14.453: Problem Set #2

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1 Perfect Risk Sharing

Consider a finite group I of individuals. Income for each individual is determined each period as a function of the current state of nature[1] $s_t \in S$ (where S is a finite set): $y_t^i(s_t)$. Denote aggregate income by $Y_t(s_t) \equiv \sum_{i \in I} y_t^i(s_t)$. Let utility for individual i be given by $E \sum_{t=0}^{\infty} \beta^t u^i(c_t)$ which under our assumption on uncertainty is,

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c_t\left(s^t\right)\right) \Pr\left(s^t\right).$$

Parts (a) and (b) are just a refresher from the lectures.

(a) Assume that there is no aggregate savings technology, that the state of nature is observable and that there are no commitment problems.

Write out the Pareto problem for given Pareto weights $\{\lambda^i\}_{i \in I}$. Show that at the optimum consumption for individual i, $c_t^i(s^t)$, can be written as depending only on aggregate income in that period – once we control for $Y_t(s^t)$ consumption does not depend additionally on s^t.

(b) We now generalize the previous result. Assume there is a "storage technology": if in period $t-1$ an amount $S_t(s^{t-1}) \geq 0$ was put aside for storage, then in period t an amount $(1 + r_t(s_t)) S_t(s^{t-1})$ is available (for consumption or storage) in addition to any current income $Y_t(s^t)$. Show that a similar result as in (a) holds but that now we must condition on total consumption $C(s^t) \equiv \sum_{i \in I} c_t^i(s^t)$. (note that we impose the non-negativity constraint on storage, thus our result in a can be thought as a special case where $r_t \equiv 0$ so that at the optimum $S_t = 0$ and thus $C_t = Y_t$).

(c) Let the utility function be of the CARA form

$$u^i(c) = \frac{-1}{\gamma^i} \exp\left\{-\gamma^i c\right\}$$

show that consumption takes the form: $c_t^i = a^i C_t + b^i$ where a^i and b^i are constants and $\sum a^i = 1$ and $\sum b^i = 0$. How does the distribution of γ^i affect a^i and b^i? How does the distribution of Pareto weights λ^i affect a^i and b^i?

[1] Note that s_t summarizes the entire distribution of current income and possibly contains additional information, e.g. forecasts of future income.

(d) Let the utility function be of the CRRA form

$$u^i(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where the risk aversion σ is assumed to be the same for all individuals. Show that consumption takes the form $c_t^i = \alpha^i C_t$ with the constants α^i satisfying $\sum_{i \in I} \alpha^i = 1$. How do the constants α^i depend on the Pareto weights λ^i?

2 Risk Sharing Productivity: Consumption and Labor

Consider a social planner who maximizes the utility of a continuum of agents of measure one ex-ante identical, with preferences over consumption and leisure

$$u(c, l)$$

where we assume that utility function $u(.)$ is strictly concave, continuously differentiable and increasing in both its arguments.

Agents are heterogeneous ex-post according to their productivity. They draw one non-negative productivity parameter $\omega \in \Omega$ independently from a known continuous distribution function F. We assume that the support $\Omega = [\omega_{\min}, \omega_{\max}]$ is compact and the density function density function $f(\omega)$ is bounded above and bounded away from zero.

The resource constraint in this economy is

$$\int c(\omega) f(\omega) d\omega \leq \int (1 - l(\omega)) \omega f(\omega) d\omega + e$$

for some endowment $e \geq 0$.

Let us assume that agents' productivity is ex-post publicly observable, so that consumption and leisure can be conditioned on their ability, *i.e.* $c(\omega)$ and $l(\omega)$.

(a) Write down the social planner problem, noting that, given that the agents are a continuum of measure one, by the law of large number, the ex-ante probability coincides with the ex-post distribution.

(b) Show that, under the given assumptions, leisure is a decreasing function of the productivity parameter ω. Discuss.

(c) Show that the dependence of consumption on the productivity parameters depends on the assumption on the sign of the cross-derivative u_{cl} (short notation for $\frac{\partial^2 u(.)}{\partial c \partial l}$). Comment.

(d) Show that assuming that leisure is a normal good is sufficient (for any sign of u_{cl}) to have the utility function decreasing in the productivity. Discuss the role of risk sharing.

(e) Assume now that agents' productivity is not ex-post observable anymore. Given your result in the previous point, is the previous allocation implementable? Comment.

3 Overlapping Generations

In this problem we study an overlapping generations model under two market interpretations and isolate the conditions for Pareto optimality.

All individuals live for only two periods. Generation t (denoted by the superscript) has utility function:

$$u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right)$$

where $u : R_+ \to R$ is strictly concave and increasing in the single consumption good c.

Individuals are endowed with one unit of labor in the first period of their lives and supply it inelastically. Capital is owned by individuals and rented out to firms. Competitive firms rent capital and labor at prices r_t and w_t (in terms of time t consumption goods). At time zero all initial capital is held by the old (i.e. generation $t = -1$).

The resource constraint is,

$$k_{t+1} + c_t^t + c_t^{t-1} \leq F\left(k_t, 1\right) + \left(1 - \delta\right) k_t,$$

where F is a constant returns to scale (CRS) production function and $\delta \in (0, 1]$.

