Replicator Dynamics

Nash makes sense (*arguably*) if…

-Uber-rational

-Calculating

Such as Auctions…

Or Oligopolies…

Image courtesy of afagen on Flickr. CC BY NC-SA

Image courtesy of longislandwins on Flickr. CC-BY

But why would game theory matter for our puzzles?

Norms/rights/morality are not chosen; rather…

We believe we have rights!

We feel angry when uses service but doesn't pay

But…

From where do these feelings/beliefs come?

In this lecture, we will introduce replicator dynamics

The replicator dynamic is a simple model of evolution and prestige-biased learning in games

Today, we will show that replicator leads to Nash

We consider a large population, N, of players

Each period, a player is randomly matched with another player and they play a two-player game

Each player is assigned a strategy. Players cannot choose their strategies

We can think of this in a few ways, e.g.:

- Players are "born with" their mother's strategy (ignore sexual reproduction)
- Players "imitate" others' strategies

Note:

Rationality and consciousness don't enter the picture.

Suppose there are two strategies, A and B.

We start with:

Some number, N_A, of players assigned strategy A
And some number, N_B, of players assigned strategy B

We denote the proportion of the population playing strategy A as X_A, so:

$x_A = N_A/N$

$x_B = N_B/N$

The state of the population is given by (x_A, x_B) where $x_A \geq 0$, $x_B \geq 0$, and $x_A + x_B = 1$.

Since players interacts with another randomly chosen player in the population, a player's EXPECTED payoff is determined by the payoff matrix and the proportion of each strategy in the population.

For example, consider the coordination game:

	A	B
A	a, a	b, c
B	c, b	d, d

$a > c$

$b < d$

And the following starting frequencies:

$x_A = .75$

$x_B = .25$

Payoff for player who is playing A is f_A

Since f_A depends on x_A and x_B we write $f_A(x_A, x_B)$

$f_A(x_A, x_B)$ = (probability of interacting with A player)*$U_A(A,A)$
+ (probability of interacting with B player)*$U_A(A,B)$

= x_A*a + x_B*b

= .75*a + .25*b

We interpret payoff as rate of reproduction (fitness).

The average fitness, f, of a population is the weighted average of the two fitness values.

$$f(x_A, x_B) = x_A*f_A(x_A, x_B) + x_B*f_B(x_A, x_B)$$

How fast do x_A and x_B grow?

Recall $x_A = N_A / N$

First, we need to know how fast does N_A grows

$$\text{Let } \dot{N}_A = dN_A/dt$$

Each individual reproduces at a rate f_A, and there are N_A of them. So:

$$\dot{N}_A = N_A * f_A(x_A, x_B)$$

Next we need to know how fast N grows. By the same logic:

$$\dot{N} = N * f(x_A, x_B)$$

By the quotient rule, and with a little simplification…

This is the replicator equation:

$$\dot{x}_A = x_A * (f_A(x_A, x_B) - f(x_A, x_B))$$

Current frequency of strategy

Own fitness relative to the average

Growth rate of A

$$\dot{x}_A = x_A * (f_A(x_A, x_B) - f(x_A, x_B))$$

Current frequency of strategy

Own fitness relative to the average

Because that's how many As can reproduce

This is our key property. More successful strategies grow faster

$$\dot{x}_A = x_A * (f_A(x_A, x_B) - f(x_A, x_B))$$

If:

$x_A > 0$: The proportion of As is non-zero

$f_A > f$: The fitness of A is above average

Then:

$\dot{x}_A > 0$: A will be increasing in the population

The steady states are

$x_A = 0$

$x_A = 1$

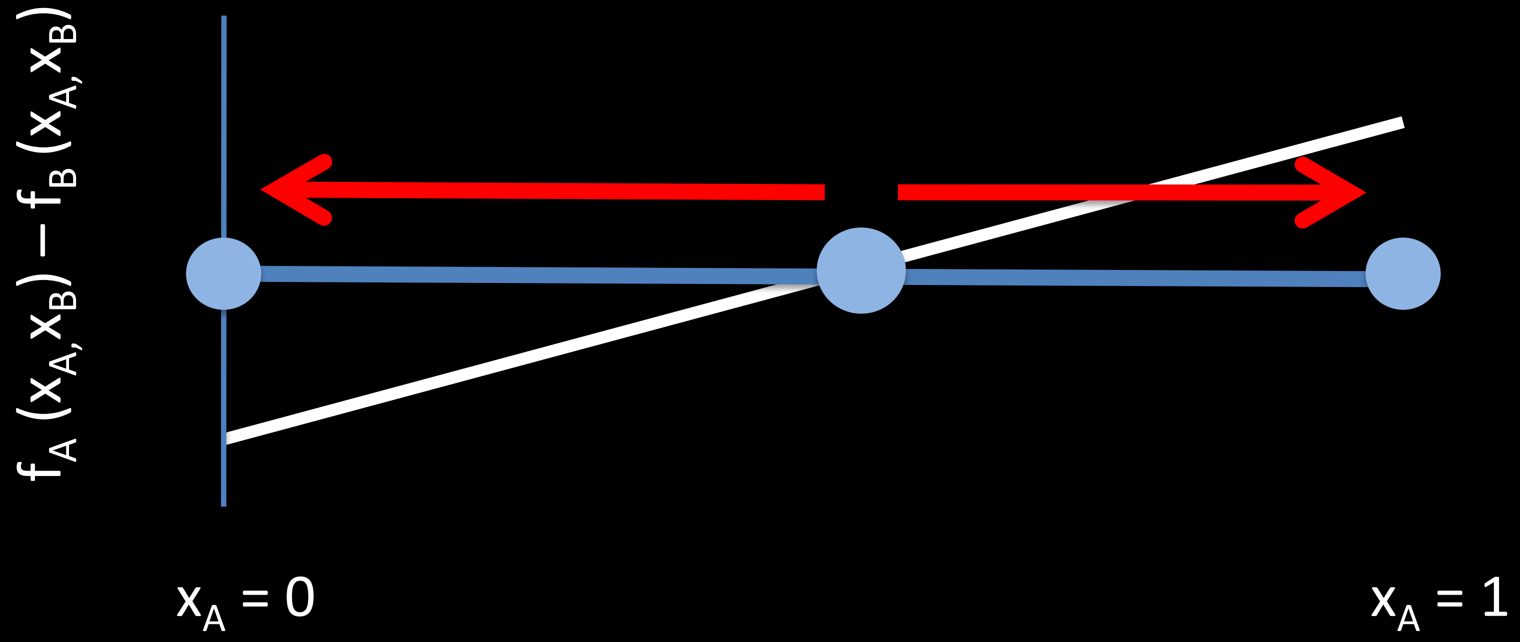
x_A such that $f_A(x_{A,}x_B) = f_B(x_{A,}x_B)$

Recall the payoffs of our (coordination) game:

	A	B
A	a, a	b, c
B	c, b	d, d

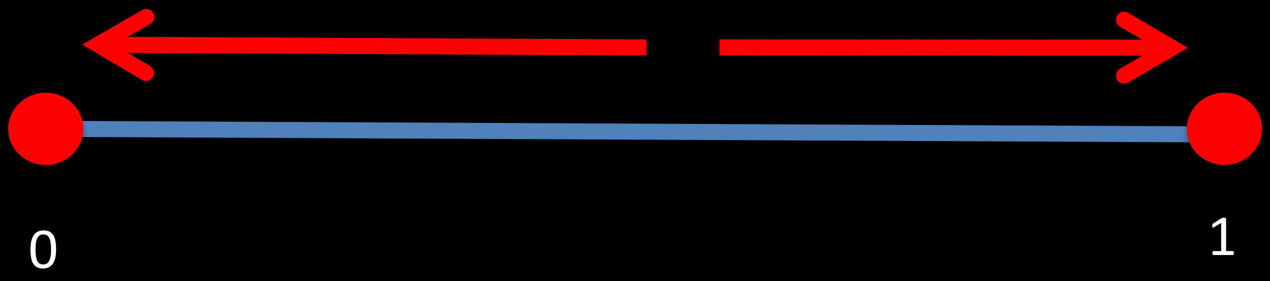
$a > c$
$b < d$

= “asymptotically stable” steady states
i.e., steady states s.t. the dynamics point toward it



