

Problem Set 3: Hsieh and Klenow (QJE 2009)

This exercise goes through careful derivations of equations in the paper.

(a) (1 point) Given the production function given in formula (3) of the paper, solve the cost minimization problem

$$P_s Y_s = \min \sum_{i=1}^{M_s} P_{si} Y_{si}$$

subject to

$$Y_s = \left(\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

Determine P_s .

Solution: Set up the Lagrangian

$$L_s = \sum_{i=1}^{M_s} P_{si} Y_{si} + \lambda_s \left(Y_s^{\frac{\sigma-1}{\sigma}} - \sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right),$$

where λ_s is the multiplier on the constraint. Taking first-order conditions yields

$$\frac{\partial L_s}{\partial Y_{si}} = P_{si} - \lambda_s \frac{\sigma-1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} = 0.$$

This shows that total costs are given by

$$\sum_{i=1}^{M_s} P_{si} Y_{si} = \lambda_s \frac{\sigma-1}{\sigma} \sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \tag{9}$$

and that the demand function is given by

$$Y_{si}^{\frac{\sigma-1}{\sigma}} = \left(\frac{1}{P_{si}} \lambda_s \frac{\sigma-1}{\sigma} \right)^{\sigma-1}. \tag{10}$$

We are now going to solve for the multiplier λ_s . The demand function (10) implies

$$Y_s = \left(\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \left(\lambda_s \frac{\sigma-1}{\sigma} \right)^{\sigma} \left(\sum_{i=1}^{M_s} \left(\frac{1}{P_{si}} \right)^{\sigma-1} \right)^{\frac{\sigma}{\sigma-1}} \tag{11}$$

This implies that the multiplier λ_s is given by

$$\frac{\sigma-1}{\sigma} \lambda_s = Y_s^{\frac{1}{\sigma}} \left(\sum_{i=1}^{M_s} \left(\frac{1}{P_{si}} \right)^{\sigma-1} \right)^{\frac{-1}{\sigma-1}}. \tag{12}$$

Plugging this in (9) yields

$$\begin{aligned}
\sum_{i=1}^{M_s} P_{si} Y_{si} &= \lambda_s \frac{\sigma - 1}{\sigma} Y_s^{\frac{\sigma-1}{\sigma}} \\
&= Y_s^{\frac{1}{\sigma}} \left(\sum_{i=1}^{M_s} \left(\frac{1}{P_{si}} \right)^{\sigma-1} \right)^{-\frac{1}{\sigma-1}} Y_s^{\frac{\sigma-1}{\sigma}} \\
&= Y_s \left(\sum_{i=1}^{M_s} \left(\frac{1}{P_{si}} \right)^{\sigma-1} \right)^{-\frac{1}{\sigma-1}}
\end{aligned} \tag{13}$$

So, finally we arrive at

$$P_s = \left(\sum_{i=1}^{M_s} P_{si}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \tag{14}$$

P_s is the price index to buy the *composite* good Y_s . More specifically: if the M individual goods have prices $[P_{si}]_{i=1}^M$, then one unit of the composite good $Y_s = \left(\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$ costs P_s , if each individual variety Y_{si} is bought in the efficient proportions. P_s is sometimes referred to as the ideal price index and in many models P_s is used as the numeraire.

(b) (1 point) Show that the profit maximization of firm i in industry s is

$$\max_{Y_{si}, L_{si}, K_{si}} (1 - \tau_{Y_{si}}) \lambda_s \frac{\sigma - 1}{\sigma} (Y_{si})^{\frac{\sigma-1}{\sigma}} - w L_{si} - (1 + \tau_{K_{si}}) R K_{si} \tag{15}$$

subject to $Y_{si} = A_{si} K_{si}^{\alpha_s} L_{si}^{1-\alpha_s}$.

