Solutions for Problem Set #2
Macro 14.453

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1 Perfect Risk Sharing

Consider a finite group I of individuals. Income for each individual is determined each period as a function of the current state of nature[1] $s_t \in S$ (where S is a finite set): $y_t^i(s_t)$. Denote aggregate income by $Y_t(s_t) \equiv \sum_{i\in I} y_t^i(s_t)$. Let utility for individual i be given by $E\sum_{t=0}^{\infty}\beta^t u^i(c_t)$ which under our assumption on uncertainty is,

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t u\left(c_t\left(s^t\right)\right)\Pr\left(s^t\right).$$

Parts (a) and (b) are just a refresher from the lectures.

1.1 Part (a)

Assume that there is no aggregate savings technology, that the state of nature is observable and that there are no commitment problems.

Write out the Pareto problem for given Pareto weights $\left\{\lambda^i\right\}_{i\in I}$. Show that at the optimum consumption for individual i, $c_t^i\left(s^t\right)$, can be written as depending only on aggregate income in that period – once we control for $Y_t\left(s^t\right)$ consumption does not depend additionally on s^t.

[1] Note that s_t summarizes the entire distribution of current income and possibly contains additional information, e.g. forcasts of future income.

The Pareto problem for given Pareto weights $\left\{\lambda^i\right\}_{i\in I}$ is

$$\max_{\left\{c_t^i(s^t)\right\}} \sum_{i\in I} \lambda^i \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t u^i\left(c_t^i\left(s^t\right)\right) \Pr\left(s^t\right)$$

$$s.t. \quad Y_t\left(s^t\right) \equiv \sum_{i\in I} y_t^i\left(s_t\right) \geq \sum_{i\in I} c_t^i\left(s^t\right)$$

for all t and $s^t \in S^t$.

Note that since the constraint is static $\forall t, s^t$, we can rewrite the problem as

$$\max_{\left\{c_t^i(s^t)\right\}} \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \Pr\left(s^t\right) \sum_{i\in I} \lambda^i u^i\left(c_t^i\left(s^t\right)\right)$$

$$s.t. \;\; Y_t\left(s^t\right) \geq \sum_{i\in I} c_t^i\left(s^t\right)$$

and so we can equivalently write (not that the constraint turns out to be binding since it is not optimal to waste resources):

$$\sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \Pr\left(s^t\right) \max_{\left\{c_t^i(s^t)\right\}} \sum_{i\in I} \lambda^i u^i\left(c_t^i\left(s^t\right)\right)$$

$$s.t. \quad Y_t\left(s^t\right) = \sum_{i\in I} c_t^i\left(s^t\right)$$

Note that the problem does not depend on s^t but through $Y_t\left(s^t\right)$ so it must be that the optimal consumption depends also from s^t only through $Y_t\left(s^t\right)$, *i.e.*

$$c_t^i\left(s^t\right) = f^i\left(Y_t\left(s^t\right), \cdot\right)$$

In particular we can show that

$$c_t^i\left(s^t\right) = f^i\left(Y_t\left(s^t\right), \lambda^1, ..., \lambda^I\right)$$

In fact the Lagrangian turns out to be

$$L = \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \Pr\left(s^t\right) \left\{ \sum_{i\in I} \lambda^i u^i\left(c_t^i\left(s^t\right)\right) + \mu\left(s^t\right)\left[Y_t\left(s^t\right) - \sum_{i\in I} c_t^i\left(s^t\right)\right]\right\}$$

where we chose, to meke things easier, a normalized lagrangian multiplier for each constraint equal to $\beta^t \Pr(s^t)\,\mu(s^t)$.

