned} 9 = y_2(9) &= A\int_5^9 x_2(\tau)d\tau \\ &= A\int_5^9 \sin\left(\frac{\pi\tau}{3}\right)d\tau \\ &= -\frac{A}{\pi/3}\cos\left(\frac{\pi\tau}{3}\right)\Big|_5^9 \\ &= \frac{9A}{2\pi}, \end{aligned}$$

so $A = 2\pi$.

There are three ranges to consider in computing $y_2(t)$. For $t < 0$, there is no overlap between $x_2(\tau)$ and $h(t-\tau)$ and hence $y_2(t) = 0$. For $0 \le t < 4$, there is partial overlap and $y_2(t)$ is given by

$$y_2(t) = 2\pi\int_0^t \sin\left(\frac{\pi\tau}{3}\right)d\tau = -\frac{2\pi}{\pi/3}\cos\left(\frac{\pi\tau}{3}\right)\Big|_0^t = 6\left(1-\cos\left(\frac{\pi t}{3}\right)\right).$$

For $t \ge 4$, the overlap is total and we have

$$y_2(t) = 2\pi\int_{t-4}^t \sin\left(\frac{\pi\tau}{3}\right)d\tau = 6\left(\cos\left(\frac{\pi(t-4)}{3}\right) - \cos\left(\frac{\pi t}{3}\right)\right).$$

Hence

$$y_2(t) = \begin{cases} 0, & t < 0, \\ 6\left(1 - \cos\left(\frac{\pi t}{3}\right)\right), & 0 \le t < 4, \\ 6\left(\cos\left(\frac{\pi(t-4)}{3}\right) - \cos\left(\frac{\pi t}{3}\right)\right), & t \ge 4. \end{cases}$$

6. DT approximation of a CT system

Let H_{C1} represent a **causal** CT system that is described by

$$\dot{y}_C(t) + 3y_C(t) = x_C(t)$$

where $x_C(t)$ represents the input signal and $y_C(t)$ represents the output signal.

$$x_C(t) \longrightarrow \boxed{H_{C1}} \longrightarrow y_C(t)$$

a. Determine the pole(s) of H_{C1}.

The the Laplace transform of the differential equation to get

$$sY_C(s) + 3Y_C(s) = X_C(s)$$

and solve for $Y_C(s)/X_C(s) = 1/(s+3)$. The pole is at $s = -3$.

Your task is to design a **causal** DT system H_{D1} to approximate the behavior of H_{C1}.

$$x_D[n] \longrightarrow \boxed{H_{D1}} \longrightarrow y_D[n]$$

Let $x_D[n] = x_C(nT)$ and $y_D[n] = y_C(nT)$ where T is a constant that represents the time between samples. Then approximate the derivative as

$$\frac{dy_C(t)}{dt} = \frac{y_C(t+T) - y_C(t)}{T}\,.$$

b. Determine an expression for the pole(s) of H_{D1}.

Take the Z transform of the difference equation

$$\frac{y_D[n+1] - y_D[n]}{T} + 3y_D[n] = x_D[n]$$

to obtain

$$\frac{zY_D(z) - Y_D}{T} + 3Y_D(z) = X_D(z)$$

Solving

$$(z - 1 + 3T)Y_D(z) = TX_D(z)$$

so that

$$H_D(z) = \frac{Y_D(z)}{X_D(z)} = \frac{T}{z - 1 + 3T}\,.$$

There is a pole at $z = 1 - 3T$.

c. Determine the range of values of T for which H_{D1} is stable.

Stability requires that the pole be inside the unit circle

$$-1 < 1 - 3T < 1$$

or

$$-2 < -3T < 0$$

so that

$$0 < T < \frac{2}{3}\,.$$

Now consider a second-order **causal** CT system H_{C2}, which is described by

$$\ddot{y}_C(t) + 100 y_C(t) = x_C(t)\,.$$

d. Determine the pole(s) of H_{C2}.

Take the Laplace transform of the differential equation to get

$$s^2 Y_C + 100 Y_C = X_C$$

and solve for $Y_C/X_C = 1/(s^2+100)$. There are poles at $s = \pm j10$.