Solution: Given the demand curve (10)

$$P_{si} = \lambda_s \frac{\sigma - 1}{\sigma} Y_{si}^{-\frac{1}{\sigma}},$$

after-tax revenue of firm si is

$$(1 - \tau_{Y_{si}}) P_{si} Y_{si} = (1 - \tau_{Y_{si}}) \lambda_s \frac{\sigma - 1}{\sigma} (Y_{si})^{\frac{\sigma-1}{\sigma}}.$$

Hence, (15) is the appropriate profit maximization problem. Note that λ_s is an endogenous variable as seen in (12). It depends on the prices of all other firms and on aggregate demand (the market size Y_s). Note also that λ_s is taken as given by firm si , although it depends on P_{si} (and hence on Y_{si} via the demand function). This is the essence of monopolistic competition - firms recognize their market power in their variety i , but they take economywide aggregates as given.

(c) Use the solution to the firm maximization problem and the expression of P_s to derive the formula (15) in the paper.

Solution First of all, sorry we had to make you go through this, but take it as a lesson for life ... and enjoy the fact that you are never going to do this again. To derive the result, we use the

following alternative definitions of sectoral average marginal products of labor and capital (instead of the ones proposed in page 1409)

$$\begin{aligned}\frac{1}{\overline{MPRL}_s} &= \sum_{i=1}^{M_s} \frac{1}{MPRL_{si}} \frac{P_{si}Y_{si}}{P_sY_s} \\ \frac{1}{\overline{MPRK}_s} &= \sum_{i=1}^{M_s} \frac{1}{MPRK_{si}} \frac{P_{si}Y_{si}}{P_sY_s}\end{aligned}\tag{16}$$

Profit maximization on the firms' behalf implies the first order conditions

$$\begin{aligned}(1 - \tau_{Y_{si}}) \lambda_s \frac{\sigma - 1}{\sigma} \frac{\sigma - 1}{\sigma} (1 - \alpha_s) Y_{si}^{\frac{\sigma-1}{\sigma}} &= wL_{si} \\ (1 - \tau_{Y_{si}}) \lambda_s \frac{\sigma - 1}{\sigma} \frac{\sigma - 1}{\sigma} \alpha_s Y_{si}^{\frac{\sigma-1}{\sigma}} &= (1 + \tau_{K_{si}}) RK_{si}.\end{aligned}$$

Using the definition of λ_s (see (10)), we get that

$$(1 - \tau_{Y_{si}}) \frac{\sigma - 1}{\sigma} (1 - \alpha_s) P_{si}Y_{si} = wL_{si}$$

and

$$(1 - \tau_{Y_{si}}) \frac{\sigma - 1}{\sigma} \alpha_s P_{si}Y_{si} = (1 + \tau_{K_{si}}) RK_{si}$$

First, look at firms' labor demands. Using equation (10) in the paper $MPRL_{si} = \frac{w}{1 - \tau_{Y_{si}}}$, we obtain

$$\begin{aligned}L_{si} &= \frac{1}{MPRL_{si}} \frac{\sigma - 1}{\sigma} (1 - \alpha_s) P_{si}Y_{si} \\ &= \frac{1}{MPRL_{si}} \frac{P_{si}Y_{si}}{P_sY_s} \frac{\sigma - 1}{\sigma} (1 - \alpha_s) P_sY_s \\ &= \frac{1}{MPRL_{si}} \frac{P_{si}Y_{si}}{P_sY_s} \frac{\sigma - 1}{\sigma} (1 - \alpha_s) \theta_s PY,\end{aligned}$$

where the last equality $P_sY_s = \theta_s PY$ follows from the Cobb-Douglas structure of final demand - expenditure shares across sectors are equal to the share parameter θ_s . Using this, we get

