6.003 Homework #7 Solutions

Problems

1. Second-order systems

The impulse response of a second-order CT system has the form

$$h(t) = e^{-\sigma t}\cos(\omega_d t + \phi)u(t)$$

where the parameters σ, ω_d, and ϕ are related to the parameters of the characteristic polynomial for the system: $s^2 + Bs + C$.

a. Determine expressions for σ and ω_d (not ϕ) in terms of B and C.

Express the impulse response in terms of complex exponentials:

$$h(t) = \frac{1}{2}e^{-\sigma t}\left(e^{j\omega_d t + j\phi} + e^{-j\omega_d t - j\phi}\right)u(t) = \frac{1}{2}e^{j\phi}e^{(-\sigma + j\omega_d)t}u(t) + \frac{1}{2}e^{-j\phi}e^{(-\sigma - j\omega_d)t}u(t)$$

The impulse response is a weighted sum of modes of the form $e^{s_0 t}$ and $e^{s_1 t}$ where s_0 and s_1 are the poles. Thus the poles of the system are at $s = -\sigma \pm j\omega_d$. The characteristic polynomial has the form $s^2 + Bs + C = (s + \sigma + j\omega_d)(s + \sigma - j\omega_d) = (s+\sigma)^2 + \omega_d^2$. Thus $B = 2\sigma$ and $C = \sigma^2 + \omega_d^2$. Solving, we find that

$$\sigma = \frac{B}{2}$$

$$\omega_d = \sqrt{C - \frac{1}{4}B^2}\,.$$

b. Determine
– the time required for the envelope $e^{-\sigma t}$ of $h(t)$ to diminish by a factor of e,
– the period of the oscillations in $h(t)$, and
– the number of periods of oscillation before $h(t)$ diminishes by a factor of e.
Express your results as functions of B and C only.

The time to decay by a factor of e is

$$\frac{1}{\sigma} = \frac{2}{B}\,.$$

The period is

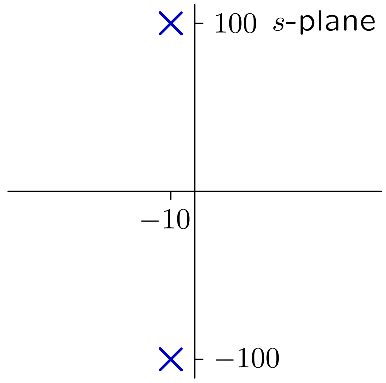
$$\frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{C - \frac{1}{4}B^2}}\,.$$

The number of periods before diminishing a factor of e is

$$\frac{\frac{2}{B}}{\frac{2\pi}{\sqrt{C - \frac{1}{4}B^2}}} = \frac{\sqrt{C - \frac{1}{4}B^2}}{\pi B}\,.$$

Notice that this last answer is equivalent to Q/π where $Q = \dfrac{\omega_d}{2\sigma}$.

c. Estimate the parameters in part b for a CT system with the following poles:



From the plot $\sigma = 10$ and $\omega_d = 100$.

The time to decay by a factor of e is 0.1.

The period is $\dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{100} = 0.0628$.

The number of cycles before decaying by e is $\dfrac{10}{2\pi} \approx 1.6$

The unit-sample response of a second-order DT system has the form

$$h[n] = r_0^n \cos(\Omega_0 n + \Phi)u[n]$$

where the parameters r_0, Ω_0, and Φ are related to the parameters of the characteristic polynomial for the system: $z^2 + Dz + E$.

d. Determine expressions for r_0 and Ω_0 (not Φ) in terms of D and E.

Express the unit-sample response in terms of complex exponentials:

$$h[n] = r_0^n \left(\frac{1}{2}e^{j\Omega_0 n + j\Phi} + \frac{1}{2}e^{-j\Omega_0 n - j\Phi}\right) u[n] = \frac{1}{2}e^{j\Phi} r_0^n e^{j\Omega_0 n} u[n] + \frac{1}{2}e^{-j\Phi} r_0^n e^{-j\Omega_0 n} u[n]$$

The poles have the form $z = r_0 e^{j\Omega_0}$ and $z = r_0 e^{-j\Omega_0}$. The characteristic equation is $z^2 + Dz + E = (z - r_0 e^{j\Omega_0})(z - r_0 e^{-j\Omega_0}) = z^2 - 2r_0 \cos \Omega_0 + r_0^2$. Thus $D = -2r_0 \cos \Omega_0$ and $E = r_0^2$. Solving, we find that

$$r_0 = \sqrt{E}$$

$$\Omega_0 = \cos^{-1} \frac{-D}{2r_0} = \cos^{-1} \frac{-D}{2\sqrt{E}}$$

e. Determine
– the length of time required for the envelope r_0^n of $h[n]$ to diminish by a factor of e.
– the period of the oscillations (i.e., $\frac{2\pi}{\Omega_0}$) in $h[n]$, and
– the number of periods of oscillation in $h[n]$ before it diminishes by a factor of e.
Express your results as functions of D and E only.