Design a **causal** DT system H_{D2} to approximate the behavior of H_{C2}. Approximate derivatives as before:

$$\dot{y_C}(t) = \frac{dy_C(t)}{dt} = \frac{y_C(t+T) - y_C(t)}{T} \quad \text{and}$$

$$\frac{d^2 y_C(t)}{dt^2} = \frac{\dot{y_C}(t+T) - \dot{y_C}(t)}{T}\,.$$

e. Determine an expression for the pole(s) of H_{D2}.

$$\begin{aligned}\frac{d^2 y_C(t)}{dt^2} &= \frac{\dot{y_C}(t+T) - \dot{y_C}(t)}{T} = \frac{\frac{y_C(t+2T)-y_C(t+T)}{T} - \frac{y_C(t+T)-y_C(t)}{T}}{T} \\ &= \frac{y_C(t+2T) - 2y_C(t+T) + y_C(t)}{T^2}\,.\end{aligned}$$

Substituting to find the difference equation, we get

$$\frac{y_D[n+2] - 2y_D[n+1] + y_D[n]}{T^2} + 100 y_D[n] = x_D[n]\,.$$

Take the Z transform to find that

$$(z^2 - 2z + 1 + 100T^2) Y_D(z) = T^2 X_D(z)$$

or

$$\frac{Y_D(z)}{X_D(z)} = \frac{T^2}{z^2 - 2z + 1 + 100T^2}\,.$$

The poles are at

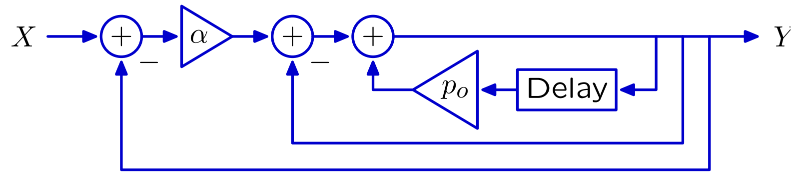
$$z = 1 \pm \sqrt{1 - 1 - 100T^2} = 1 \pm j10T$$

f. Determine the range of values of T for which H_{D2} stable.

The poles are always outside the unit circle. The system is always unstable.

7. Feedback

Consider the system defined by the following block diagram.



a. Determine the system functional $\frac{Y}{X}$.

We can use Black's equation (previous problem) to find the system functional for the innermost loop:

$$H_1 = \frac{1}{1 - p_0\mathcal{R}}\,.$$

Then apply Black's equation for a second time to find the system functional for the next loop:

$$H_2 = \frac{H_1}{1+H_1} = \frac{\frac{1}{1-p_0\mathcal{R}}}{1+\frac{1}{1-p_0\mathcal{R}}} = \frac{1}{2-p_0\mathcal{R}}\,.$$

Repeat for the outermost loop:

$$H_3 = \frac{\alpha H_2}{1+\alpha H_2} = \frac{\frac{\alpha}{2-p_0\mathcal{R}}}{1+\frac{\alpha}{2-p_0\mathcal{R}}} = \frac{\alpha}{2+\alpha-p_0\mathcal{R}}\,.$$

b. Determine the number of closed-loop poles.

The denominator is a first order polynomial in $\mathcal{R}$. Therefore, there is a single pole. It is located at $z = \frac{p_0}{2+\alpha}$.

c. Determine the range of gains (α) for which the closed-loop system is stable.

The closed-loop system will be stable iff the closed-loop pole is inside the unit circle:

$$|z| = \left|\frac{p_0}{2+\alpha}\right| < 1$$

which implies that $|2+\alpha| > |p_0|$. This will be true if $\alpha > |p_0| - 2$ or if $\alpha < -|p_0| - 2$.

