

14.451 Problem Set 2

Fall 2009

Due on October 2

1 An adjustment cost model

Consider a firm with the technology

$$y_t = f(k_t)$$

where $f : R_+ \rightarrow R$ is continuously differentiable and strictly increasing. Capital is the only input in this firm and the initial capital stock k_0 is given. The firm maximizes the discounted sum of future profits

$$\sum_{t=0}^{\infty} \beta^t \pi_t,$$

and profits are given by

$$\pi_t = f(k_t) - i_t - \xi \frac{1}{2} (i_t)^2,$$

where i_t denotes investment and is given by

$$i_t = k_{t+1} - (1 - \delta) k_t$$

(δ is the depreciation rate of capital). The expression $\xi/2 (i_t)^2$ represents an adjustment cost associated to investment and disinvestment.

Suppose that there is an upper bound \bar{k} on the capital stock that can be used in the firm, so $k_0 \in [0, \bar{k}]$ and k_t must be chosen in $[0, \bar{k}]$ for all $t = 1, 2, \dots$

1. Show that all the assumptions are satisfied to apply dynamic programming with bounded returns (in SLP: Assumptions 4.3 and 4.4.)
2. Show that the value function is increasing.

3. Show that an alternative version of Theorem 4.7 can be proved if Assumptions 4.5 and 4.6 are replaced by the following assumption (in SLP notation):

Assumption. For each pair x', x'' with $x'' \geq x'$ (with at least one strict inequality) and each $y' \in \Gamma(x')$ there is a $y'' \geq y'$ such that $F(x'', y'') > F(x', y')$ and $y'' \in \Gamma(x'')$.

(If you want, you can do part 3 first and apply your theorem in part 2, or do part 2 first to get your intuition straight and then generalize).

2 Working and resting

Time is discrete and infinite. In each period you can choose one of two actions: work or rest.

The payoff from working is as follows: if last time you rested was in period $t - n - 1$ and then you worked in periods $t - n, t - n + 1, \dots, t - 1$ (with no interruption), your payoff is given by $r(n)$ (r is a function $r: \mathbb{Z}^+ \rightarrow \mathbb{R}$ with \mathbb{Z}^+ the set of non-negative integers).

If you rest your payoff is 0. At the beginning of time suppose you just came from a period of rest (at -1) so your initial payoff if working is $r(0)$.

Your objective is to maximize the discounted sum of your payoffs with discount factor $\beta \in (0, 1)$.

The function $r(n)$ is positive at 0, decreasing for all $n \geq 0$ and negative $r(n) < 0$ for all $n \geq \bar{n} > 0$.

1. Setup the problem in recursive form. (What is a good state variable here?)
2. Suppose the function $r(n)$ is bounded. Argue that the methods of bounded dynamic programming can be applied (even though X is not a subset of R^l and continuity assumptions clearly make no sense in our context.) Prove, in particular, the analogue of Theorem 4.6.
3. Prove, using inductive arguments, that the value function is always positive and is non-increasing in n .
4. Show that the optimal policy yields a cycle of n^* periods of work and 1 period of rest, for some n^* . Prove that $0 < n^* \leq \bar{n}$.
5. Suppose $r(n) = 3 - 2n$. Prove that for $\beta < 1/2$ the optimal policy yields $n^* = 2$ and for $\beta > 1/2$ yields $n^* = 1$. Why does a more patient agent work less?
- 6*. (Optional) Can you prove that an increase in β never leads to a larger n^* (for any decreasing function $r(\cdot)$)?

3 Non-differentiabilities

Consider the value function coming from the problem

$$V(x) = \max_{y \in [-2, 2]} f(x, y)$$

where the function f is defined as

$$f(x, y) = \max \left\{ -(y+1)^2, x - (y-1)^2 \right\}. \quad (1)$$

1. Show that $V(x)$ is non-differentiable at $x = 0$ and that the optimal policy is not continuous even though there are no corner solutions here (unlike in our example in class).
2. Does it matter that the function f is non-differentiable? Can you find an example where f is everywhere differentiable, optimal solutions are always interior and V is non differentiable? (A graphical argument on a variant of function (1) is enough).
3. Now suppose

$$f(x, y) = \max \left\{ -(y-1)^2, x - (y-1)^2 \right\}.$$

Show that the policy is continuous, yet the value function is non-differentiable. Does it matter that f is non-differentiable here?

MIT OpenCourseWare
<http://ocw.mit.edu>

14.451 Dynamic Optimization Methods with Applications
Fall 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.