The time to diminish by a factor of e is $r_0^n = \frac{1}{e}$. Taking the log of both sides yields $n \ln r_0 = -1$ so that the time is

$$-\frac{1}{\ln r_0} = -\frac{1}{\ln \sqrt{E}}$$

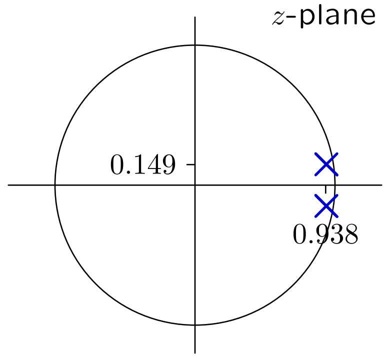
The period is $\frac{2\pi}{\Omega_0}$ which is

$$\frac{2\pi}{\cos^{-1}\frac{-D}{2\sqrt{E}}}$$

The number of periods before the response diminishes by e is

$$\frac{\frac{-1}{\ln r_0}}{\frac{2\pi}{\cos^{-1}\frac{-D}{2\sqrt{E}}}} = \frac{\frac{-1}{\ln\sqrt{E}}\cos^{-1}\frac{-D}{2\sqrt{E}}}{2\pi}$$

f. Estimate the parameters in part e for a DT system with the following poles:



From the plot $\Omega_0 = \tan^{-1}\frac{0.149}{0.938} \approx 0.16$ radians and $r_0 = \sqrt{0.149^2 + 0.938^2} \approx 0.95$.

The time to decay by a factor of e is $\frac{-1}{\ln 0.95} \approx 19.5.$

The period is $\frac{2\pi}{\Omega_0} = \frac{2\pi}{0.16} \approx 39.3.$

The number of cycles before decaying by e is $\frac{19.5}{39.3} \approx 0.5$

2. Matches

The following plots show pole-zero diagrams, impulse responses, Bode magnitude plots, and Bode angle plots for six causal CT LTI systems. Determine which corresponds to which and fill in the following table.

Pole-zero diagram 1 has a single pole at zero. The impulse response of a system with a single pole at zero is a unit step function (3). We evaluate the frequency response by considering frequencies along the $j\omega$ axis. As we move away from the pole at the origin the log-magnitude decays linearly (5). The phase is constant since the angle between the pole and any point along positive side of the $j\omega$ axis remains constant at $\pi/2$. The angle of the frequency response is therefore $-\pi/2$ (4).

Pole-zero diagram 4 has a single pole at at $s = -1$. The impulse response has the form $e^{st}u(t) = e^{-t}u(t)$ (2). As we move along the $j\omega$ axis, we move away from the pole at the origin, and the log-magnitude will eventually decay linearly. Because the pole is not exactly at the origin, this decay is not significant until $\omega = 1$ (6). The phase starts at 0, and eventually moves to $-\pi/2$. Note that as we move farther up the $j\omega$ axis, this system behaves like the system of diagram 1 (2).

Pole-zero diagram 3 adds a zero at the origin. A zero at the origin corresponds to taking the derivative, so we take the impulse response of pole-zero diagram 4 (2) and take its derivative (4). When ω is small, the zero is dominant. As we move away from $\omega = 0$, the effect of the zero diminishes and the log-magnitude increases linearly. For sufficiently large ω we are far enough that the zero and pole appear to cancel each other, and the magnitude becomes a constant (3). A zero at the origin means that we take the phase response of pole-zero diagram 4 (2) and add $\pi/2$ to it (6).

Pole-zero diagram 2 contains complex conjugate poles

$$H(s) = \frac{K}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)} = \frac{jA}{s+\sigma+j\omega_d} - \frac{jA}{s+\sigma-j\omega_d}\,.$$

The impulse response has the form

$$h(t) \propto e^{-\sigma t}(e^{j\omega_d t} - e^{-j\omega_d t}) \propto e^{-\sigma t}\sin\omega_d t$$

which is response (1). The magnitude response will eventually decay twice as fast as that of pole-zero diagram 4 (6). Since there are two poles, there will be a bump at around $\omega = 1$ (2). At the origin, the angular contributions of the two poles cancel each other out, hence the angle is zero. As we move up the $j\omega$ axis, the angles add up to $-\pi$, with each pole contributing $-\pi/2$ (3).

