14.01 Principles of Microeconomics, Fall 2007
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Lecture 5

Deriving MRS from Utility Function, Budget Constraints, and Interior Solution of Optimization

Outline

1. Chap 3: *Utility Function, Deriving MRS*
2. Chap 3: *Budget Constraint*
3. Chap 3: *Optimization: Interior Solution*

1 Utility Function, Deriving MRS

Examples of utility:

Example (Perfect substitutes).

$$U(x, y) = ax + by.$$

Example (Perfect complements).

$$U(x, y) = min\{ax, by\}.$$

Example (Cobb-Douglas Function).

$$U(x, y) = Ax^b y^c.$$

Example (One good is bad).

$$U(x, y) = -ax + by.$$

An important thing is to derive MRS.

$$MRS = -\frac{dy}{dx} = |\text{Slope of Indifference Curve}|.$$

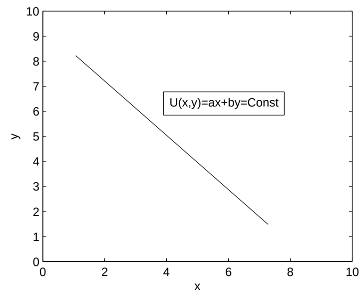

Figure 1: Utility Function of Perfect Substitutes

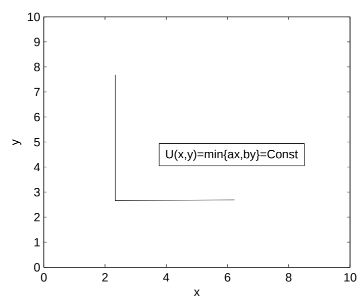

Figure 2: Utility Function of Perfect Complements

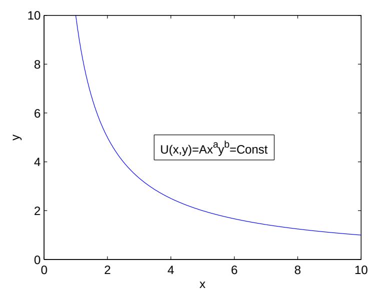

Figure 3: Cobb-Douglas Utility Function

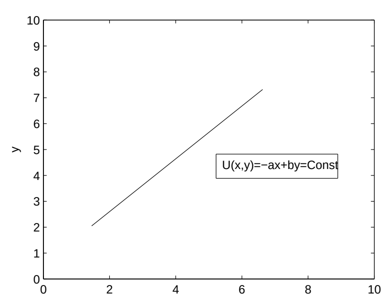

Figure 4: Utility Function of the Situation That One Good Is Bad

Because utility is constant along the indifference curve,

$$u = (x, y(x)) = C, \Longrightarrow$$
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0, \Longrightarrow$$
$$-\frac{dy}{dx} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

Thus,

$$MRS = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

Example (Sample utility function).

$$u(x, y) = xy^2.$$

Two ways to derive MRS:

- Along the indifference curve

$$xy^2 = C.$$
$$y = \sqrt{\frac{c}{x}}.$$

  Thus,

$$MRSd = -\frac{dy}{dx} = \frac{\sqrt{c}}{2x^{3/2}} = \frac{y}{2x}.$$

- Using the conclusion above

$$MRS = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{y^2}{2xy} = \frac{y}{2x}.$$

2 Budget Constraint

The problem is about how much goods a person can buy with limited income.

Assume: no saving, with income I, only spend money on goods x and y with the price P_x and P_y.

Thus the budget constraint is

$$P_x \cdot x + P_y \cdot y \leqslant I.$$

Suppose $P_x = 2$, $P_y = 1$, $I = 8$, then

$$2x + y \leqslant 8.$$

The slope of budget line is

$$-\frac{dy}{dx} = \frac{P_x}{P_y}.$$

Bundles below the line are affordable.

