14.453: Problem Set #3

Iván Werning

1 Precautionary Savings in General Equilibrium

Let utility be given by:

$$E\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t}\right)$$

where $u\left(c\right)=-\frac{1}{\gamma}\exp\left\{-\gamma c\right\}$. Assume the standard intertemporal budget constraint

$$A_{t+1}=\left(1+r\right)\left(A_{t}+y_{t}-c_{t}\right).$$

(note: we do not necessarily impose $\beta\left(1+r\right)=1$ here). Assume that y_t is *i.i.d.* across time and agents. Let $y_t=\bar{y}+\varepsilon_t$ where ε_t is i.i.d. and $E_{t-1}\varepsilon_t=0$. We do not impose a borrowing constraint on this problem, A_t can take any value, although a no-Ponzi condition should be thought as being implicitly imposed for the problem to be well defined (you will not have to think about this no-Ponzi condition explicitly for solving the problem though).

(a) Show that the consumption function,

$$c_{t}=\frac{r}{1+r}\left[A_{t}+y_{t}+\frac{1}{r}\bar{y}\right]-\pi\left(r,\gamma\right)$$

for some $\pi\left(r,\gamma\right)$ implies,

$$\Delta c_{t}=\frac{r}{1+r}\left[y_{t}-\bar{y}\right]+r\pi\left(r,\gamma\right)$$

[note that the functional form of the consumption function, as a function of A_t and current and expected income, is like the CEQ-PIH except for the constant $\pi\left(r,\gamma\right)$]

(b) Use the Euler equation and your results in (a) to show that the consumption function in (a) is optimal for some π (hint: use the Euler equation to guess and verify the optimality of the above consumption function) which depends on r and the distribution of ε.

(c) Show that $\pi\left(r,\gamma\right)>0$ if r is such that $\beta\left(1+r\right)=1$. Compare this to the CEQ-PIH case. How does π depend on the uncertainty in y_t?

(d) Assume there is a constant measure 1 of individuals in the population. Argue that for aggregate consumption and assets to be constant and finite in the long run (in the limit as $t\to\infty$) we require that $\pi\left(r,\gamma\right)=0$. What does this imply about average long-run asset holdings as a function of r and γ(denote this by $A\left(r,\gamma\right)$)? What is happening to the cross-section of consumption? Does this distribution converge?

(e) Use your results in (d) to compute the equilibrium interest rate r^e as a function of γ for $\beta=.97$, $\bar{y}=1$. Assume ε distributed normal with mean zero and standard deviation equal to σ (this distributional assumption allows you to find an explicit expression for $E\exp(-\varepsilon)$). Compute r^e for two cases $\sigma = 0.2$ and 0.4 , fixing γ =1. Compare this to the interest rate that prevails without uncertainty.

(f) Briefly discuss how you would think of calibrating γ if you really believed that preferences are CRRA $c^{1-\sigma}/(1-\sigma)$ for some known value of σ, but you want to work with this model as an approximation (because of its analytical tractability).

Speculate on whether this is likely to be a good approximation for learning about r^e. Can this be a good approximation for learning about the long-run (invariant) distribution of asset holdings?

2 Income Fluctuation Problem – Numerical Computation

This exercise asks you to compute numerically an income fluctuations problem. The problem is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to:

$$x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

where x_t represents "cash in hand" and y_t labor income. Consider the following two alternative income processes:

Process I y_t is assumed to be i.i.d. with only two possible realizations $y_l = 1.7w$ and $y_h = .5w$ with 1/2 probability each.

Process II y_t follows a two-state Markov process with transition matrix

$$\Pi = \begin{bmatrix} \pi_{HH} & \pi_{HL} \\ \pi_{LH} & \pi_{LL} \end{bmatrix}$$

where again $y_l = 1.7w$ and $y_h = .5w$ and $\pi_{HH} = .9$ and $\pi_{LL} = .5$.

Set $\beta = .97$, $r = 2\%$ and $w = 1$.