Pole-zero diagram 6 adds a zero at the origin, meaning that we take the derivative of the impulse response of pole-zero diagram 2 (1). The derivative ends up being the combination of a decaying $\cos(t)$ term minus a decaying $\sin(t)$ term (5). The zero at the origin adds a linearly increasing component to the magnitude function (4). It also adds $\pi/2$ to the phase response everywhere (5).

Pole-zero diagram 5 has complex conjugate poles and zeros at the same frequency ω. The system function has the form

$$H(s) = \frac{s^2 - \frac{\omega_0}{Q}s + \omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}\,.$$

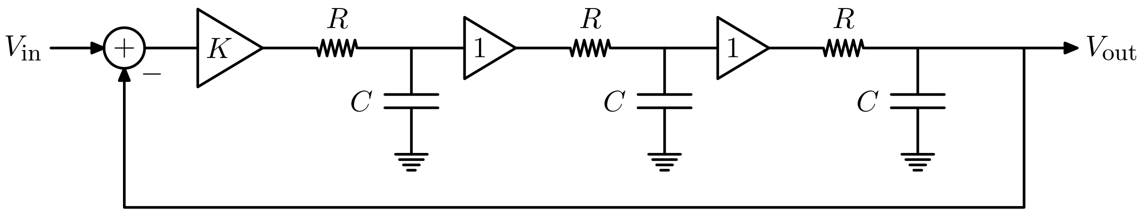
This denominator has the same form as pole-zero diagrams 2 and 6, but has an additional power of s (corresponding to differentiation) in the numerator. This leads to a response of the form in (6). The symmetry of the poles and zeros means they cancel each other's effect on magnitude (1). The phase response at $\omega = 0$ is zero, as the contributions cancel each other out. As we move past $\omega = 1$ where the conjugates are located, the phase moves in the negative direction faster, but eventually settles back at 0 as we move farther and the contributions again cancel each other out (1).

	$h(t)$	Magnitude	Angle
PZ diagram 1:	3	5	4
PZ diagram 2:	1	2	3
PZ diagram 3:	4	3	6
PZ diagram 4:	2	6	2
PZ diagram 5:	6	1	1
PZ diagram 6:	5	4	5

Engineering Design Problems

3. Desired oscillations

The following feedback circuit was the basis of Hewlett and Packard's founding patent.



a. With $R = 1\,\text{k}\Omega$ and $C = 1\mu\text{F}$, sketch the pole locations as the gain K varies from 0 to ∞, showing the scale for the real and imaginary axes. Find the K for which the system is barely stable and label your sketch with that information. What is the system's oscillation period for this K?

The closed-loop gain is

$$H(s) = \frac{\frac{K}{(1+sRC)^3}}{1+\frac{K}{(1+sRC)^3}} = \frac{K}{(1+sRC)^3+K}$$

The denominator is zero if

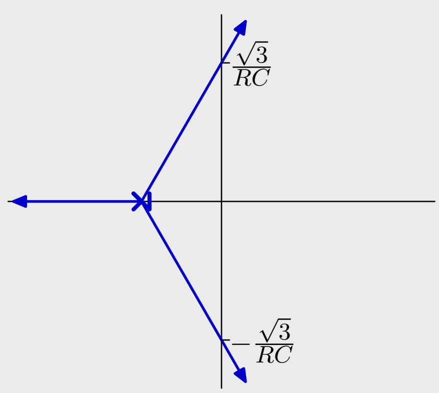
$$(1+sRC)^3 = -K$$

$$(1+sRC) = \sqrt[3]{-K}$$

$$s = \frac{-1+\sqrt[3]{-K}}{RC}$$

There are three cube roots of $-K$: $-\sqrt[3]{K}$, $\sqrt[3]{K}e^{j\pi/3}$, and $\sqrt[3]{K}e^{-j\pi/3}$ and three corresponding poles:

$$s = \frac{-1-\sqrt[3]{K}}{RC},\ \frac{-1+\sqrt[3]{K}e^{j\pi/3}}{RC},\ \text{and}\ \frac{-1+\sqrt[3]{K}e^{-j\pi/3}}{RC}$$



The point of marginal stability is where the root locus crosses the $j\omega$ axis. This occurs when the real part of $-1+\sqrt[3]{K}e^{j\pi/3}$ equals zero:

$$\sqrt[3]{K} = 2$$

so that $K = 8$. The frequency of oscillation is $\omega = \frac{\sqrt{3}}{RC}$ so the period of oscillation is

$$T = \frac{2\pi}{\omega} = \frac{2\pi RC}{\sqrt{3}}\,.$$

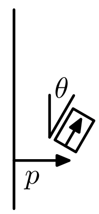
For $RC = 1\,\text{ms}$ (as given), the period $T = 3.63\,\text{ms}$.

b. How do your results change if R is increased to $10\,\text{k}\Omega$?

