14.471 Notes on Linear Taxation

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1 Overview

- Two models
  - single agent (Ramsey), no lump sum tax
  - agent heterogeneity and lump sum tax
- Two approaches
  - primal
  - dual
- Mixed Taxation

2 Single Agent Ramsey

- consumers:

$$\max_x u(x) \qquad \sum_i q_i x_i \leq 0$$

  e.g $u(c_1, c_2, \ldots, c_n, l)$ and $\sum p_i(1+\tau_i)c_i = (1-\tau^l)wl$

- CRS technology (inputs are supressed)

$$F(y) \leq 0$$

  e.g. $\sum \bar{p}_i y_i - l \leq 0$

- Remark: Production efficiency holds so that $F(y) = 0$ at optimum
  (implies intermediate inputs go untaxed)
  without CRS this result requires profit taxes (see Diamond-Mirrlees)

- First Best
  - $\text{MRS}^h_{ij} = \text{MRS}^{h'}_{ij}$
  - $\text{MRS}^h_{ij} = \text{MRT}_{ij}$
  - $F = 0$ (efficient production; with inputs this requires a marginal condition equating the relative marginal products across goods)

- Firms

$$\max_y py \qquad F(y) \leq 0$$

- government

$$\sum p_i g_i \leq \sum t_i x_i$$

- market clearing:

$$x_i + g_i = y_i \qquad \forall i$$

- note: we could have $u(c, g)$, but in what follows g is fixed, so we supress the dependence.

- Definition: A Competitive Equilibrium (CE) with taxes is p, q, c

  1. x solves the consumer's maximization

$$\max_x u(x) \qquad \sum_i q_i x_i \leq 0$$

  2. y solves the profit maximization

$$\max_y py \qquad F(y) \leq 0$$

  3. x, g, t, p satisfy the government budget constraint

$$\sum p_i g_i \leq \sum t_i x_i$$

  4. markets clear

$$x_i + g_i = y_i \qquad \forall i$$

- Result: CE $\iff$ $F(x+g) = 0$ and agent optimization (1)

- note: second condition involves x and q only

- First Best

$$\max_{x,q} u(x)$$

$$F(x+g)=0$$

- Second Best

$$\max_{x,q} u(x)$$

$$F(x+g)=0$$

$$x \in \arg\max_x u(x) \qquad q \cdot x \leq 0$$

- we have two variables x, q but they are related through the last condition

- At this point, from consumer maximization we can approach things from...

  - primal: solve q as a function of x
  - dual: solve x as a function of q

- both approaches are useful

2.1 Dual

- define

  $$V(q,I) = \max_x u(x) \qquad q \cdot x \leq I$$

  and let $x_i(q,I)$ denote the solution (Marshallian/uncompensated demand)

  $$e(q,v) \equiv \min_x q \cdot x \qquad u(x) = v$$

  and let $x_i^c(q,v) = e_{q_i}(q,v)$ denote the solution (Hicks/compensated demand)

- we abuse notation: $V(q) = V(q,0)$

- Second Best:

  $$\max_q V(q) \qquad \text{s.t.} \qquad F(x(q,0)+g)=0$$

- property:

  $$x^c(q,V(q)) = x(q,0)$$

- equivalently

$$\max_q V(q) \qquad \text{s.t.} \qquad F(x^c(q, V(q) + g) = 0$$

2.2 Optimality condition

- We have the first order condition

$$\frac{\partial V}{\partial q_j}(q,0) - \kappa \sum_i \frac{\partial F}{\partial y_i}\left(\frac{\partial x_i^c}{\partial q_j} + \frac{\partial x_i^c}{\partial v}\frac{\partial V}{\partial q_j}\right) = 0$$

- By Roy's identity $\frac{\partial V}{\partial q_j} = -x_j \frac{\partial V}{\partial I}$:

$$-\frac{1}{\kappa} x_j \frac{\partial V}{\partial I} - \sum_i \frac{\partial F}{\partial y_i}\left(\frac{\partial x_i^c}{\partial q_j} - x_j \frac{\partial x_i^c}{\partial v}\frac{\partial V}{\partial I}\right) = 0$$