$$\begin{aligned}L_s &= \sum_i L_{si} = \sum_i \left(\frac{1}{MPRL_{si}} \frac{P_{si}Y_{si}}{P_sY_s} \frac{\sigma - 1}{\sigma} (1 - \alpha_s) \theta_s PY \right) \\ &= \frac{\sigma - 1}{\sigma} (1 - \alpha_s) \theta_s PY \sum_i \left(\frac{1}{MPRL_{si}} \frac{P_{si}Y_{si}}{P_sY_s} \right) \\ &\equiv \frac{\sigma - 1}{\sigma} (1 - \alpha_s) \theta_s PY \frac{1}{\overline{MPRL}_s},\end{aligned}$$

where \overline{MPRL}_s is defined in (16). Rearranging terms yields

$$\frac{L_s}{\frac{1}{\overline{MPRL}_s} (1 - \alpha_s) \theta_s} = \frac{\sigma - 1}{\sigma} PY.\tag{17}$$

Using the labor market clearing condition $L = \sum_s L_s$, (17) implies that

$$L_s = L \frac{\frac{1}{\overline{MPRL}_s} (1 - \alpha_s) \theta_s}{\sum \frac{1}{\overline{MPRL}_{s'}} (1 - \alpha_{s'}) \theta_{s'}} \quad (12' \text{ HK})$$

Similarly, consider the firms' capital demands

$$\begin{aligned} K_{si} &= \frac{1}{R} \frac{1 - \tau_{Y_{si}}}{1 + \tau_{K_{si}}} \frac{\sigma - 1}{\sigma} \alpha_s P_{si} Y_{si} \\ &= \frac{1}{\overline{MRPK}_{si}} \frac{\sigma - 1}{\sigma} \alpha_s P_{si} Y_{si} \end{aligned}$$

Therefore, by definition of \overline{MPRK}_s , we have

$$K_s = \sum K_{si} = \frac{1}{\overline{MPRK}_s} \frac{\sigma - 1}{\sigma} \alpha_s \theta_s P Y.$$

We also obtain

$$K_s = K \frac{\frac{1}{\overline{MPRK}_s} \alpha_s \theta_s}{\sum \frac{1}{\overline{MPRK}_{s'}} \alpha_{s'} \theta_{s'}} \quad (13' \text{ HK})$$

From the definition of the sectoral average marginal product of labor and capital (see (17)) we have

$$\overline{MPRL}_s L_s = \frac{\sigma - 1}{\sigma} (1 - \alpha_s) P_s Y_s;$$

so

$$\frac{P_s Y_s}{L_s} = \frac{\sigma}{\sigma - 1} \frac{1}{1 - \alpha_s} \overline{MPRL}_s. \quad (18)$$

Similarly

$$\overline{MPRK}_s K_s = \frac{\sigma - 1}{\sigma} \alpha_s P_s Y_s;$$

so

$$\frac{P_s Y_s}{K_s} = \frac{\sigma}{\sigma - 1} \frac{1}{\alpha_s} \overline{MPRK}_s. \quad (19)$$

We are now in the position to derive (15) of the paper. First of all note that by definition (see (14) in the paper)

$$TFP_s = \frac{Y_s}{K_s^{\alpha_s} L_s^{1 - \alpha_s}}.$$

Using (18) and (19) this implies that

$$\begin{aligned} TFP_s &= \left(\frac{Y_s P_s}{K_s} \right)^{\alpha_s} \left(\frac{Y_s P_s}{L_s} \right)^{1 - \alpha_s} \frac{1}{P_s} \\ &= \frac{\sigma}{\sigma - 1} \left(\frac{\overline{MPRK}_s}{\alpha_s} \right)^{\alpha_s} \left(\frac{\overline{MPRL}_s}{1 - \alpha_s} \right)^{1 - \alpha_s} \frac{1}{P_s} \\ &= \overline{TFPR}_s \frac{1}{P_s}. \end{aligned}$$