Increasing R by a factor of 10 increases the period T by a factor of 10, to $T = 36.3\,\text{ms}$. It has no effect of the critcal value of $K = 8$.

4. Robotic steering

Design a steering controller for a car that is moving forward with constant velocity V.



You can control the steering-wheel angle $w(t)$, which causes the angle $\theta(t)$ of the car to change according to

$$\frac{d\theta(t)}{dt} = \frac{V}{d}w(t)$$

where d is a constant with dimensions of length. As the car moves, the transverse position $p(t)$ of the car changes according to

$$\frac{dp(t)}{dt} = V\sin\big(\theta(t)\big) \approx V\theta(t)\,.$$

Consider three control schemes:

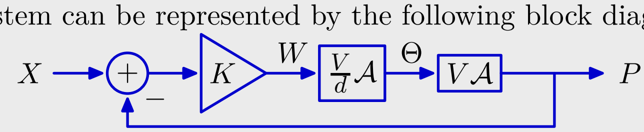
a. $w(t) = Ke(t)$

b. $w(t) = K_v\dot{e}(t)$

c. $w(t) = Ke(t) + K_v\dot{e}(t)$

where $e(t)$ represents the difference between the desired transverse position $x(t) = 0$ and the current transverse position $p(t)$. Describe the behaviors that result for each control scheme when the car starts with a non-zero angle ($\theta(0) = \theta_0$ and $p(0) = 0$). Determine the most acceptable value(s) of K and/or K_v for each control scheme or explain why none are acceptable.

Part a. This system can be represented by the following block diagram:



We are given a set of initial conditions — $p(0) = 0$ and $\theta(0) = \theta_0$ — and we are asked to characterize the response $p(t)$. Initial conditions are easy to take into account when a system is described by differential equations. However, feedback is easiest to analyze for systems expressed as operators or (equivalently) Laplace transforms. Therefore we first calculate the closed-loop system function,

$$H(s)\frac{Y(s)}{X(s)} = \frac{K\frac{V}{d}V\frac{1}{s^2}}{1 + K\frac{V}{d}V\frac{1}{s^2}} = \frac{K\frac{V^2}{d}}{s^2 + K\frac{V^2}{d}}$$

which has two poles: $\pm j\omega_0$ where $\omega_0 = V\sqrt{\frac{K}{d}}$. We can convert the system function to a differential equation:

$$\ddot{p}(t) + K\frac{V^2}{d}p(t) = K\frac{V^2}{d}x(t)$$

and then find the solution when $x(t) = 0$,

$$\ddot{p}(t) + K\frac{V^2}{d}p(t) = 0$$

so that $p(t) = C\sin\omega_0 t$ since $p(0) = 0$.

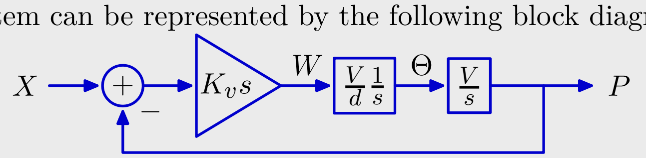
From $p(t)$ we can calculate $\theta(t) = \dot{p}(t)/V = \frac{C}{V}\omega_0\cos\omega_0 t$. From the initial condition $\theta(0) = \theta_0$, it follows that $C = V\theta_0/\omega_0$ and

$$p(t) = \frac{V\theta_0}{\omega_0}\sin\omega_0 t = \theta_0\sqrt{\frac{d}{K}}\sin V\sqrt{\frac{K}{d}}t$$

for $t > 0$.

If K is small, then the oscillations are slow, but they have a large amplitude. If K is large, then the oscillations are fast (and therefore uncomfortable for passengers), but the amplitude is small. While none of these behaviors are desireable, it would probably be best to increase K so that the amplitude of the oscillation is small enough so that the car stays in its lane.