The F.O.C. are

$$\left(c_t^i\left(s^t\right)\right) : \lambda^i u^{i\,\prime}\left(c_t^i\left(s^t\right)\right) = \mu(s^t) \quad \forall i, t, s^t$$

so that we can obtain

$$\frac{u^{i\prime}\left(c_t^i\left(s^t\right)\right)}{u^{k\prime}\left(c_t^1\left(s^t\right)\right)} = \frac{\lambda^k}{\lambda^i} \qquad \forall i, k, t, s^t \text{ with } i \neq k \tag{1}$$

so that

$$c_t^i\left(s^t\right) = f^i(c_t^k\left(s^t\right), \lambda^i, \lambda^k) \qquad \forall i, k, t, s^t \text{ with } i \neq k$$

Plug the consumption function for any $i \neq k$ into the constraint $\forall t, s^t$ we can derive

$$Y\left(s_t\right) = \sum_{i \neq 1} f^i(c_t^k\left(s^t\right), \lambda^i, \lambda^k) + c_t^k\left(s^t\right)$$

which implicetely defines

$$c_t^k\left(s^t\right) = f^i\left(Y_t\left(s^t\right), \lambda^1, ..., \lambda^I\right)$$

which is what we wanted to show. Note that you can repeat this procedure for any $k \in I$.

1.2 Part (b)

We now generalize the previous result. Assume there is a "storage technology": if in period $t-1$ an amount $S_t\left(s^{t-1}\right) \geq 0$ was put aside for storage, then in period t an amount $\left(1 + r_t\left(s_t\right)\right) S_t\left(s^{t-1}\right)$ is available (for consumption or storage) in addition to any current income $Y_t\left(s^t\right)$. Show that a similar result as in (a) holds but that now we must condition on total consumption $C\left(s^t\right) \equiv \sum_{i \in I} c_t^i\left(s^t\right)$. (note that we impose the non-negativity constraint on storage, thus our result in a can be thought as a special case where $r_t \equiv 0$ so that at the optimum $S_t = 0$ and thus $C_t = Y_t$).

The Pareto problem is now

$$\max_{\{c_t^i(s^t), \{S_t(s^t)\}} \sum_{i \in I} \lambda^i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u^i\left(c_t^i\left(s^t\right)\right) \Pr\left(s^t\right)$$

$$s.t. \quad S_{t+1}(s^t) + \sum_{i \in I} c_t^i\left(s^t\right) \leq Y\left(s_t\right) + (1 + r(s^t))S_t(s^{t-1}) \;\; \forall t, s^t$$
$$S_t(s^t) \geq 0 \;\; \forall t, s^t$$

Note that we can define

$$C_t\left(s^t\right) \equiv Y\left(s_t\right) + (1 + r(s^t))S_t(s^{t-1}) - S_{t+1}(s^t) \quad \forall t, s^t$$

Then we can fix $C_t\left(s^t\right)$ and solve the problem exactly as in the previous point except that $Y_t\left(s^t\right)$ is substitued by $C_t\left(s^t\right)$ and then solve for the optimal $S_{t+1}(s^t)$ using the optimal consumption functions determined before.

So, proceeding as in the previous point we get

$$c_t^k\left(s^t\right) = f^i\left(C_t\left(s^t\right), \lambda^1, ..., \lambda^I\right) \qquad \forall k, t, s^t$$

where $C_t\left(s^t\right)$ is endogenously determined.

1.3 Part (c)

Let the utility function be of the CARA form

$$u^i\left(c\right) = \frac{-1}{\gamma^i} \exp\left\{-\gamma^i c\right\}$$

show that consumption takes the form: $c_t^i = a^i C_t + b^i$ where a^i and b^i are constants and $\sum a^i = 1$ and $\sum b^i = 0$. How does the distribution of γ^i affect a^i and b^i? How does the distribution of Pareto weights λ^i affect a^i and b^i?

Using equation (1) we derived in point (a) and using the fact that with our preference specification the marginal utility is given by

$$u^{i\prime}\left(c\right) = \exp\left\{-\gamma^i c\right\}$$

we get

$$\exp\left\{\gamma^k c_t^k\left(s^t\right) - \gamma^i c_t^i\left(s^t\right)\right\} = \frac{\lambda^k}{\lambda^i} \qquad \forall i, k, t, s^t$$

and taking log

$$c_t^i\left(s^t\right) = \frac{\gamma^k}{\gamma^i} c_t^k\left(s^t\right) - \frac{1}{\gamma^i} \log\left(\frac{\lambda^k}{\lambda^i}\right) \qquad \forall i, k, t, s^t$$

