

14.452 Economic Growth: Lecture 1, Stylized Facts of Economic Growth and Development and Introduction to the Solow Model

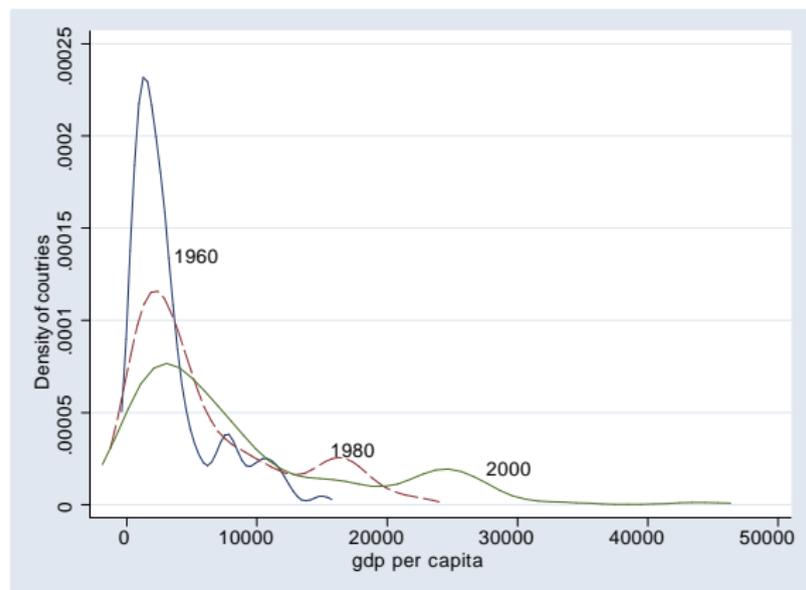
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Cross-Country Income Differences

- There are very large differences in income per capita and output per worker across countries today.



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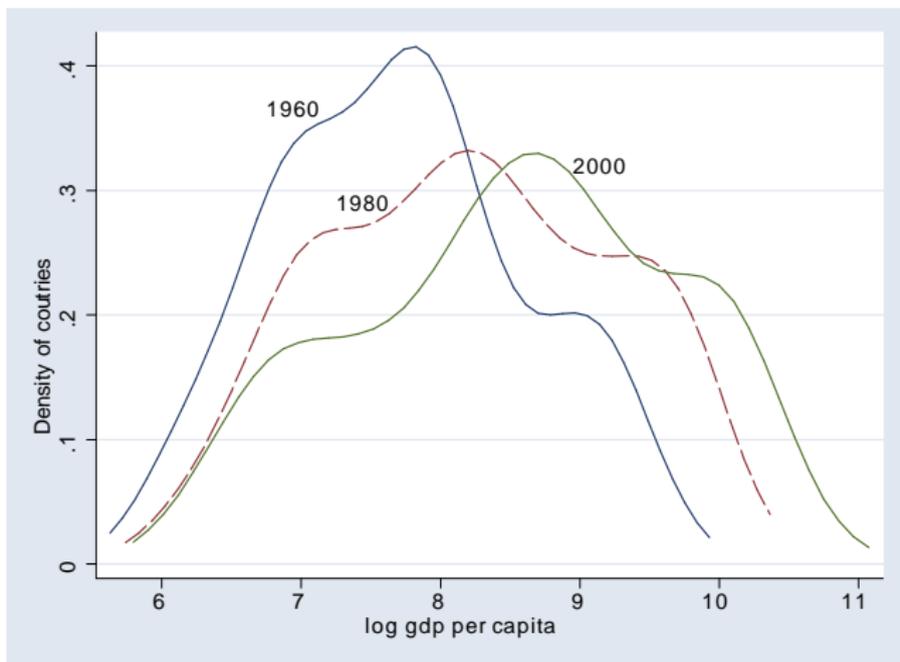
Figure 1.1 in Acemoglu, Daron. *Introduction to Modern Economic Growth*. Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Distribution of PPP-adjusted GDP per capita.

Cross-Country Income Differences

- Part of the spreading out of the distribution in the Figure is because of the increase in average incomes.
- More natural to look at the log of income per capita when growth is approximately proportional:
 - when $x(t)$ grows at a proportional rate, $\log x(t)$ grows linearly,
 - if $x_1(t)$ and $x_2(t)$ both grow by 10%, $x_1(t) - x_2(t)$ will also grow, while $\log x_1(t) - \log x_2(t)$ will remain constant.
- The next Figure shows a similar pattern, but now the spreading-out is more limited.

Cross-Country Income Differences



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Figure 1.2 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

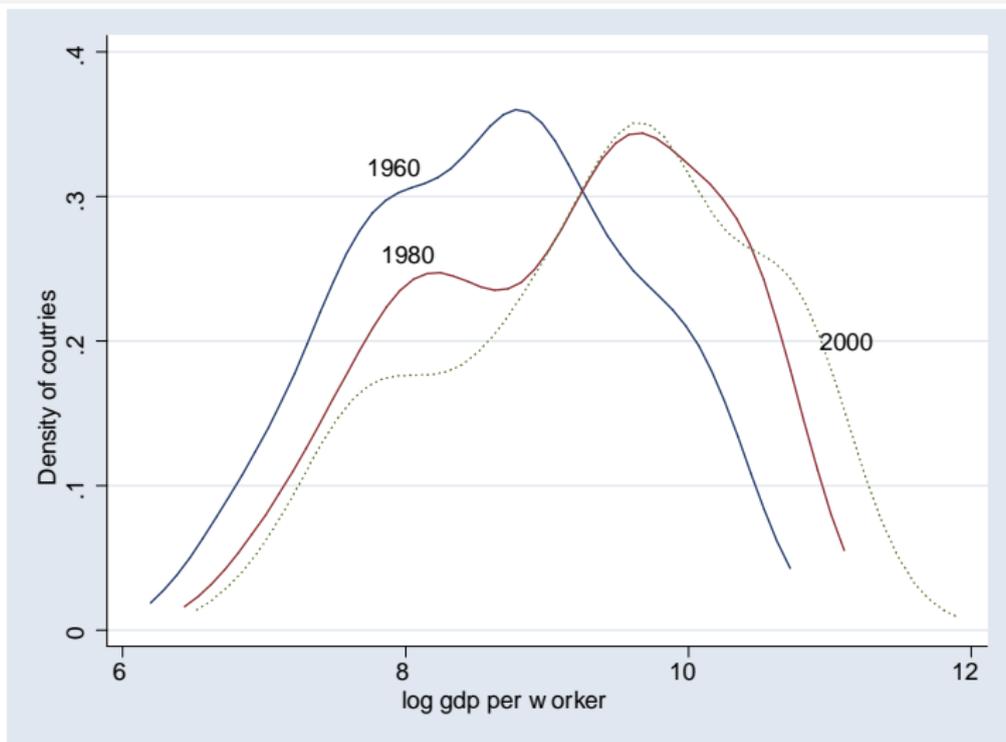
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Estimates of the distribution of countries according to log GDP per capita (PPP-adjusted) in 1960, 1980 and 2000.