- Now use that $\frac{\partial x_i^c}{\partial v}\frac{\partial V}{\partial q_j} = \frac{\partial x_i}{\partial I}$ and $p_i = \frac{\partial F}{\partial y_i}$ to get

$$-\frac{1}{\kappa} x_j \frac{\partial V}{\partial I} - \sum_i p_i \frac{\partial x_i^c}{\partial q_j} + x_j \sum_i p_i \frac{\partial x_i}{\partial I} = 0$$

- Now we know that $\sum_i q_i \frac{\partial x_j^c}{\partial q_i} = 0$ and that $\frac{\partial x_j^c}{\partial q_i} = \frac{\partial x_i^c}{\partial q_j}$ by symmetry so that

$$-\sum_i p_i \frac{\partial x_i^c}{\partial q_j} = \sum_i t_i \frac{\partial x_i^c}{\partial q_j}$$

- Also, we know that $\sum_i q_i \frac{\partial x_i}{\partial I} = 1$ so that

$$\sum_i p_i \frac{\partial x_i}{\partial I} = 1 - \sum_i t_i \frac{\partial x_i}{\partial I}$$

- Thus, we obtain

$$\sum_i t_i \frac{\partial x_i^c}{\partial q_j} = -x_j \theta$$

where

$$\theta \equiv -\frac{1}{\kappa}\frac{\partial V}{\partial I} + 1 - \sum_i t_i \frac{\partial x_i}{\partial I}$$

- or equivalently (using symmetry)

$$\sum_i t_i \frac{\partial x_j^c}{\partial q_i} = -x_j \theta.$$

- interpretation:
  - each good is "discouraged" by a common percentage θ, i.e. interpret (falsely) as an estimate of how much good x_j fell due to taxation.
  - DWL$= e(q, V(q)) - \sum t_i x_i^c(p, V(q))$

$$\frac{1}{x_i p_i} \frac{\partial \text{DWL}}{\partial \tau_i} = \text{constant}$$

intuitive: marginal DWL is proportional to revenue base (mg cost = mg benefit)

2.3 Primal

- Primal solves q from x
- consumer optimization

$$x \in \arg\max_x u(x) \qquad q \cdot x \leq 0$$

- necessary and sufficient conditions: $\exists \lambda > 0$ s.t. (assuming local non-satiation)

$$q_i = \lambda u_i(x)$$

$$q \cdot x = 0$$

thus (imlementability condition)

$$\sum u_i(x) x = 0$$

- Result: reverse is also true: if $\sum u_i(x)x = 0$ then $\exists q$ such that $x \in \arg\max_x u(x)$ s.t. $q \cdot x \leq 0$.
- Second best

$$\max u(x)$$

$$F(x+g) = 0$$

$$\sum u_i(x)x = 0$$

- Lagrangian:

$$L = u(x) + \mu \sum u_i(x)x_i - \gamma F(x+g)$$

- FOC

$$(1+\mu)u_i(x) + \mu \sum_j u_{ij}(x)x_j = \gamma F_i(x+g)$$

- implication

$$\frac{F_i(x+g)}{F_k(x+g)} = \frac{u_i(x)}{u_k(x)} \frac{1 + \mu + \mu \sum_j \frac{u_{ij}(x)}{u_i(x)} x_j}{1 + \mu + \mu \sum_j \frac{u_{kj}(x)}{u_k(x)} x_j}$$

- since

$$\frac{F_i(x+g)}{F_k(x+g)} = \frac{p_i}{p_k} \qquad \frac{u_i(x)}{u_k(x)} = \frac{q_i}{q_k}$$

- tax rate (where $q_i = \tau_i p_i$)

$$\frac{\tau_i}{\tau_k} = \frac{1 + \mu + \mu \sum_j \frac{u_{ij}(x)}{u_i(x)} x_j}{1 + \mu + \mu \sum_j \frac{u_{kj}(x)}{u_k(x)} x_j}$$

- exercise: show that if $U(G(x_1, x_2, \ldots, x_n), x_0)$ and G is homogeneous of degree 1 then $\tau_1 = \tau_2 = \cdots = \tau_n$.