(a) *Sequential Trade.* Consider the sequential competitive market arrangement where individuals in generation t face the budget constraints,

$$\begin{aligned} c_t^t + k_{t+1} &= w_t \\ c_{t+1}^t &= R_{t+1} k_{t+1} \end{aligned}$$

with $R_t \equiv (1 - \delta + r_t)$.

Given k_0, define a competitive equilibrium for this market arrangement for c_t^0, $\left\{\left(c_t^t, c_{t+1}^t\right), k_{t+1}\right\}_{t=0}^{\infty}$ and prices $\{r_t, w_t\}_{t=0}^{\infty}$.

(b) *Time-0 trade.* Now consider the complete market arrangement where we imagine all generations (the born and yet unborn) and firms meeting at time zero and competitively trading in claims for future consumption, labor and capital. We generalize the notation and specialize it to interpret our overlapping generations model.

Generation t faces the budget constraint:

$$\sum_{s=0}^{\infty} q_s^0 \left(c_s^t + k_{s+1}^t - R_s k_s^t\right) \leq \sum_{s=0}^{\infty} q_s^0 w_s \bar{n}_s^t + R_0 \bar{k}_0^t$$

where we have normalize $q_0^0 = 1$ and R_s is as before.

Notation: $\bar{n}_s^t$ and $\bar{k}_0^t$ represent endowment of labor in period s and initial capital owned by generation t. Thus, in our OLG model: $n_t^s = 1$ for $s = t$ and $n_t^s = 0$ for $s \neq t$; $\bar{k}_0^{-1} = k_0$ and $\bar{k}_0^s = 0$ for $s \neq -1$.

Think of each generation-t as having a utility function U^t defined over the entire consumption stream $\{c_s^t\}_{s=0}^{\infty}$. Of course, in our OLG model: $U^t\left(\{c_s^t\}_{s=0}^{\infty}\right) \equiv u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right)$.

Define an equilibrium for c_t^0, $\left\{\left(c_t^t, c_{t+1}^t\right), k_{t+1}\right\}_{t=0}^{\infty}$ and prices $\{q_t^0\}_{t=0}^{\infty}$ and $\{r_t, w_t\}_{t=0}^{\infty}$ using the standard Walrasian setup.

Show that equilibria must satisfy the arbitrage condition: $q_t^0/q_{t+1}^0 = R_{t+1}$. Argue that the sequential market equilibrium in part (a) is a time-0 market equilibrium, as outlined here, for appropriately chosen prices $\{q_t^0\}$.

(c) Consider the special case of log utility, $u(c) = \log c$, and Cobb-Douglas production function $F(n_t, k_t) = A k_t^{\alpha} n_t^{1-\alpha}$. Characterize the entire equilibrium allocation c_t^0, $\left\{\left(c_t^t, c_{t+1}^t\right), k_{t+1}\right\}_{t=0}^{\infty}$ and prices $\{r_t, w_t\}_{t=0}^{\infty}$. Solve for the steady state level of capital k_{ss}.

Show that the equilibrium is not Pareto efficient *if* steady state capital higher than the golden rule $k_g = \arg\max_k \{F(k,1) - \delta k\}$. Show that there are parameters values for which $k_g < k_{ss}$.

(Hint: construct an allocation that increases everyone's consumption for all periods t high enough such that at the original equilibrium allocation capital remains higher than the golden rule, i.e. $k_s > k_g$ for $s \geq t$).

(d) In terms of the time-0, complete-market arrangement. Why does the First Welfare Theorem fail to apply? (Hint: argue that the condition

$$\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} q_t^0 w_t \bar{n}_t^s < \infty$$

is necessary for the proof of the first welfare theorem and that it is not satisfied here). Show that the welfare theorem does apply if $k_{ss} \leq k_g$.

(e) The previous points show that if $k_{ss} \leq k_g$ then there are Pareto weights $\{\lambda_t\}$ for generations such that the equilibrium allocation maximizes:

$$\sum_{t=0}^{\infty} \lambda_t \left[u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right)\right] + \lambda_{-1} \beta u\left(c_1^0\right) = \sum_{t=0}^{\infty} \left[\lambda_t u\left(c_t^t\right) + \lambda_{t-1} \beta u\left(c_t^{t-1}\right)\right]$$

subject to $c_t^t + c_t^{t-1} + k_{t+1} \leq F(k_t, 1) + (1-\delta) k_t$.

Show that if $k_0 = k_{ss}$ then $\lambda_t / \lambda_{t-1} = \alpha$ for some $\alpha < 1$.

(f) Show that,

$$\begin{aligned} U(C) \equiv & \max [\alpha u(c^y) + \beta u(c^o)] = \phi u(C) \\ & \text{s.t. } c^y + c^o = C \end{aligned}$$

for some $\phi > 0$.

Use this and the results above to show that when $k_{ss} \leq k_g$ the equilibrium allocation with $k_0 = k_{ss}$ solves the representative agent problem,

$$\max_{C,k'} \sum_{t=0}^{\infty} \alpha^t u(C_t)$$

subject to, $C_t + k_{t+1} \leq F(k_t, 1) + (1-\delta) k_t$ and $k_0 = k_{ss}$.

How does α compare to β?