Cross-Country Income Differences

- Theory is easier to map to data when we look at output (GDP) per worker.
- Moreover, key sources of difference in economic performance across countries are national policies and institutions.
- The next Figure looks at the unweighted distribution of countries according to (PPP-adjusted) GDP per worker
 - “workers”: total economically active population according to the definition of the International Labour Organization.
- Overall, two important facts:
 - Large amount of inequality in income per capita and income per worker across countries.
 - Slight but noticeable increase in inequality across nations (though not necessarily across individuals in the entire world).

Cross-Country Income Differences



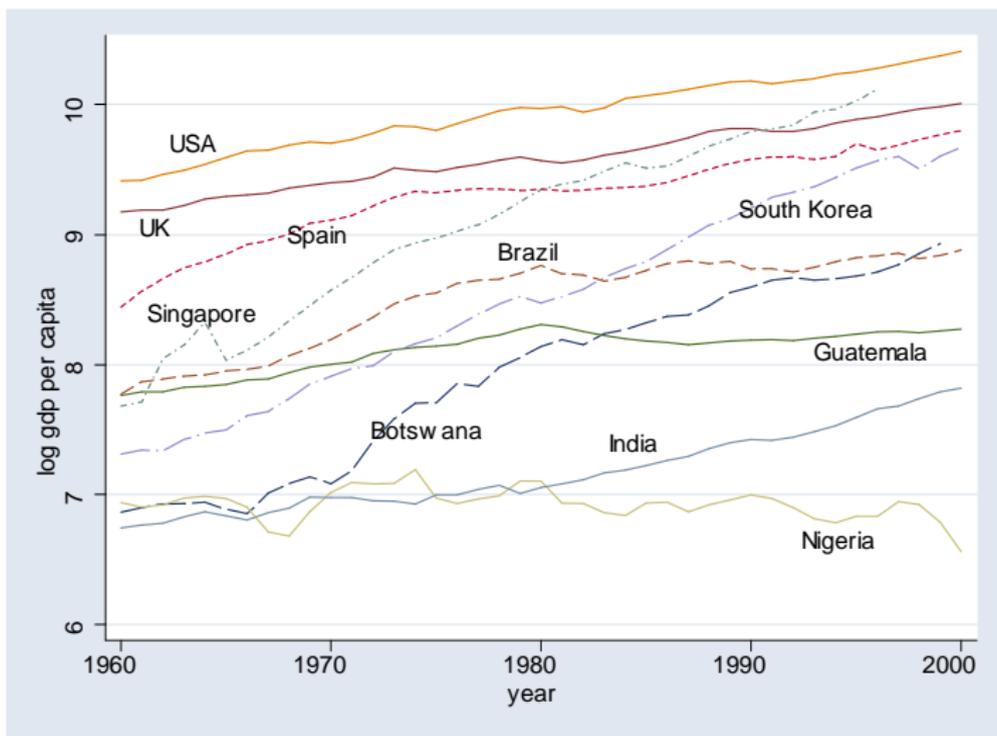
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Figure 1.4 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Distribution of log GDP per worker (PPP-adjusted).

Economic Growth and Income Differences



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Figure 1.8 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

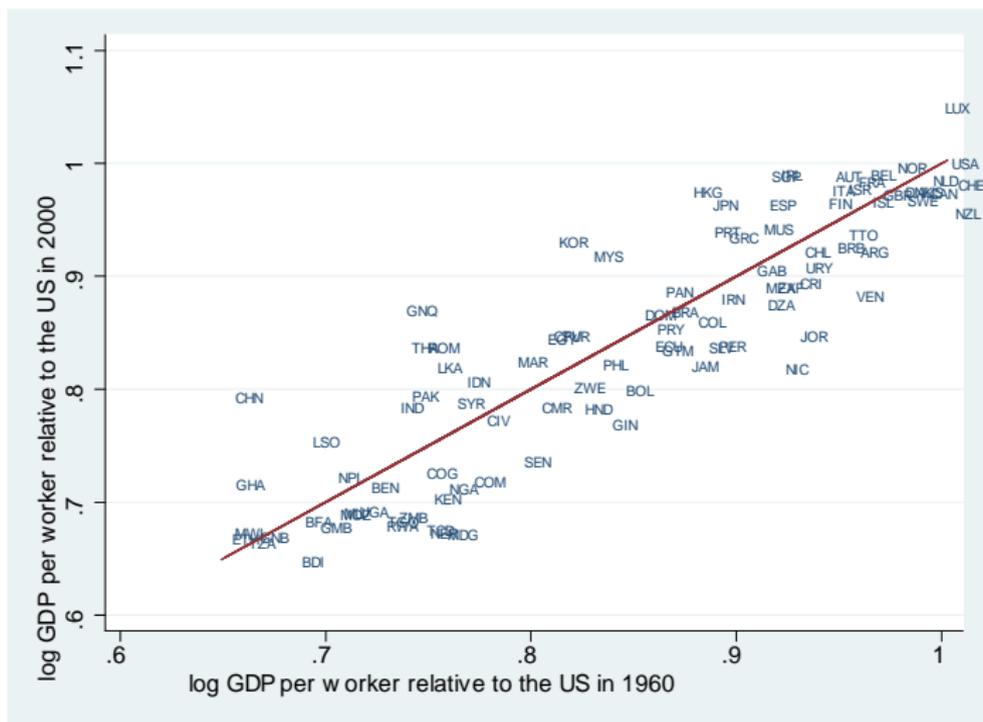
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Figure: The evolution of income per capita 1960-2000.