2.4 Many Agents Dual

- Second Best (dual)

$$\max_{q,I} \sum \lambda^h V^h(q,I)\pi^h \qquad \text{s.t.} \qquad F(\sum_h x^{c,h}(q, V^h(q,I))\pi^h + g) = 0$$

- note about I:
  - we can impose $I = 0$;
  - typically we do not want to: captures a lump sum transfer/tax
  - if we allow I free then productive efficiency is obvious
- more generally
  - Pareto problem not convex
  - cannot maximize weighted utility
  - but pareto weights for local optimality condition
- Define Lagrangian

$$L = \sum_h \lambda^h V^h(q,I)\pi^h - \gamma F(\sum_h x^{c,h}(q, V^h(q,I))\pi^h + g)$$

- FOCs: (using same identities as before)

$$-\sum_h \lambda^h x_j^h \frac{\partial V^h}{\partial I}\pi^h - \gamma \sum_{h,i} F_i \left[\frac{\partial x_i^{c,h}}{\partial q_j} - \frac{\partial x_i^{c,h}}{\partial I} x_i^h \right] \pi^h = 0$$

$$\sum_h \lambda^h \frac{\partial V^h}{\partial I}\pi^h - \gamma \sum_{h,i} F_i \frac{\partial x_i^h}{\partial I}\pi^h = 0$$

- notation:
  - population average: $\mathbb{E}_h[\cdot] = \sum_h [\cdot]\pi^h$
  - adjusted pareto weight: $\beta^h \equiv \frac{\lambda^h}{\gamma}\frac{\partial V^h}{\partial I}$

- we arrive at the condition

$$\mathbb{E}_h\left[\sum_l t_l \frac{\partial x_j^{c,h}}{\partial q_l}\right] = X_j \mathbb{E}_h\left[\frac{x_j^h}{X_j}\left(-1+\beta^h+\sum_l t_l \frac{\partial x_l^h}{\partial I}\right)\right]$$

- Note that if we have homothetic and separable preferences then

$$\frac{x_j^h}{X_j}$$

is independent of j. So from here we can see a uniform tax result.

- if we have a lump sum then:

$$\mathbb{E}_h\left[-1+\beta^h+\sum_l t_l \frac{\partial x_l^h}{\partial I}\right] = 0$$

so we can write

$$\mathbb{E}_h\left[\sum_l t_l \frac{\partial x_j^{c,h}}{\partial q_l}\right] = X_j Cov_h\left[\frac{x_j^h}{X_j}, \hat{\beta}^h\right]$$

where $\hat{\beta}^h = \beta^h + \sum_l t_l \frac{\partial x_l^h}{\partial I}$.

- We get two intuitive cases:
  - $\hat{\beta}^h$ is constant;
  - $\frac{x_j^h}{X_j}$ is independent of j. Then back to regular case.
- Q: Pareto inefficiency?
- A: If $\#agents < \#goods$ maybe cannot find β^h that solve these equations
- Suppose utility is

$$U^i(G(x_1,\ldots,x_{N_1}), H(x_{N_1+1},\ldots,x_N))$$

and G, H are h.o.d. 1

- Result: tax uniformly within each group.
- Proof: treat goods $(x_1, x_2, \ldots, x_{N_1})$ and $(x_{N_1}, x_2, \ldots, x_N)$ as inputs into production of G and H.

3 Mixed Taxation: Atkinson-Stiglitz

- Notation:
  $x \in R^m$ consumption goods
  $Y \in \mathbb{R}$ labor (in efficiency units)
  B budget set

- Given B consumers solve:

$$(x^i, Y^i) \in \arg\max_{(x,Y)\in B} U^i(x, Y)$$

- Technology (linear)

$$\sum_{i,j} p_j x_j^i \pi^i \leq \sum_i Y^i$$

- Feasibility. previous 2 conditions hold.

- if B^i allowed to be dependent on i then we can get the first best (Welfare theorem)

- ...but here B is independent of i so we are in the second best

- Assume:

$$u^i(x, Y) = U^i(G(x), Y)$$

- Result: uniform taxation is efficient (Atkinson-Stiglitz).

$$B_{AS} \equiv \{(x, Y) | p \cdot x \leq Y - T(Y)\}$$

  Indeed, anything else is Pareto inefficient!

- Exercise to get to result...

  1. start from B_0 that uses commodity taxes
  2. create new B that is "better"

  Here "better": save resources and same utility. (Why better?)