Budget line can shift:

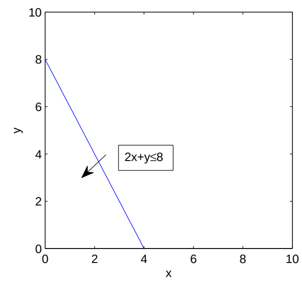

Figure 5: Budget Constraint

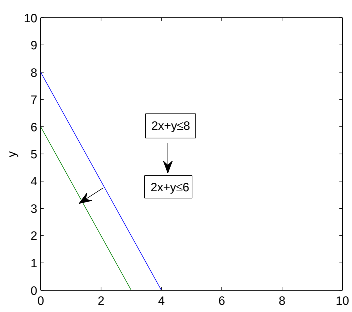

Figure 6: Budget Line Shifts Because of Change in Income

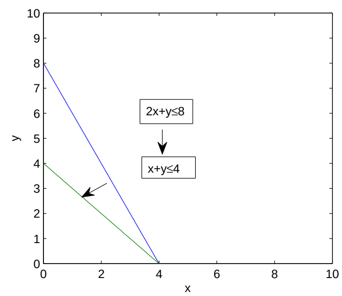

Figure 7: Budget Line Rotates Because of Change in Price

- Change in Income Assume $I' = 6$, then $2x + y = 6$. The budget line shifts right which means more income makes the affordable region larger.
- Change in Price Assume $P_x' = 2$, then $2x + 2y = 8$. The budget line changes which means lower price makes the affordable region larger.

3 Optimization: Interior Solution

Now the consumer's problem is: how to be as happy as possible with limited income. We can simplify the problem into language of mathematics:

$$\max_{x,y} U(x, y) \text{ subject to} \left\{ \begin{array}{c} xP_x + yP_y \leqslant I \\ x \geqslant 0 \\ y \geqslant 0 \end{array} \right\}.$$

Since the preference has non-satiation property, only (x, y) on the budget line can be the solution. Therefore, we can simplify the inequality to an equality:

$$xP_x + yP_y = I.$$

First, consider the case where the solution is interior, that is, $x > 0$ and $y > 0$. Example solutions:

- Method 1

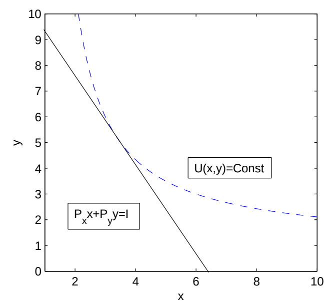

Figure 8: Interior Solution to Consumer's Problem

From Figure 8, the utility function reaches its maximum when the indifferent curve and constraint line are tangent, namely:

$$\frac{P_x}{P_y} = MRS = \frac{\partial u/\partial x}{\partial u/\partial y} = \frac{u_x}{u_y}.$$

– If

$$\frac{P_x}{P_y} > \frac{u_x}{u_y},$$

then one should consume more y, less x.

– If

$$\frac{P_x}{P_y} < \frac{u_x}{u_y},$$

then one should consume more x, less y. Intuition behind $\frac{P_x}{P_y} = MRS$: $\frac{P_x}{P_y}$ is the market price of x in terms of y, and MRS is the price of x in terms of y valued by the individual. If $P_x/P_y > MRS$, x is relatively expensive for the individual, and hence he should consume more y. On the other hand, if $P_x/P_y < MRS$, x is relatively cheap for the individual, and hence he should consume more x.

- Method 2: Use Lagrange Multipliers

$$L(x, y, \lambda) = u(x, y) - \lambda(xP_x + yP_y - I).$$

In order to maximize u, the following first order conditions must be satisfied:

$$\frac{\partial L}{\partial x} = 0 \Longrightarrow \frac{u_x}{P_x} = \lambda,$$
$$\frac{\partial L}{\partial y} = 0 \Longrightarrow \frac{u_y}{P_y} = \lambda,$$
$$\frac{\partial L}{\partial \lambda} = 0 \Longrightarrow xP_x + yP_y - I = 0.$$

Thus we have

$$\frac{P_x}{P_y} = \frac{u_x}{u_y}.$$

- Method 3

  Since $xP_x + yP_y + I = 0$,

$$y = \frac{I - xP_x}{P_y}.$$

  Then the problem can be written as

$$\max_{x,y} u(x,y) = u(x, \frac{I - xP_x}{P_y}).$$

  At the maximum, the following first order condition must be satisfied:

$$u_x + u_y(\frac{\partial y}{\partial x}) = u_x + u_y(-\frac{P_x}{P_y}) = 0.$$

$$\Longrightarrow$$

$$\frac{P_x}{P_y} = \frac{u_x}{u_y}.$$