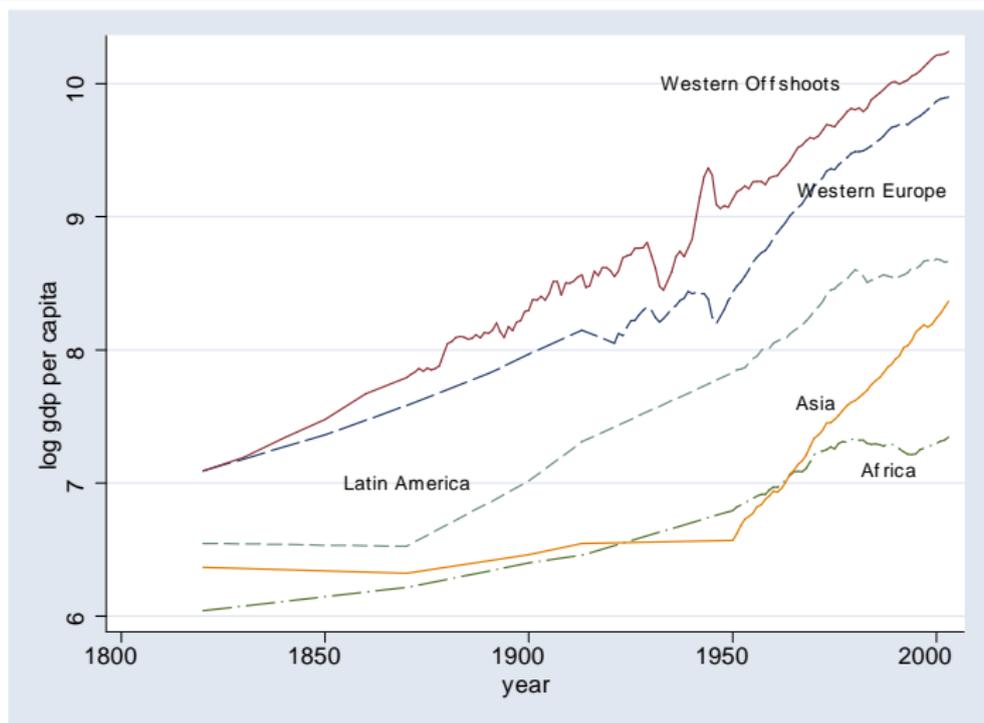
Economic Growth and Income Differences

- Why is the United States richer in 1960 than other nations and able to grow at a steady pace thereafter?
- How did Singapore, South Korea and Botswana manage to grow at a relatively rapid pace for 40 years?
- Why did Spain grow relatively rapidly for about 20 years, but then slow down? Why did Brazil and Guatemala stagnate during the 1980s?
- What is responsible for the disastrous growth performance of Nigeria?
 - Central questions for understanding how the capitalist system works and the origins of economic growth.
 - Central questions also for policy and welfare, since differences in income related to living standards, consumption and health.
- Our first task is to develop a coherent framework to investigate these questions and as a byproduct we will introduce the workhorse models of dynamic economic analysis and macroeconomics.

Origins of Income Differences and World Growth



Origins of Income Differences and World Growth



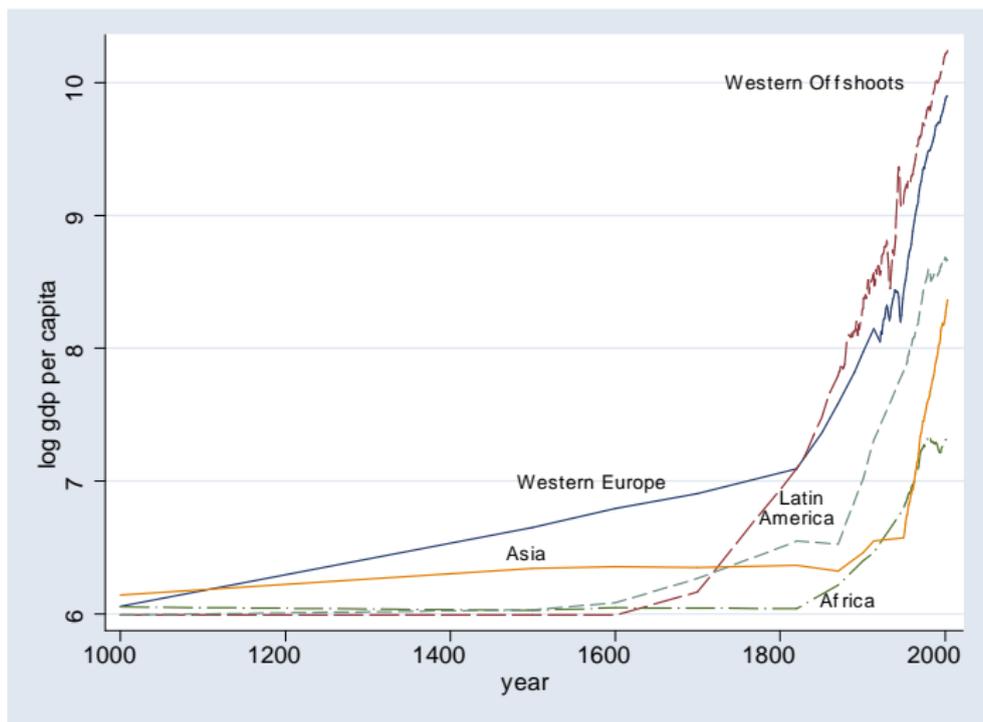
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Figure 1.10 in Acemoglu, Daron. *Introduction to Modern Economic Growth*

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Figure: Evolution of GDP per capita 1820-2000.

Origins of Income Differences and World Growth



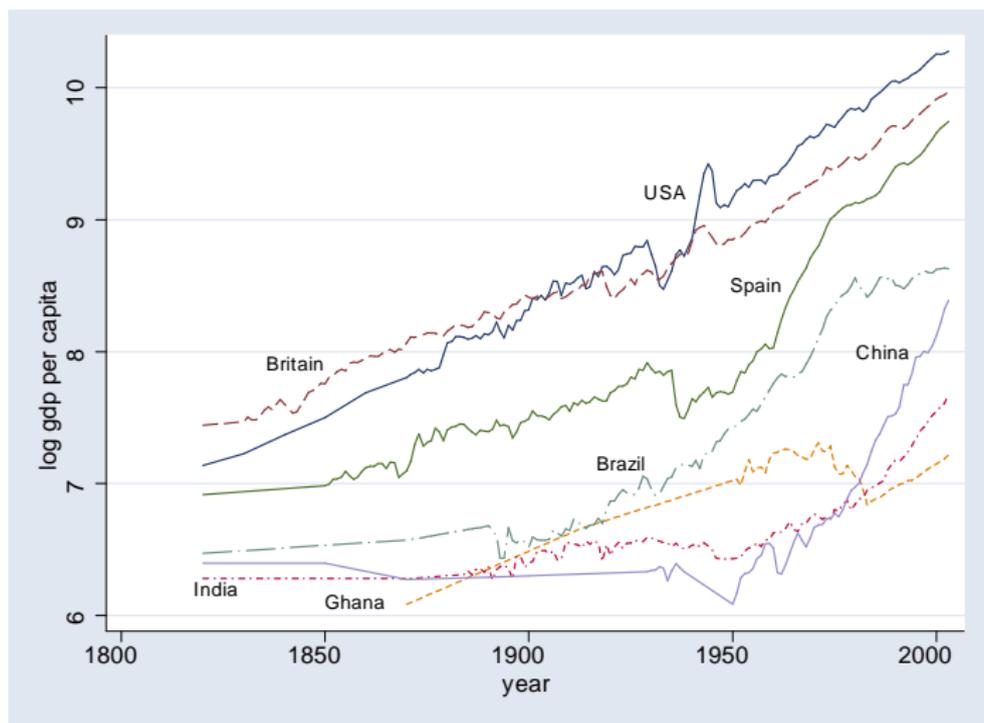
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Figure 1.11 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

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Figure: Evolution of GDP 1000-2000.