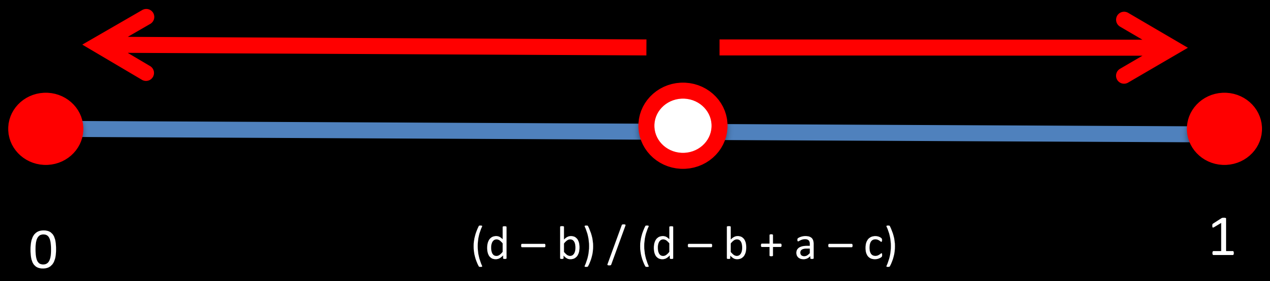
What were the pure Nash equilibria of the coordination game?

	A	B
A	a, a	b, c
B	c, b	d, d


0
1

And the mixed strategy equilibrium is:

$$x_A = (d - b) / (d - b + a - c)$$


0
(d – b) / (d – b + a – c)
1

Replicator teaches us:

We end up at Nash
(…if we end)

AND not just any Nash
(e.g. not mixed Nash in coordination)

Let’s generalize this to three strategies:

R

P

S

Now…

N_R is the number playing R

N_P is the number playing P

N_S is the number playing S

Now…

x_R is the proportion playing R

x_P is the proportion playing P

x_S is the proportion playing S

The state of population is
(x_R, x_S, x_P) where $x_R \geq 0$, $x_P \geq 0$, $x_S \geq 0$,
and $x_R + x_S + x_P = 1$

For example,
Consider the
Rock-Paper-Scissors
Game:

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

With starting frequencies:

$x_R = .25$
$x_P = .25$
$x_S = .5$

Fitness for player playing R is f_R

$f_R(x_R,x_P,x_S)$ = (probability of interacting with R player)*$U_R(R,R)$
+ (probability of interacting with P player)*$U_R(R,P)$
+ (probability of interacting with S player)*$U_R(R,S)$

= .25*0 + .25*-1 + .5*1
= .25

In general, fitness for players with strategy R is:

$$f_R(x_R,x_P,x_S) = x_R*0 + x_P*-1 + x_S*1$$

The average fitness, f, of the population is:

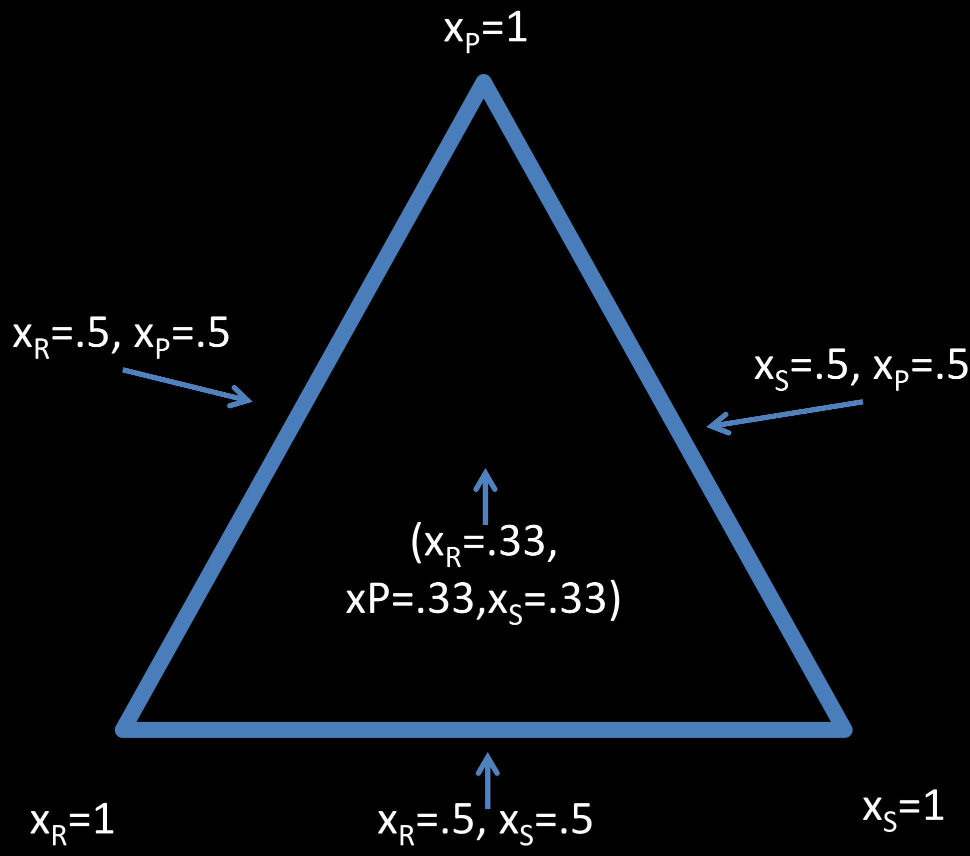
$$f(x_R,x_P,x_S) = x_R*f_R(x_R,x_P,x_S) + x_P*f_P(x_R,x_P,x_S) + x_S*f_S(x_R,x_P,x_S)$$