Part b. The system can be represented by the following block diagram:



The closed-loop system function is

$$H(s) = \frac{K_v s\frac{V^2}{d}\frac{1}{s^2}}{1 + K_v s\frac{V^2}{d}\frac{1}{s^2}} = \frac{K_v s\frac{V^2}{d}}{s(s + K_v\frac{V^2}{d})}.$$

The closed-loop poles are at $s = 0$ and $s = -\frac{K_v}{d}V^2$.

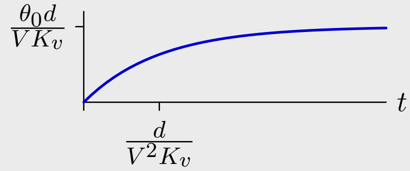
Since $p(0) = 0$, the form of $p(t)$ is given by

$$p(0) = C\left(1 - e^{-\frac{K_v}{d}V^2 t}\right)$$

for $t > 0$. We can find C by relating C to the initial value of $\theta(t) = \dot{p}(t)/V$. Since $\theta(0) = \theta_0$, $\dot{p}(0) = V\theta_0$. Therefore $C = \frac{1}{K_v\frac{V}{d}}$, so that

$$p(t) = \frac{\theta_0}{K_v\frac{V}{d}}\left(1 - e^{-\frac{K_v}{d}V^2 t}\right)$$

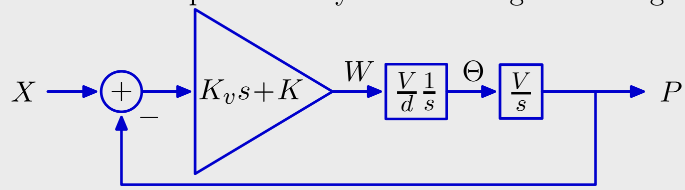
for $t > 0$ as shown below.



We would like to make K_v large because large K_v leads to fast convergence. Large values of K_v also lead to smaller steady-state errors in $p(t)$.

There are no oscillations in $p(t)$ with the velocity sensor, which is an advantage over results with the position sensor in part a. However, there is now a steady-state error in $p(t)$, which is worse. Fortunately the steady-state error can be made small with large K_v.

Part c. The system can be represented by the following block diagram:



The closed-loop system function is

$$H(s) = \frac{(K_v s + K)\frac{V^2}{d}\frac{1}{s^2}}{1 + (K_v s + K)\frac{V^2}{d}\frac{1}{s^2}} = \frac{(K_v s + K)\frac{V^2}{d}}{s^2 + (K_v s + K)\frac{V^2}{d}} = \frac{(K_v s + K)\frac{V^2}{d}}{s^2 + \frac{1}{Q}s\omega_0 + \omega_0^2}$$

This second-order system has a resonant frequency $\omega_0 = \sqrt{\frac{K}{d}V^2}$ and a quality factor $Q = \frac{K}{K_v}\frac{1}{\omega_0}$.

There is an enormous variety of acceptable solutions to this problem, since there are many values of K and K_v that can work. Here, we focus on one line of reasoning based on our normalization of second-order system in terms of Q and ω_0.

To avoid excessive oscillations, we would like Q to be small. Try $Q = 1$. Then

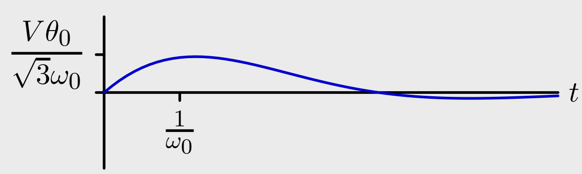
$$H(s) = \frac{(K_v s + K)\frac{V^2}{d}}{s^2 + \omega_0 s + \omega_0^2} .$$

Then $p(t)$ has the form

$$p(t) = Ce^{-\omega_0 t/2} \sin\left(\frac{\sqrt{3}}{2}\omega_0 t\right) .$$

As before, we can use the intial condition of $\theta(0) = \theta_0$ to determine C. In general, $\theta(t) = \dot{p}(t)/V$ so $\dot{p}(0) = \theta_0 V = C\sqrt{3}\omega_0/2$. Therefore

$$p(t) = \frac{2V\theta_0}{\sqrt{3}\omega_0} e^{-\omega_0 t/2} \sin\left(\frac{\sqrt{3}}{2}\omega_0 t\right) .$$



Increasing Q would reduce the overshoot but slow the response. We could compensate for the slowing of the response by increasing ω_0.

Performance can be adjusted to be better than either part a or part b. By adjusting Q and ω_0 we can get convergence of $p(t)$ to zero with minimum oscillation.

Although the steady-state value of the error is zero and the oscillation is minimized, there is still a transient behavior, which could momentarily move the car into the other lane!