Origins of Income Differences and World Growth



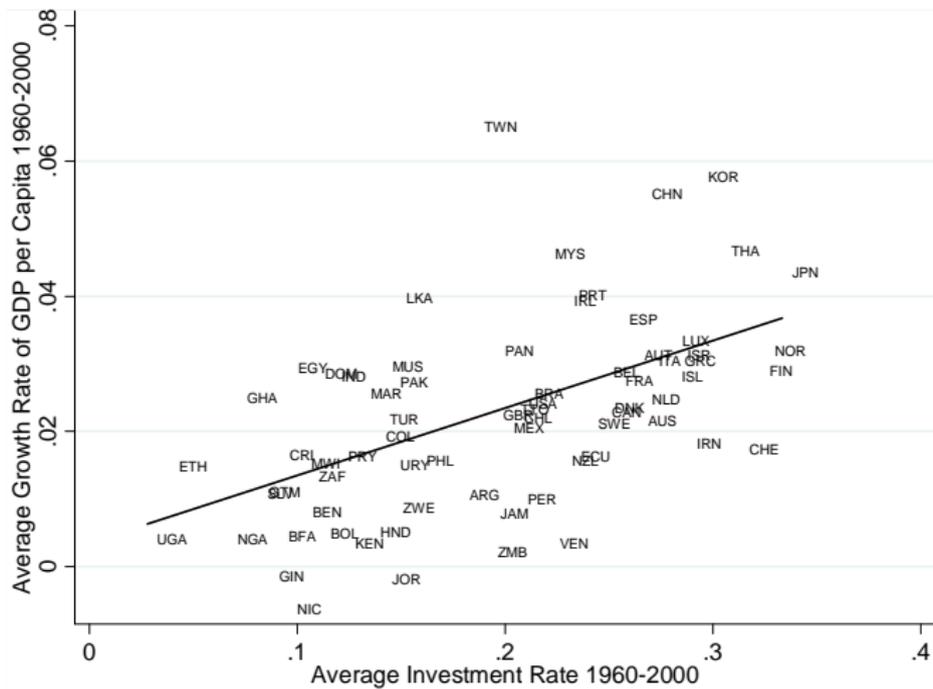
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Figure 1.12 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

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Figure: Evolution of income per capita in various countries.

Correlates of Economic Growth



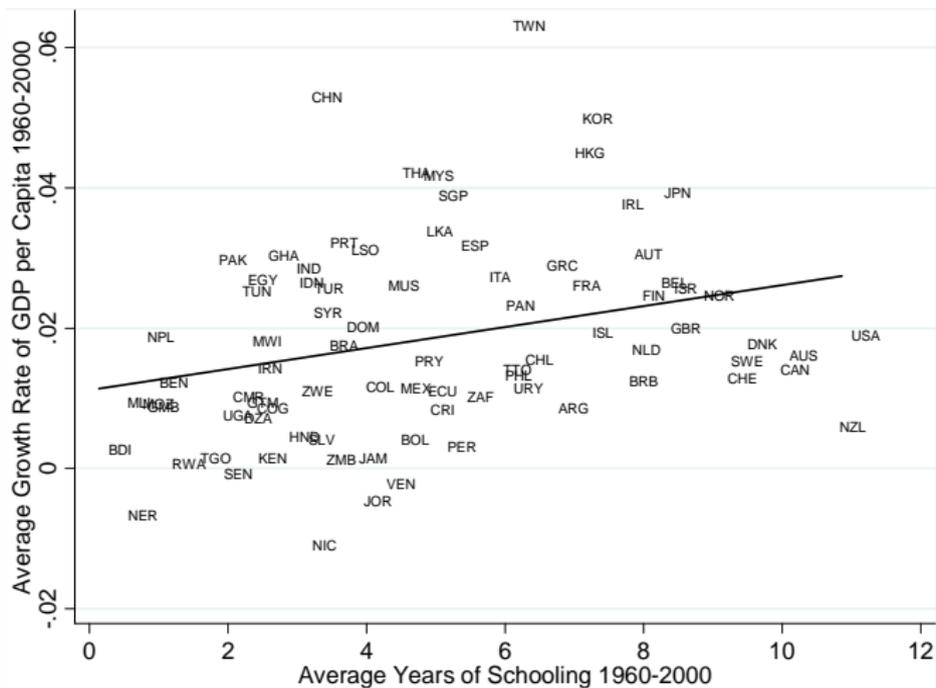
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Figure 1.15 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

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Figure: Average investment to GDP ratio and economic growth.

Correlates of Economic Growth



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Figure 1.16 in Acemoglu, Daron. [Introduction to Modern Economic Growth](#).

Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Schooling and economic growth.

From Correlates to Fundamental Causes

- Correlates of economic growth, such as physical capital, human capital and technology, will be our first topic of study.
- But these are only *proximate causes* of economic growth and economic success:
 - why do certain societies fail to improve their technologies, invest more in physical capital, and accumulate more human capital?
- Return to Figure above to illustrate this point further:
 - how did South Korea and Singapore manage to grow, while Nigeria failed to take advantage of the growth opportunities?
 - If physical capital accumulation is so important, why did Nigeria not invest more in physical capital?
 - If education is so important, why our education levels in Nigeria still so low and why is existing human capital not being used more effectively?
- The answer to these questions is related to the *fundamental causes* of economic growth.

From Correlates to Fundamental Causes

- We can think of the following list of potential fundamental causes:
 - luck (or multiple equilibria)
 - geographic differences
 - institutional differences
 - cultural differences
- An important caveat should be noted: discussions of geography, institutions and culture can sometimes be carried out without explicit reference to growth models or even to growth empirics. But it is only by formulating parsimonious models of economic growth and confronting them with data that we can gain a better understanding of both the proximate and the fundamental causes of economic growth.

Solow Growth Model

- Develop a simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model*
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar model emphasized potential dysfunctional aspects of growth: e.g, how growth could go hand-in-hand with increasing unemployment.
- Solow model demonstrated why the Harrod-Domar model was not an attractive place to start.
- At the center of the Solow growth model is the *neoclassical* aggregate production function.

The Economic Environment of the Basic Solow Model

- Study of economic growth and development therefore necessitates dynamic models.
- Despite its simplicity, the Solow growth model is a dynamic general equilibrium model (though many key features of dynamic general equilibrium models, such as preferences and dynamic optimization are missing in this model).

Households and Production I

- Closed economy, with a unique final good.
- Discrete time running to an infinite horizon, time is indexed by $t = 0, 1, 2, \dots$
- Economy is inhabited by a large number of households, and for now households will not be optimizing.
- This is the main difference between the Solow model and the *neoclassical growth model*.
- To fix ideas, assume all households are identical, so the economy admits *a representative household*.

Households and Production II

- Assume households save a constant exogenous fraction s of their disposable income
- Same assumption used in basic Keynesian models and in the Harrod-Domar model; at odds with reality.
- Assume all firms have access to the same production function: economy admits a **representative firm**, with a representative (or aggregate) production function.
- Aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \quad (1)$$

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- $A(t)$ is a *shifter* of the production function (1). Broad notion of technology.
- Major assumption: technology is **free**; it is publicly available as a non-excludable, non-rival good.

Key Assumption

Assumption 1 (Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale) The production function $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is twice continuously differentiable in K and L , and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Moreover, F exhibits constant returns to scale in K and L .

- Assume F exhibits *constant returns to scale* in K and L . I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

Review

Definition Let K be an integer. The function $g : \mathbb{R}^{K+2} \rightarrow \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

Theorem (Euler's Theorem) Suppose that $g : \mathbb{R}^{K+2} \rightarrow \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y . Then

$$mg(x, y, z) = g_x(x, y, z)x + g_y(x, y, z)y$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^K$.