Replicator is *still*:

$$\dot{x}_R = x_R * (f_R(x_R,x_P,x_S) - f(x_R,x_P,x_S))$$

Current frequency of strategy

Own fitness relative to the average


x_P=1
x_R=.5, x_P=.5
x_S=.5, x_P=.5
(x_R=.33,
xP=.33,x_S=.33)
x_R=1
x_R=.5, x_S=.5
x_S=1

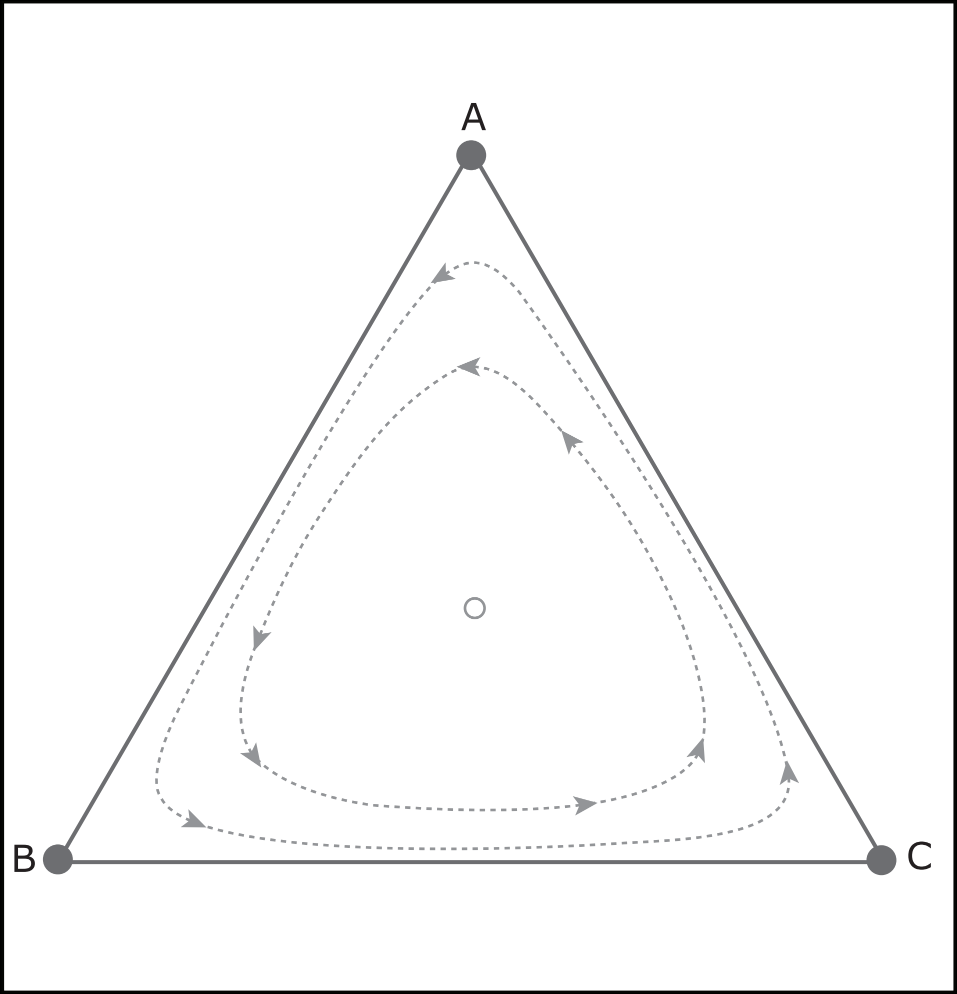


Image by MIT OpenCourseWare.