Plugging this back into the constraint

$$c_t^k\left(s^t\right)+\sum_{i\neq k}\left[\frac{\gamma^k}{\gamma^i}c_t^k\left(s^t\right)-\frac{1}{\gamma^i}\log\left(\frac{\lambda^k}{\lambda^i}\right)\right]=C(s^t)$$

which implies

$$c_t^k\left(s^t\right)=a^kC_t\left(s^t\right)+b^k$$

where

$$a^k=\frac{1}{\sum_{i\in I}\frac{\gamma^k}{\gamma^i}}$$

and

$$b^k=\frac{\sum_{k\in I}\frac{1}{\gamma^i}\log\left(\frac{\lambda^k}{\lambda^i}\right)}{\sum_{k\in I}\frac{\gamma^k}{\gamma^i}}$$

Notice that

$$\sum_{k\in I}\alpha^k=\frac{\sum_{k\in I}\frac{1}{\gamma^k}}{\sum_{i\in I}\frac{1}{\gamma^i}}=1,$$

and

$$\begin{aligned}\sum_{k\in I}b^k &= \left(\frac{1}{\sum_{i\in I}\frac{1}{\gamma^i}}\right)\left(\sum_{k\in I}\frac{1}{\gamma^k}\sum_{i\in I}\frac{1}{\gamma^i}\left[\log\left(\lambda^k\right)-\log\left(\lambda^i\right)\right]\right)= \\ &= \left(\frac{1}{\sum_{i\in I}\frac{1}{\gamma^i}}\right)\left(\sum_{i\in I}\sum_{k\in I}\frac{\log\left(\lambda^k\right)}{\gamma^k\gamma^i}-\sum_{i\in I}\sum_{k\in I}\frac{\log\left(\lambda^i\right)}{\gamma^k\gamma^i}\right)=0\end{aligned}$$

Note that the γ^i affect negatively both the slope and the shift terms, a^i and b^i, mening that the more risk averse individuals get as smaller fraction of the total income because of its riskiness and so get also less transfers since they are bearing less risk. Moreover the distribution of the λ^ι does not affect α^i , but affect positively b^i , as expected.

1.4 Part (d)

(Let the utility function be of the CRRA form

$$u^i\left(c\right)=\frac{c^{1-\sigma}}{1-\sigma}$$

where the risk aversion σ is assumed to be the same for all individuals. Show that consumption takes the form $c_t^i = \alpha^i C_t$ with the constants α^i satisfying $\sum_{i \in I} \alpha^i = 1$. How do the constants α^i depend on the Pareto weights λ^i?

Using equation (1) we derived in point (a) and using the fact that with our preference specification the marginal utility is given by

$$u^{i\prime}(c) = c^{-\sigma}$$

we get

$$\left(\frac{c_t^i(s^t)}{c_t^k(s^t)}\right)^{-\sigma} = \frac{\lambda^k}{\lambda^i} \qquad \forall i, k, t, s^t$$

so that

$$c_t^i(s^t) = c_t^k(s^t)\left(\frac{\lambda^k}{\lambda^i}\right)^{-\frac{1}{\sigma}} \qquad \forall i, k, t, s^t$$

By plugging this into the budget constraint we get

$$\sum_{i \in I} c_t^k(s^t)\left(\frac{\lambda^k}{\lambda^i}\right)^{-\frac{1}{\sigma}} = C(s^t)$$

and so

$$c_t^k(s^t) = \alpha^k C(s^t)$$

where

$$\alpha^k = \frac{1}{\sum_{i \in I}\left(\frac{\lambda^i}{\lambda^k}\right)^{\frac{1}{\sigma}}}$$

Note that

$$\sum \alpha^i = \frac{\sum_{i \in I}\left(\frac{1}{\lambda^i}\right)^{\frac{1}{\sigma}}}{\sum_{i \in I}\left(\frac{1}{\lambda^i}\right)^{\frac{1}{\sigma}}} = 1$$

and the constants α^i depend positively on the Pareto weights λ^i as expected.