Moreover, $g_x(x, y, z)$ and $g_y(x, y, z)$ are themselves homogeneous of degree $m - 1$ in x and y .

Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive, so ours will be a prototypical *competitive general equilibrium model*.
- Households own all of the labor, which they supply inelastically.
- Endowment of labor in the economy, $\bar{L}(t)$, and all of this will be supplied regardless of the price.
- The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t) \quad (2)$$

for all t , where $L(t)$ denotes the demand for labor (and also the level of employment).

- More generally, should be written in complementary slackness form.
- In particular, let the *wage rate* at time t be $w(t)$, then the labor market clearing condition takes the form

$$L(t) \leq \bar{L}(t), w(t) \geq 0 \text{ and } (L(t) - \bar{L}(t)) w(t) = 0$$

Market Structure, Endowments and Market Clearing II

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive.
- Households also own the capital stock of the economy and rent it to firms.
- Denote the *rental price of capital* at time t be $R(t)$.
- Capital market clearing condition:

$$K^s(t) = K^d(t)$$

- Take households' initial holdings of capital, $K(0)$, as given
- $P(t)$ is the price of the final good at time t , normalize the price of the final good to 1 *in all periods*.
- Build on an insight by Kenneth Arrow (Arrow, 1964) that it is sufficient to price *securities* (assets) that transfer one unit of consumption from one date (or state of the world) to another.

Market Structure, Endowments and Market Clearing III

- Implies that we need to keep track of an *interest rate* across periods, $r(t)$, and this will enable us to normalize the price of the final good to 1 in every period.
- *General equilibrium economies*, where different commodities correspond to the same good at different dates.
- The same good at different dates (or in different states or localities) is a different commodity.
- Therefore, there will be *an infinite number of commodities*.
- Assume capital depreciates, with “exponential form,” at the rate δ : out of 1 unit of capital this period, only $1 - \delta$ is left for next period.
- Loss of part of the capital stock affects the interest rate (rate of return to savings) faced by the household.
- *Interest rate* faced by the household will be $r(t) = R(t) - \delta$.

Firm Optimization I

- Only need to consider the problem of a *representative firm*:

$$\max_{L(t) \geq 0, K(t) \geq 0} F[K(t), L(t), A(t)] - w(t)L(t) - R(t)K(t). \quad (3)$$

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem.
- Equivalently, *cost minimization problem*.
- Features worth noting:
 - Problem is set up in terms of aggregate variables.
 - Nothing multiplying the F term, price of the final good has normalized to 1.
 - Already imposes competitive factor markets: firm is taking as given $w(t)$ and $R(t)$.
 - Concave problem, since F is concave.

Firm Optimization II

- Since F is differentiable, first-order necessary conditions imply:

$$w(t) = F_L[K(t), L(t), A(t)], \quad (4)$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \quad (5)$$

- Note also that in (4) and (5), we used $K(t)$ and $L(t)$, the amount of capital and labor used by firms.
- In fact, solving for $K(t)$ and $L(t)$, we can derive the capital and labor demands of firms in this economy at rental prices $R(t)$ and $w(t)$.
- Thus we could have used $K^d(t)$ instead of $K(t)$, but this additional notation is not necessary.

Firm Optimization III

Proposition Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t)L(t) + R(t)K(t).$$

- **Proof:** Follows immediately from Euler Theorem for the case of $m = 1$, i.e., constant returns to scale.
- Thus firms make no profits, so ownership of firms does not need to be specified.

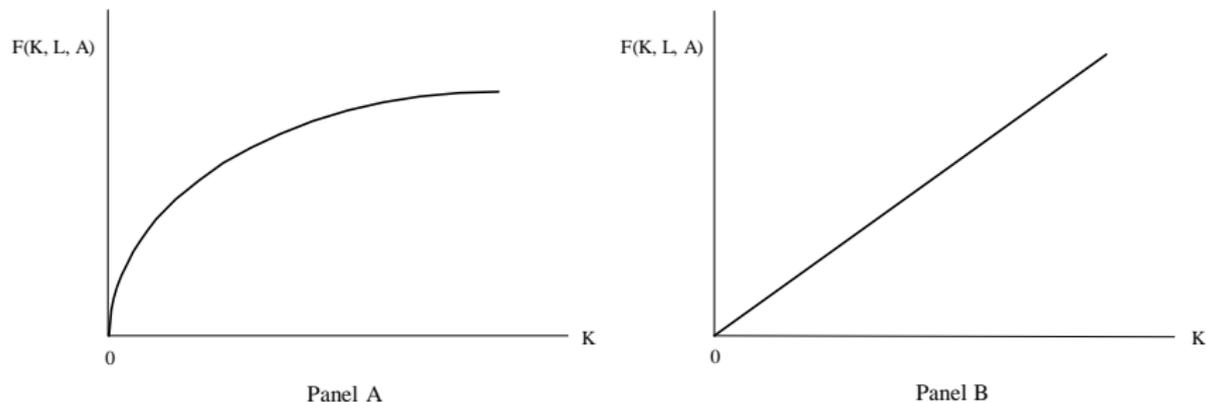
Second Key Assumption

Assumption 2 (Inada conditions) F satisfies the Inada conditions

$$\begin{aligned}\lim_{K \rightarrow 0} F_K(\cdot) &= \infty \text{ and } \lim_{K \rightarrow \infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ all } A \\ \lim_{L \rightarrow 0} F_L(\cdot) &= \infty \text{ and } \lim_{L \rightarrow \infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ all } A.\end{aligned}$$

- Important in ensuring the existence of *interior equilibria*.
- It can be relaxed quite a bit, though useful to get us started.

Production Functions



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Figure 2.1 in Acemoglu, Daron. *Introduction to Modern Economic Growth*. Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

Fundamental Law of Motion of the Solow Model I

- Recall that K depreciates exponentially at the rate δ , so

$$K(t+1) = (1 - \delta) K(t) + I(t), \quad (6)$$

where $I(t)$ is investment at time t .

- From national income accounting for a closed economy,

$$Y(t) = C(t) + I(t), \quad (7)$$

- Using (1), (6) and (7), any *feasible* dynamic allocation in this economy must satisfy

$$K(t+1) \leq F[K(t), L(t), A(t)] + (1 - \delta) K(t) - C(t)$$

for $t = 0, 1, \dots$

- Behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably.