You are given the basic matlab codes that should allow you to understand how to compute the solution to the income fluctuation problem. In fact you are given two codes, one that uses the simple grid method (explained in SL ch.4) and the other which uses splines which sometimes is more efficient, but you are not required to look at it. You are supposed to modify the simple code in order to match the new parameters and the two different income processes. You are required to hand in your code.

Perform all calculations below for both the alternative income processes and for $\sigma = .8$ **and** $\sigma = 1.8$.

We solve this problem by iterating on the Bellman equation

$$v(a,y) = \max_{a \geq 0} \left\{ \frac{((1+r)a + y - a')^{1-\sigma}}{1-\sigma} + \beta E[v(a',y') \mid y] \right\}$$

Notice that in the *i.i.d.* we could have simplified the problem by reducing the state at cash-in-hand only, but since we are working also with a permanent process for income, we need a two dimensional state vector anyway.

(a) Solve the optimal consumption problem obtaining the consumption function $c(x)$. Plot the function for consumption, asset holdings $a'(z) = z - c(z)$, and cash-in-hand for tomorrow (for both realizations of tomorrow's income shock), that is, $z'(a, y') = (1 + r) a'(z) + y'$ for both possible values of y'. How do your results differ between the two alternative income processes?

(b) Use this policy function together with random shocks to simulate the evolution of cash-in-hand, income, consumption and assets for 200 periods. Plot the simulated series for consumption, income and assets. Is consumption smoother than income? How high are asset holdings? For what fraction of periods is the agent liquidity constrained (i.e. $x_t - c_t = 0$)? How do your results depend on σ? Compare the two alternative income processes

(c) Compute a longer simulation (around 11001 periods), throw away the first 1001 periods (that is, from $t = 0$ to $t = 1000$ included) and calculate the average asset holdings with the remaining periods. That is, based on the simulated sequence for $\{z_t\}_{t=0}^{11000}$ compute $\frac{1}{10000} \sum_{t=1001}^{11000} a'(z_t)$ where $a'(z)$ is the policy function found in a.

(d) (Carroll, 1997) Modify the income process to have the following characteristic: there is a small probability π of income being zero. If this event does not occur income is drawn from the same distribution as before.

Argue that, with the preferences above, the borrowing constraint will never bind: we always have $a_t = x_t - c_t > 0$ (you do not have to compute. If we allow for some borrowing, so that we replace the constraint $a_t \geq 0$ with $a_t \equiv x_t - c_t \geq -b$ for some positive $b > 0$, argue that this condition will never bind and that in fact $a_t > 0$.

(e) Now you will compute the equilibrium for this model as in Aiyagari (1994). Using the Cobb-Douglas technology $F(K, L) = K^\alpha L^{1-\alpha}$ solve the following system for $K(r)$ and $w(r)$:

$$\begin{aligned} r &= F_k(K, 1) - \delta \qquad (1) \\ w &= F_L(K, 1) \end{aligned}$$

Here $K(r)$ and $w(r)$ represent the level of capital demanded by firms and the associated wage if the interest rate at a steady state were equal to r and labor supply and demand are equal.

For any proposed value of r (equivalently one can propose a value for K and obtain the implied proposed r using (1)) we can use the implied value for the wage $w(r)$ and solve the individual's income fluctuations problem given r and $w(r)$. Then using the method in (c) we can calculate the average asset holdings, denote this by $A(r)$. Think of $A(r)$ as representing the steady state supply of capital by individuals when the interest rate equals to r and the wage is $w(r)$.

We are looking for a value for r such that $K(r) = A(r)$, i.e. that the quantity demanded and supplied of capital at a steady state are equal. Using $\alpha = .33$, $\delta = 8\%$ and the parameters given above plot $A(r)$ against $K(r)$ by computing $A(r)$ for several values of r (pick values in the interval $0 \leq r < 1/\beta - 1$). Find the value of r and K at which both curves intersect.