Notice not asymptotically stable

It cycles

Will show this in HW

	R	P	S
R	0	-1	2
P	2	0	-1
S	-1	2	0

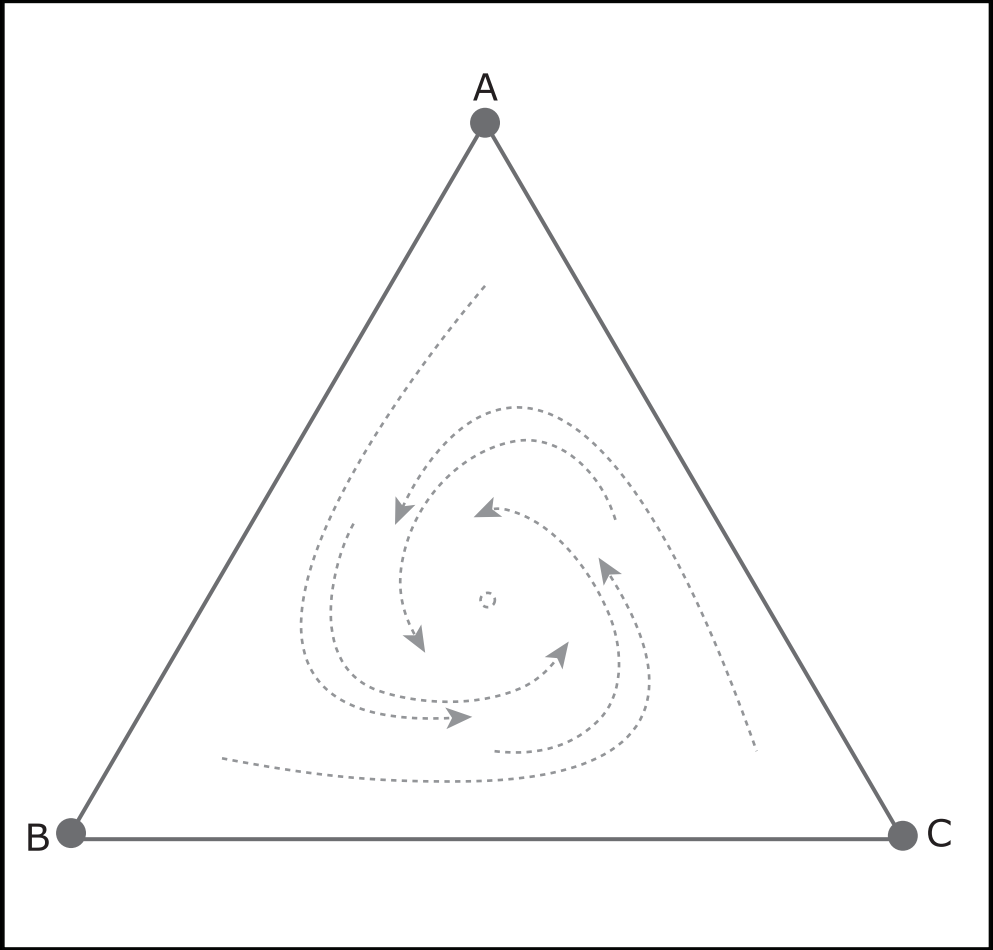


Image by MIT OpenCourseWare.

Note now is asymptotically stable

Will solve for Nash and show this is what dynamics look like in HW

For further readings, see:

Nowak Evolutionary Dynamics Ch. 4

Weibull Evolutionary Game Theory Ch. 3

Some notes:

Can be extended to any number of strategies

Doesn't always converge, but when does converges to Nash

We will later use this to provide evidence that dynamics predict behavior better than Nash

14.11 Insights from Game Theory into Social Behavior
Fall 2013