Fundamental Law of Motion of the Solow Model II

- Note not derived from the maximization of utility function: welfare comparisons have to be taken with a grain of salt.
- Since the economy is closed (and there is no government spending),

$$S(t) = I(t) = Y(t) - C(t).$$

- Individuals are assumed to save a constant fraction s of their income,

$$S(t) = sY(t), \quad (8)$$

$$C(t) = (1 - s)Y(t) \quad (9)$$

- Implies that the supply of capital resulting from households' behavior can be expressed as

$$K^s(t) = (1 - \delta)K(t) + S(t) = (1 - \delta)K(t) + sY(t).$$

Fundamental Law of Motion of the Solow Model III

- Setting supply and demand equal to each other, this implies $K^s(t) = K(t)$.
- From (2), we have $L(t) = \bar{L}(t)$.
- Combining these market clearing conditions with (1) and (6), we obtain *the fundamental law of motion* the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1 - \delta)K(t). \quad (10)$$

- Nonlinear *difference equation*.
- Equilibrium of the Solow growth model is described by this equation together with laws of motion for $L(t)$ (or $\bar{L}(t)$) and $A(t)$.

Definition of Equilibrium I

- Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model.
- Households do not optimize, but firms still maximize and factor markets clear.

Definition In the basic Solow model for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $K(t)$ satisfies (10), $Y(t)$ is given by (1), $C(t)$ is given by (9), and $w(t)$ and $R(t)$ are given by (4) and (5).

- Note an equilibrium is defined as an entire path of allocations and prices: *not* a static object.

Equilibrium Without Population Growth and Technological Progress I

- Make some further assumptions, which will be relaxed later:
 - There is no population growth; total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L(t) = L$.
 - No technological progress, so that $A(t) = A$.
- Define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \quad (11)$$

- Using the constant returns to scale assumption, we can express output (income) per capita, $y(t) \equiv Y(t) / L$, as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \quad (12)$$

Equilibrium Without Population Growth and Technological Progress II

- Note that $f(k)$ here depends on A , so I could have written $f(k, A)$; but A is constant and can be normalized to $A = 1$.
- From Euler Theorem,

$$\begin{aligned}R(t) &= f'(k(t)) > 0 \text{ and} \\w(t) &= f(k(t)) - k(t)f'(k(t)) > 0.\end{aligned}\tag{13}$$

- Both are positive from Assumption 1.

Example: The Cobb-Douglas Production Function I

- Very special production function and many interesting phenomena are ruled out, but widely used:

$$\begin{aligned} Y(t) &= F[K(t), L(t), A(t)] \\ &= AK(t)^\alpha L(t)^{1-\alpha}, \quad 0 < \alpha < 1. \end{aligned} \quad (14)$$

- Satisfies Assumptions 1 and 2.
- Dividing both sides by $L(t)$,

$$y(t) = Ak(t)^\alpha,$$

- From equation (13),

$$\begin{aligned} R(t) &= \frac{\partial Ak(t)^\alpha}{\partial k(t)}, \\ &= \alpha Ak(t)^{-(1-\alpha)}. \end{aligned}$$

Example: The Cobb-Douglas Production Function II

- Alternatively, in terms of the original production function (14),

$$\begin{aligned}R(t) &= \alpha AK(t)^{\alpha-1} L(t)^{1-\alpha} \\ &= \alpha Ak(t)^{-(1-\alpha)},\end{aligned}$$

- Similarly, from (13),

$$\begin{aligned}w(t) &= Ak(t)^\alpha - \alpha Ak(t)^{-(1-\alpha)} \times k(t) \\ &= (1-\alpha) AK(t)^\alpha L(t)^{-\alpha},\end{aligned}$$

Equilibrium Without Population Growth and Technological Progress I

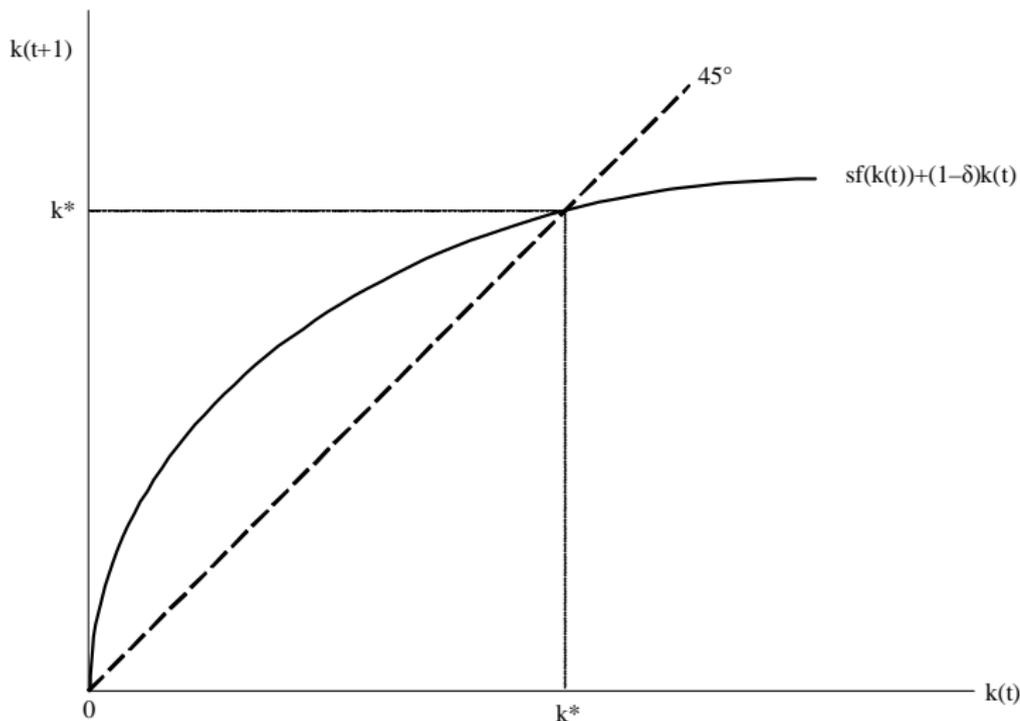
- The per capita representation of the aggregate production function enables us to divide both sides of (10) by L to obtain:

$$k(t+1) = sf(k(t)) + (1 - \delta)k(t). \quad (15)$$

- Since it is derived from (10), it also can be referred to as the *equilibrium difference equation* of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio $k(t)$.

Definition A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k(t) = k^*$ for all t .

- The economy will tend to this steady state equilibrium over time (but never reach it in finite time).



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Figure 2.2 in Acemoglu, Daron. [Introduction to Modern Economic Growth](#).

Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Determination of the steady-state capital-labor ratio in the Solow model without population growth and technological change.