- really can start from any arbitrary B_0

- Note: "two stage" budgeting (given any B)...
  Define:

$$b = \{(g, Y) | \exists x \text{ s.t. } g = G(x) \text{ and } (x, Y) \in B\}$$

then agents solve (outer stage):

$$\arg\max_{g,Y\in b} U^i(g,Y)$$

- Idea: given B_0 we have some b_0. We change B_1 but keep implied $b_1 = b_0$. Then we get the same choices of Y^i and the same utility for each agent. Good choice:

$$B_1 = B_{AS} \equiv \{(x,Y)|\exists g \text{ s.t. } p\cdot x \le e^G(g,p) \text{ and } (g,Y)\in b_0\}$$

  where $e^G(g,p) \equiv \min_x p\cdot x$ s.t. $g = G(x)$, is the expenditure function for G.

- Equivalently if we define

$$\hat{b} \equiv \{(y,Y)|\exists g \text{ s.t. } y = e^G(g,p) \text{ and } (g,Y)\in b\}$$

  then

$$B_{AS} \equiv \{(x,Y)|p\cdot x \le y \text{ and } (y,Y)\in \hat{b}\}$$

  which has an obvious income tax interpretation.

- This will save resources as long as x choices change. Why?

4 Pigouvian Taxation

- now assume
  - single agent
  - lump sum taxation
  - but externalities
- utility

$$u(x,\bar{x})$$

  concave in both x and $\bar{x}$

- technology

$$F(x+g) = 0$$

- in equilibrium

$$\bar{x} = x$$

- agent solves (takes $\bar{x}$ as given)

$$\max_x u(x,\bar{x}) \qquad q\cdot x = I$$

$$\Rightarrow u_x(x^e,x^e) = \lambda q$$

$$\Rightarrow \frac{q_i}{q_j} = \frac{u_{x_i}(x^e,x^e)}{u_{x_j}(x^e,x^e)}$$

- Social optimum

$$\max_x u(x,x) \qquad F(x+g)=0$$

$$\Rightarrow u_x(x^*,x^*) + u_{\bar{x}}(x^*,x^*) = \gamma F_x(x^*+g)$$

$$\Rightarrow \frac{p_i}{p_j} = \frac{F_{x_i}}{F_{x_j}} = \frac{u_{x_i}(x^*,x^*) + u_{\bar{x}_i}(x^*,x^*)}{u_{x_j}(x^*,x^*) + u_{\bar{x}_j}(x^*,x^*)}$$

- To make

$$x^e = x^*$$

a necessary condition is that both conditions hold, implying

$$\frac{p_i/q_i}{p_j/q_j} = \frac{1+\frac{u_{\bar{x}_i}(x^*,x^*)}{u_{x_i}(x^*,x^*)}}{1+\frac{u_{\bar{x}_j}(x^*,x^*)}{u_{x_j}(x^*,x^*)}}$$

- Theorem: if p and q to satisfy this equation, then there exists an income I (i.e. lump sum tax/transfer) so that the agent chooses $x = x^*$.
  Proof: (sketch) Use Lagrangian sufficiency theorem.

5 Application to Intertemporal Taxation

- neoclassical growth model
- simplifying assumptions
  - single agent first
  - no uncertainty

- technology

$$c_t + g_t + k_{t+1} \leq F(k_t, L_t) + (1 - \delta)k_t$$

where F is CRS

- preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t, L_t)$$

- budget constraints

  - agents

$$c_t + k_{t+1} + q_{t,t+1}B_{t+1} \leq (1 - \tau_t)w_t L_t + R_t k_t + (1 - \kappa_t^B)B_t$$

    where

$$R_t = 1 + \kappa_t(r_t - \delta)$$

    we also need some no-ponzi conditions

$$q_{0,t} = q_{0,1}q_{1,2} \cdots q_{t-1,t}$$

$$\lim_{T \to \infty} q_{0,T}B_T \geq 0$$

  - government:

$$g_t + B_t \leq \tau_t w_t L_t + \kappa_t r_t k_t + q_{t,t+1}B_{t+1}$$

- without loss of generality:

$$\kappa_t^B = 0 \qquad t = 1, 2, \ldots$$

- Firms:

$$\max_{K_t, L_t}\{F(K_t, L_t) - w_t L_t - r_t K_t\}$$

necessary and sufficient conditions

$$F_L(K_t, L_t) = w_t$$
$$F_K(K_t, L_t) = r_t$$

- Definition of an equilibrium:

  - agents maximize given prices and taxes

– firms maximize

– government budget constraint satisfied

– market clears: goods, capital and bonds

- adding up both budget constraints gives

$$g_t + c_t + k_{t+1} \leq w_t L_t + (1 + r_t - \delta) k_t = F(k_t, L_t) + (1 - \delta) k_t$$

which is just the resource constraint

- solving B_t forward

$$\sum_{t=0}^{\infty} q_{0,t} (c_t - (1 - \tau_t) w_t L_t - R_t k_t + k_{t+1}) \leq (1 - \kappa_0^B) B_0$$

unless

$$q_{t+1} R_{t+1} = \frac{q_{0,t+1}}{q_{0,t}} R_{t+1} = 1 \qquad t = 0, 1, \ldots$$

there is an arbitrage

- cancelling:

$$\sum_{t=0}^{\infty} q_{0,t} (c_t - (1 - \tau_t) w_t L_t) \leq R_0 k_0 + (1 - \kappa_0^B) B_0$$

- now we can just apply the primal approach

- implementability condition:

$$\sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t + u_{L,t} L_t) = u_{c,0} (R_0 k_0 + (1 - \kappa_0^B) B_0)$$

- Lagrangian

$$L \equiv \sum_{t=0}^{\infty} \beta^t W(c_t, L_t; \mu) - \mu u_{c,0} (R_0 k_0 + (1 - \kappa_0^B) B_0)$$

where

$$W(c, L; \mu) \equiv u(c, L) + \mu \left(u_c(c, L) c + u_L(c, L) L \right)$$

- optimality conditions obtained from

$$\max L \qquad \text{s.t. resource constraint}$$

- first order conditions:

$$-\frac{W_L(c_t, L_t; \mu)}{W_c(c_t, L_t; \mu)} = F_L(K_t, L_t)$$

$$W_c(c_t, L_t; \mu) = \beta R^*_{t+1} W_c(c_{t+1}, L_{t+1}; \mu)$$

where $R^*_{t+1} \equiv F_k(k_{t+1}, L_{t+1}) + 1 - \delta$ is the social rate of return

- for agent

$$w_t(1 - \tau_t) = -\frac{u_L(c_t, L_t)}{u_c(c_t, L_t)}$$

$$u_c(c_t, L_t) = \beta R_t u_c(c_{t+1}, L_{t+1})$$

- implications

$$1 - \tau_t = \frac{u_L(c_t, L_t)}{u_c(c_t, L_t)} \frac{W_c(c_t, L_t; \mu)}{W_L(c_t, L_t; \mu)}$$

$$\frac{R_{t+1}}{R^*_{t+1}} = \frac{u_c(c_t, L_t)}{u_c(c_{t+1}, L_{t+1})} \frac{W_c(c_{t+1}, L_{t+1}; \mu)}{W_c(c_t, L_t; \mu)}$$

- results:
  - a form of labor tax smoothing:
    * the entire sequence of g_t has an impact on the tax through μ
    * no special role for current g_t, conditional on current allocation
    * clearer in special cases: if

$$u(c, L) = \frac{c^{1-\sigma}}{1-\sigma} - \alpha \frac{L^\gamma}{\gamma}$$

      with $\sigma > 0$ and $\gamma > 1$ then

$$\tau_t = \bar{\tau}$$

  - at a steady state the tax on capital is zero (Chamley-Judd):

$$c_t \to \bar{c} \qquad L_t \to \bar{L}$$

$$\Rightarrow \frac{R_{t+1}}{R^*_{t+1}} \to 1$$

  - initial tax on capital and bonds:
    * equivalent to a lump sum tax
    * optimal to expropriate

        * if upper bound on tax rates, then they will binding

- last result leads to time inconsistency:
    - plan to...
        * tax initial capital highly
        * tax future capital at zero
    - will plan be carried out? can we commit to it?
        * if not, and reoptimize once and for all then raise capital again
        * if reoptimize all the time (discretion): expect high taxes, which lowers welfare
- with heterogeneous agents
    - allow a lump sum (poll) tax
    - first two results hold: tax smoothing and Chamley-Judd
    - the last conclusion less clear:
        * Pareto analysis
        * depends on distribution of assets and redistributive intent
    - even if capital levy is optimal, it may be bounded, and correct intuition is not based on a lump sum tax
    - time inconsistency also more subtle: in general not time consistent, but depends on evolution of wealth

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