Now recall that

$$P_s = \left(\sum_i P_{si}^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

which implies that

$$\frac{1}{P_s} = \left(\sum_i \left(\frac{A_{si}}{TFPR_{si}} \right)^{\sigma-1} \right)^{\frac{1}{\sigma-1}},$$

as

$$TFPR_{si} = \frac{P_{si} Y_{si}}{K_{si}^{\alpha_s} L_{si}^{1-\alpha_s}} = P_{si} A_{si}.$$

Hence,

$$\begin{aligned} TFP_s &= \overline{TFPR}_s \frac{1}{P_s} \\ &= \overline{TFPR}_s \left(\sum_i \left(\frac{A_{si}}{TFPR_{si}} \right)^{\sigma-1} \right)^{\frac{1}{\sigma-1}} \\ &= \left(\sum_i \left(A_{si} \overline{TFPR}_s \right)^{\sigma-1} \right)^{\frac{1}{\sigma-1}}, \end{aligned}$$

which is the required equation.

(d) There is a large literature trying to link the distortions $(\tau_{Y_{si}}, \tau_{K_{si}})$ to financial frictions individual firms face. To see the relation between these exogenous taxes and credit constraints, suppose that there are no taxes (i.e. $\tau_{Y_{si}} = \tau_{K_{si}} = 0$) but firm i faces a credit constraint of the form

$$wL_{si} + \zeta RK_{si} \leq W(z_{si}, \eta),$$

where z_{si} is a firm characteristic (e.g. wealth), η parametrizes the financial system and ζ parametrizes how much of capital expenses can be pledged. Suppose that W is increasing in both argument, i.e. wealthy firms are less constrained and better financial system are associated with higher values of η . Derive the firm's factor demands taking factor prices as given and aggregate demand as given. What are the firm-specific "taxes" in this framework? Which firms face high "output-taxes $\tau_{Y_{si}}$ "? Under what conditions would a researcher conclude that $\tau_{K_{si}} = 0$?

Solution: The firm solves the problem given in part (b) with $\tau_{Y_{si}} = \tau_{K_{si}} = 0$ but facing the credit constraint. I.e. the profit maximization problem is given by

$$\max_{Y_{si}, L_{si}, K_{si}} \lambda_s \frac{\sigma-1}{\sigma} (Y_{si})^{\frac{\sigma-1}{\sigma}} - wL_{si} - RK_{si}$$

subject to $Y_{si} = A_{si} L_{si}^{1-\alpha_s} K_{si}^{\alpha_s}$ and the constraint

$$wL_{si} + RK_{si} \leq W(z_{si}, \eta).$$

Letting $\mu(z_{si}, \eta)$ be the multiplier on the credit constraint, the first order conditions characterizing the firm's factor demand are given by

$$\begin{aligned} \frac{1}{1 + \mu(z_{si}, \eta)} \lambda_s \left(\frac{\sigma - 1}{\sigma} \right)^2 \alpha_{si} Y_{si}^{\frac{\sigma-1}{\sigma}} &= RK_{si} \left(\frac{1 + \zeta \mu(z_{si}, \eta)}{1 + \mu(z_{si}, \eta)} \right) \\ \frac{1}{1 + \mu(z_{si}, \eta)} \lambda_s \left(\frac{\sigma - 1}{\sigma} \right)^2 (1 - \alpha_{si}) Y_{si}^{\frac{\sigma-1}{\sigma}} &= wL_{si} \end{aligned}$$

Substituting the expression for λ_s , factor demands are

$$\begin{aligned} \frac{1}{1 + \mu(z_{si}, \eta)} \left(\frac{\sigma - 1}{\sigma} \right) \alpha_{si} P_{si} Y_{si} &= RK_{si} \left(\frac{1 + \zeta \mu(z_{si}, \eta)}{1 + \mu(z_{si}, \eta)} \right) \\ \frac{1}{1 + \mu(z_{si}, \eta)} \left(\frac{\sigma - 1}{\sigma} \right) (1 - \alpha_{si}) P_{si} Y_{si} &= wL_{si}. \end{aligned}$$