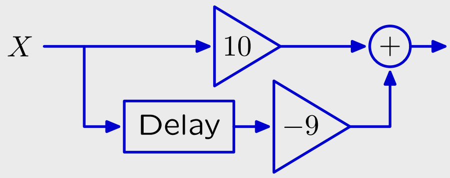
8. Finding a system

a. Determine the difference equation and block diagram representations for a system whose output is $10, 1, 1, 1, 1, \ldots$ when the input is $1, 1, 1, 1, 1, \ldots$.

Notice that $Y = 10X - 9\mathcal{R}X$. This relation suggests the following difference equation

$$y[n] = 10x[n] - 9x[n-1]$$

and block diagram



b. Determine the difference equation and block diagram representations for a system whose output is $1, 1, 1, 1, 1, \ldots$ when the input is $10, 1, 1, 1, 1, \ldots$.

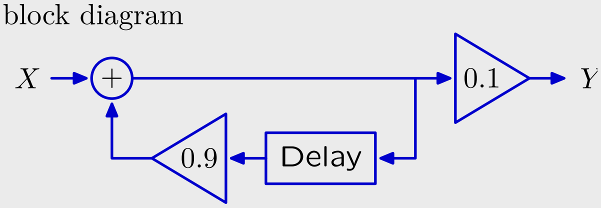
The difference equation for the inverse relation can be obtained by interchanging y and x in the previous difference equation to get

$$x[n] = 10y[n] - 9y[n-1].$$

So

$$y[n] = \frac{9y[n-1] + x[n]}{10},$$

which has this block diagram

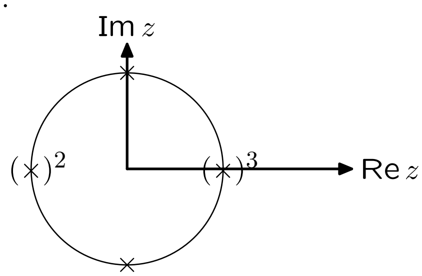


c. Compare the difference equations in parts a and b. Compare the block diagrams in parts a and b.

The difference equations for parts a and b have exactly the same structure. The only difference is that the roles of x and y are reversed. The block diagrams have similar parts (1 delay, 1 adder, 2 gains), but the topologies are completely different. The first is acyclic and the second is cyclic.

9. Lots of poles

All of the poles of a system fall on the unit circle, as shown in the following plot, where the '2' and '3' means that the adjacent pole, marked with parentheses, is a repeated pole of order 2 or 3 respectively.



Which of the following choices represents the order of growth of this system's unit-sample response for large n? Give the letter of your choice plus the information requested.

a. $y[n]$ is periodic. If you choose this option, determine the period.

b. $y[n] \sim An^k$ (where A is a constant). If you choose this option, determine k.

c. $y[n] \sim Az^n$ (where A is a constant). If you choose this option, determine z.

d. None of the above. If you choose this option, determine a closed-form asymptotic expression for $y[n]$.

A partial fraction expansion of the system functional will have terms of the following forms:

$$\frac{1}{1-\mathcal{R}}\,,\quad \left(\frac{1}{1-\mathcal{R}}\right)^2,\quad \left(\frac{1}{1-\mathcal{R}}\right)^3,\quad \frac{1}{1+\mathcal{R}^2}\,,\quad \frac{1}{1+\mathcal{R}}\,,\quad \text{and}\quad \left(\frac{1}{1+\mathcal{R}}\right)^2.$$

The third one will have the fastest growth for large n. Its expansion has the form

$$(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\times(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\times(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)\,.$$

Multiplying the first two:

	1	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\cdots$
1	1	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\cdots$
$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\mathcal{R}^4$	$\cdots$
$\mathcal{R}^2$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\mathcal{R}^4$	$\mathcal{R}^5$	$\cdots$
$\mathcal{R}^3$	$\mathcal{R}^3$	$\mathcal{R}^4$	$\mathcal{R}^5$	$\mathcal{R}^6$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Group same powers of $\mathcal{R}$ by following reverse diagonals:

$$1+2\mathcal{R}+3\mathcal{R}^2+4\mathcal{R}^3+\cdots$$

Multiplying this by the last term:

	1	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\cdots$
1	1	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\cdots$
$2\mathcal{R}$	$2\mathcal{R}$	$2\mathcal{R}^2$	$2\mathcal{R}^3$	$2\mathcal{R}^4$	$\cdots$
$3\mathcal{R}^2$	$3\mathcal{R}^2$	$3\mathcal{R}^3$	$3\mathcal{R}^4$	$3\mathcal{R}^5$	$\cdots$
$4\mathcal{R}^3$	$4\mathcal{R}^3$	$4\mathcal{R}^4$	$4\mathcal{R}^5$	$4\mathcal{R}^6$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

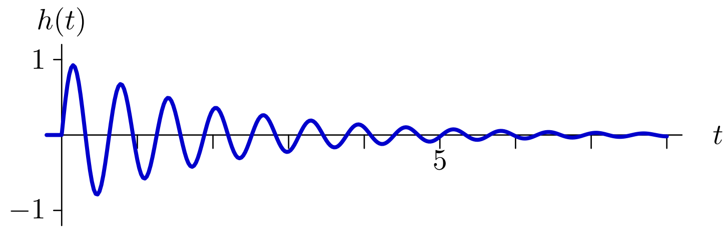
Group same powers of $\mathcal{R}$ by following reverse diagonals:

$$1 + 3\mathcal{R} + 6\mathcal{R}^2 + 10\mathcal{R}^3 + \cdots$$

This expression grows with $(n+1)(n+2)/2$ which is on the order of n^2. Thus b is the correct solution with $k = 2$.

10. Relation between time and frequency responses

The impulse response of an LTI system is shown below.



If the input to the system is an eternal cosine, i.e., $x(t) = \cos(\omega t)$, then the output will have the form

$$y(t) = C\cos(\omega t + \phi)$$

The impulse response has the form of a decaying sinusoid. The time constant of decay is approximately 2, so the exponential part has the form $e^{-t/2}$. The sinusoid has approximately 8 periods in 5 time units so $8\frac{2\pi}{\omega_d} = 5$. Solving this, we find that $\omega_d \approx 10$. The impulse response therefore has the form

$$h(t) = e^{-t/2}\sin(10t)u(t)\,.$$

There are two poles associated with such a response and no zeros. The poles have real parts of $-\sigma = -\frac{1}{2}$ and imaginary parts of $\pm j10$. The characteristic equation is $(s-p_0)(s-p_1) = (s+\frac{1}{2}+j10)(s+\frac{1}{2}-j10) = s^2+s+100.25 = s^2+\frac{\omega_0}{Q}s+\omega_0^2$. Thus $\omega_0 \approx 10$ and $Q \approx 10$.

The system function is the Laplace transform of the impulse response,

$$H(s) = \frac{\omega_d}{s^2+\frac{\omega_0}{Q}s+\omega_0^2} \approx \frac{10}{s^2+s+100}$$

a. Determine ω_m, the frequency ω for which the constant C is greatest. What is the value of C when $\omega = \omega_m$?

The gain of the system is largest at a frequency $\omega_m = \sqrt{\omega_0^2 - 2\sigma^2} \approx 10$. The gain is then approximately $Q \approx 10$ times the DC gain, which is $\approx \frac{1}{10}$. Thus $C \approx 1$.

b. Determine ω_p, the frequency ω for which the phase angle ϕ is $-\frac{\pi}{4}$. What is the value of C when $\omega = \omega_p$?

The phase angle varies from 0 when $\omega = 0$ to $-\pi$ as $\omega \to \infty$. The phase angle is equal to $-\frac{\pi}{2}$ when $\omega = \omega_0$ [notice that when $\omega = \omega_0$ the ω_0^2 term in the denominator of the system function is cancelled by $s^2 = (j\omega_0)^2$]. The phase angle will be $-\frac{\pi}{4}$ when $\omega = \omega_p = \omega_0 - \sigma$ (so that the vector from the upper pole is $\sqrt{2}$ times longer at ω_p than at ω_0. At ω_p, the gain is reduced from its maximum by 3 dB (a factor of $\sqrt{2}$). Thus $C \approx \frac{1}{\sqrt{2}}$.