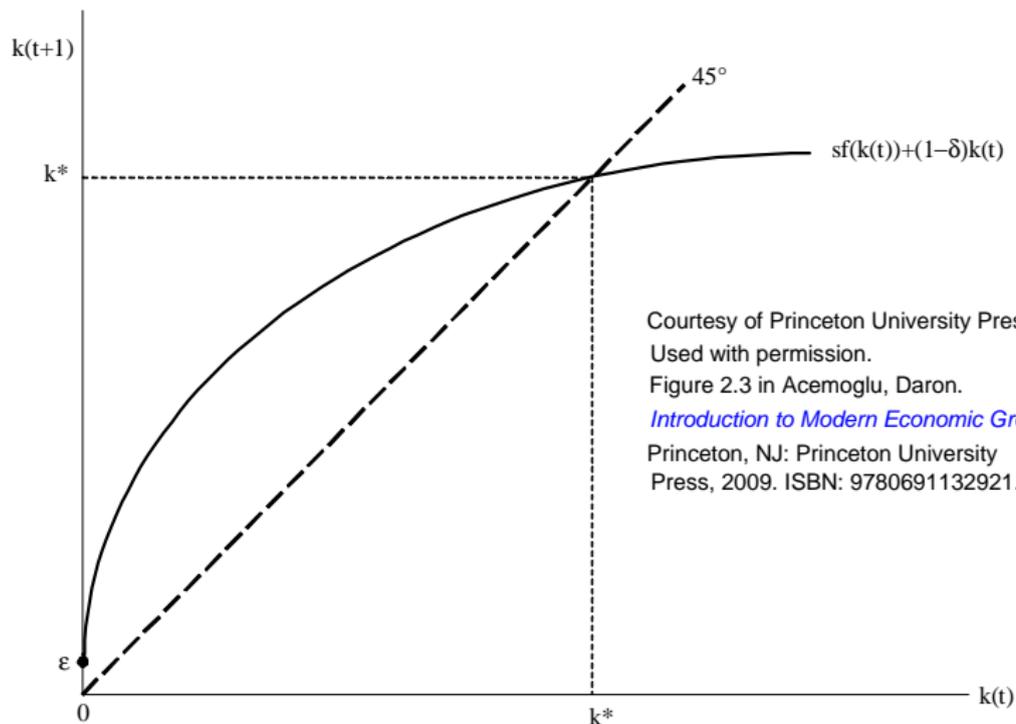
Equilibrium Without Population Growth and Technological Progress II

- Thick curve represents (15) and the dashed line corresponds to the 45° line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio k^* ,

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \quad (16)$$

- There is another intersection at $k = 0$, because the figure assumes that $f(0) = 0$.
- Will ignore this intersection throughout:
 - If capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium
 - This intersection, even when it exists, is an *unstable point*
 - It has no economic interest for us.

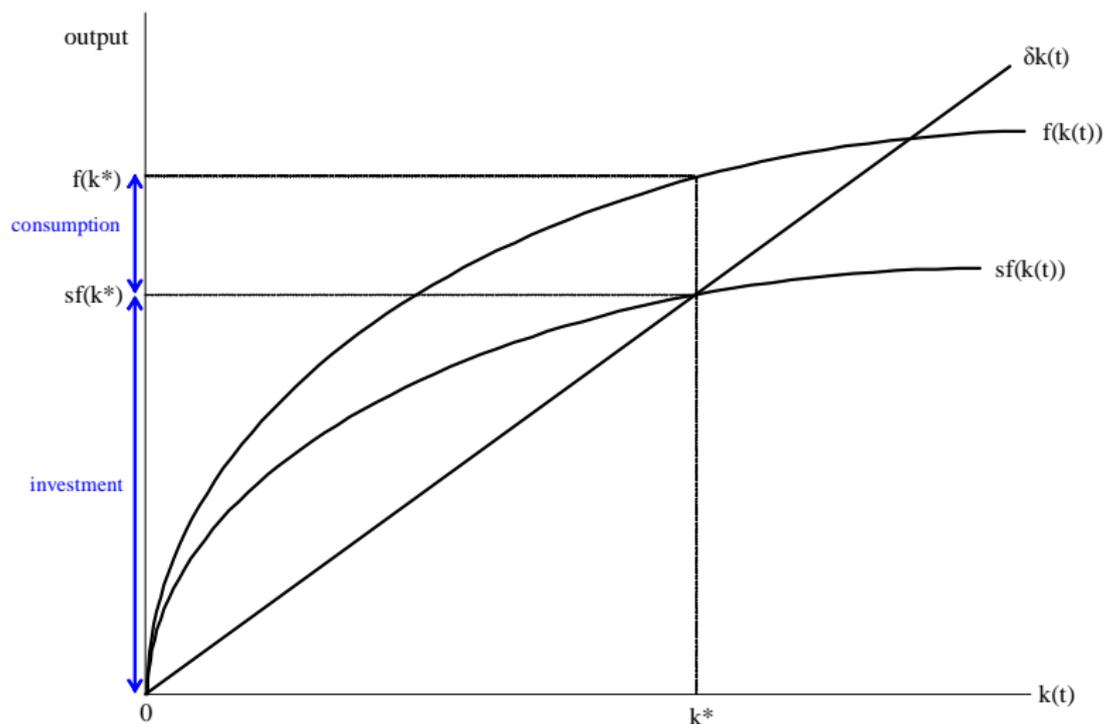
Equilibrium Without Population Growth and Technological Progress III



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Figure 2.3 in Acemoglu, Daron.
[Introduction to Modern Economic Growth](#).
Princeton, NJ: Princeton University
Press, 2009. ISBN: 9780691132921.

Equilibrium Without Population Growth and Technological Progress IV

- Alternative visual representation of the steady state: intersection between δk and the function $sf(k)$. Useful because:
 - Depicts the levels of consumption and investment in a single figure.
 - Emphasizes the steady-state equilibrium sets investment, $sf(k)$, equal to the amount of capital that needs to be “replenished”, δk .



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Figure 2.4 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Investment and consumption in the steady-state equilibrium.

Equilibrium Without Population Growth and Technological Progress V

Proposition Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio $k^* \in (0, \infty)$ is given by (16), per capita output is given by

$$y^* = f(k^*) \quad (17)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \quad (18)$$

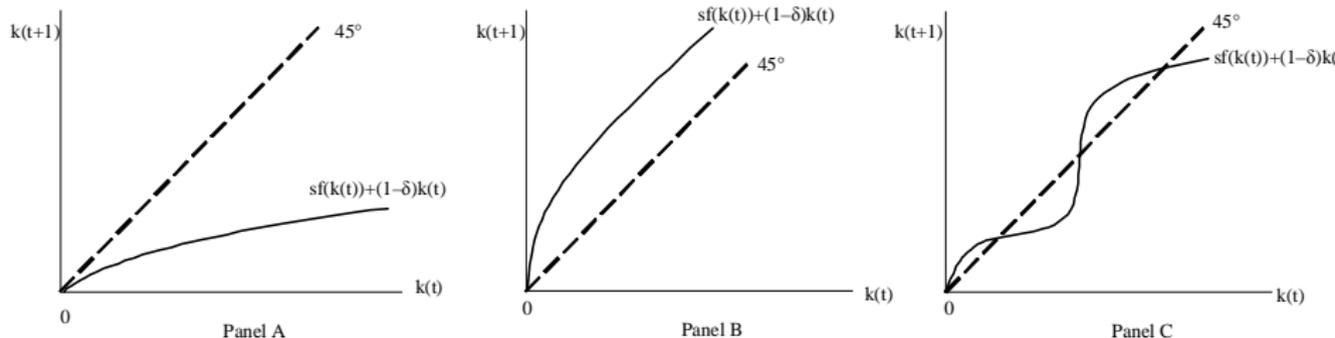
Proof of Theorem

- The preceding argument establishes that any k^* that satisfies (16) is a steady state.
- To establish existence, note that from Assumption 2 (and from L'Hospital's rule), $\lim_{k \rightarrow 0} f(k)/k = \infty$ and $\lim_{k \rightarrow \infty} f(k)/k = 0$.
- Moreover, $f(k)/k$ is continuous from Assumption 1, so by the Intermediate Value Theorem there exists k^* such that (16) is satisfied.
- To see uniqueness, differentiate $f(k)/k$ with respect to k , which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \quad (19)$$

where the last equality uses (13).

- Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (16).
- Equations (17) and (18) then follow by definition.



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Figure 2.5 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.

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Figure: Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

Equilibrium Without Population Growth and Technological Progress VI

- Figure shows through a series of examples why Assumptions 1 and 2 cannot be dispensed with for the existence and uniqueness results.
- Generalize the production function in one simple way, and assume that

$$f(k) = a\tilde{f}(k),$$

- $a > 0$, so that a is a (“Hicks-neutral”) shift parameter, with greater values corresponding to greater productivity of factors..
- Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

Equilibrium Without Population Growth and Technological Progress VII

Proposition Suppose Assumptions 1 and 2 hold and $f(k) = a\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(a, s, \delta)$ and the steady-state level of output by $y^*(a, s, \delta)$ when the underlying parameters are a , s and δ . Then we have

$$\begin{aligned} \frac{\partial k^*(\cdot)}{\partial a} &> 0, \quad \frac{\partial k^*(\cdot)}{\partial s} > 0 \quad \text{and} \quad \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \\ \frac{\partial y^*(\cdot)}{\partial a} &> 0, \quad \frac{\partial y^*(\cdot)}{\partial s} > 0 \quad \text{and} \quad \frac{\partial y^*(\cdot)}{\partial \delta} < 0. \end{aligned}$$

Equilibrium Without Population Growth and Technological Progress VIII

- **Proof of comparative static results:** follows immediately by writing

$$\frac{\tilde{f}(k^*)}{k^*} = \frac{\delta}{as'}$$

which holds for an open set of values of k^* . Now apply the implicit function theorem to obtain the results.

- For example,

$$\frac{\partial k^*}{\partial s} = \frac{\delta (k^*)^2}{s^2 w^*} > 0$$

where $w^* = f(k^*) - k^* f'(k^*) > 0$.

- The other results follow similarly.

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- Same comparative statics with respect to a and δ immediately apply to c^* as well.
- But c^* will not be monotone in the saving rate (think, for example, of $s = 1$).
- In fact, there will exist a specific level of the saving rate, s_{gold} , referred to as the “golden rule” saving rate, which maximizes c^* .
- But cannot say whether the golden rule saving rate is “better” than some other saving rate.
- Write the steady state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned}c^*(s) &= (1 - s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s),\end{aligned}$$

- The second equality exploits that in steady state $sf(k) = \delta k$.

Equilibrium Without Population Growth and Technological Progress X

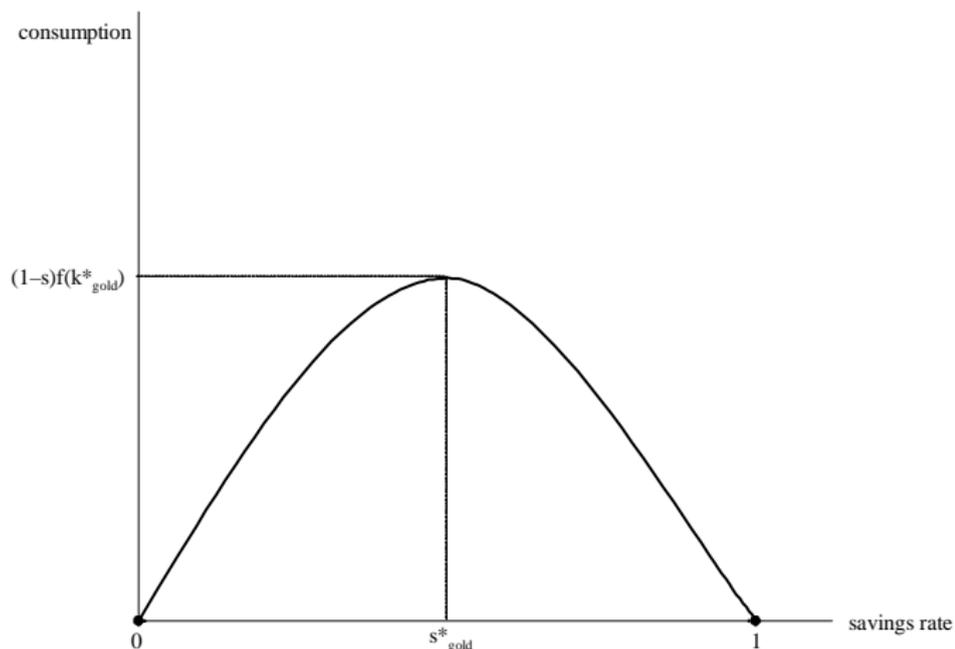
- Differentiating with respect to s ,

$$\frac{\partial c^*(s)}{\partial s} = [f'(k^*(s)) - \delta] \frac{\partial k^*}{\partial s}. \quad (20)$$

- s_{gold} is such that $\partial c^*(s_{gold}) / \partial s = 0$. The corresponding steady-state golden rule capital stock is defined as k_{gold}^* .

Proposition In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold} , with the corresponding steady state capital level k_{gold}^* such that

$$f'(k_{gold}^*) = \delta. \quad (21)$$



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Figure 2.6 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: The “golden rule” level of savings rate, which maximizes steady-state consumption.

Proof of Proposition: Golden Rule

- By definition $\partial c^*(s_{gold}) / \partial s = 0$.
- From Proposition above, $\partial k^* / \partial s > 0$, thus (20) can be equal to zero only when $f'(k^*(s_{gold})) = \delta$.
- Moreover, when $f'(k^*(s_{gold})) = \delta$, it can be verified that $\partial^2 c^*(s_{gold}) / \partial s^2 < 0$, so $f'(k^*(s_{gold})) = \delta$ indeed corresponds a local maximum.
- That $f'(k^*(s_{gold})) = \delta$ also yields the global maximum is a consequence of the following observations:
 - $\forall s \in [0, 1]$ we have $\partial k^* / \partial s > 0$ and moreover, when $s < s_{gold}$, $f'(k^*(s)) - \delta > 0$ by the concavity of f , so $\partial c^*(s) / \partial s > 0$ for all $s < s_{gold}$.
 - by the converse argument, $\partial c^*(s) / \partial s < 0$ for all $s > s_{gold}$.
 - Therefore, only s_{gold} satisfies $f'(k^*(s)) = \delta$ and gives the unique global maximum of consumption per capita.

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- When the economy is below k_{gold}^* , higher saving will increase consumption; when it is above k_{gold}^* , steady-state consumption can be increased by saving less.
- In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (*dynamic inefficiency*).
- But no utility function, so statements about “inefficiency” have to be considered with caution.
- Such dynamic inefficiency will not arise once we endogenize consumption-saving decisions.

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