The corresponding demand in Hsieh-Klenow are given by

$$(1 - \tau_{Y_{si}}) \frac{\sigma - 1}{\sigma} (1 - \alpha_s) P_{si} Y_{si} = wL_{si}$$

and

$$(1 - \tau_{Y_{si}}) \frac{\sigma - 1}{\sigma} \alpha_s P_{si} Y_{si} = (1 + \tau_{K_{si}}) RK_{si}$$

Hence, taxes play exactly the role of the Lagrange multiplier of the firm's problem. In particular, the solutions to these problem is identical if

$$1 - \tau_{Y_{si}} = \frac{1}{1 + \mu(z_{si}, \eta)}$$

and

$$1 + \tau_{K_{si}} = \frac{1 + \zeta \mu(z_{si}, \eta)}{1 + \mu(z_{si}, \eta)}.$$

Clearly, a high τ_Y firm is one where $\mu(z_{si}, \eta)$ is high, i.e. which has a high shadow value of internal funds. With parametrization given above, μ is decreasing in z_{si} (i.e. if you have more capital to pledge your shadow value of internal funds is lower) and η (as good financial institution e.g. allow you to borrow more against each dollar of collateral). Hence, poor firms and firms in underdeveloped regions face binding constraints and will be identified as firms facing high output distortions. Similarly, $\tau_{K_{si}}$ is equal to zero, whenever $\zeta = 1$, i.e. if capital and labor are "equally pledgable", the relative tax on capital is zero. If $\zeta < 1$, $\tau_K < 0$, i.e. capital is the relative unconstrained factor and the firm acts as if capital is cheap. Basically: you can borrow against capital but not against labor. As the production function allows some substitution between labor and capital, the firm will produce as higher capital intensities, which Hsieh and Klenow would identify as the firm receiving a subsidy of capital.

[Part II] This part concerns the analysis of equations in Appendix I in the paper.

(a) Show that $TFP = \overline{TFPR} \frac{w}{P\gamma}$ in which $\overline{TFPR} = \sum_{i=1}^M \frac{L_i}{L} TFP R_i$

Soln Firm i solves

$$\max_{L_i} \pi_i = (1 - \tau_i) P A_i L_i^\gamma - w L_i.$$

The FOC is

$$(1 - \tau_i) P A_i \gamma L_i^{\gamma-1} = w$$

so that firm i 's output is given by

$$Y_i = \frac{w}{P\gamma} L_i \frac{1}{1 - \tau_i}.$$

Aggregate output is given by

$$Y = \sum Y_i = \frac{w}{P\gamma} \sum_i L_i \frac{1}{1 - \tau_i}.$$

As a result,

$$\begin{aligned} TFP &= \frac{Y}{L} = \frac{w}{P\gamma} \sum \frac{L_i}{L} \frac{1}{1 - \tau_i} \\ &= \frac{w}{P\gamma} \sum \frac{L_i}{L} TFP R_i \\ &= \frac{w}{P\gamma} \overline{TFPR}. \end{aligned}$$

(b) Suppose $(1 - \tau_i) = a \frac{1}{A_i}$. Using the labor market clearing condition, show that

$$TFP = \frac{1}{M\gamma} \frac{\sum_{i=1}^M A_i}{L^{1-\gamma}}$$

independent of a . Give a concise interpretation why aggregate TFP is independent of a . What is the crucial assumption for this result?

Solution The firms' labor demand equation (20) implies

$$a P \gamma L_i^{\gamma-1} = w,$$

so that

$$L_i = \left(\frac{a P \gamma}{w} \right)^{\frac{1}{1-\gamma}}.$$

Market clearing implies

$$L = \sum_{i=1}^M L_i = M \left(\frac{a P \gamma}{w} \right)^{\frac{1}{1-\gamma}},$$

so that equilibrium wages are given by

$$w = a P \gamma \left(\frac{M}{L} \right)^{1-\gamma}.$$

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