Quasi-geostrophic ocean models

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1 Introduction

The starting point for theoretical and numerical study of the three dimensional large-scale circulation of the atmosphere and ocean is a vorticity equation that has a form very similar to that employed in the study of 2-d incompressible flow [see eqs.(6) and (7) of section 2.1 of 'vorticity equation' notes] except that the vorticity variable - which goes under the name of the 'quasi-geostrophic potential vorticity' - is a *three-dimensional* Laplacian of a streamfunction, rather than a two-dimensional Laplacian.

In this chapter we outline the assumptions underlying the quasi–geostrophic equations and describe how they can be integrated forward numerically to study the circulation of the ocean. These numerical studies played a seminal role in the development of baroclinic theories of the wind-driven ocean circulation.

2 The quasi-geostrophic equations

Underpinning the quasi-geostrophic equations are two key assumptions:

1. The absolute vorticity of a column of fluid is modified chiefly by vertical stretching or compression.

$$\frac{D}{Dt}\left(f + [curl\ \mathbf{v}]_z\right) = f\frac{\partial w}{\partial z} \tag{1}$$

where f is the Coriolis parameter (for now assumed constant), w is the vertical velocity and z is a vertical coordinate. By comparison, it is assumed that the change in vorticity due to tilting of the Taylor columns by the thermal wind is small. Note that in eq(1) it has been assumed that the vertical component of $[curl\ \mathbf{v}]_z$ is small compared to f when multiplying the stretching term, $\frac{\partial w}{\partial z}$: i.e. the Rossby number is small

$$R_o = \frac{V}{fL} << 1$$

1. where V is the speed of a typical horizontal current which varies over a horizontal scale L.

2. We suppose that the flow is nearly geostrophic. One can use the geostrophic relation to compute $[curl\ \mathbf{v}]_z$ thus:

$$[curl\ \mathbf{v}]_z = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

where the geostrophic winds are given by

$$f v_g - \frac{1}{\overline{\rho}}\frac{\partial p}{\partial x} = 0; \; f u_g + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial y} = 0 \tag{2}$$

and we have supposed that when we divide by ρ, it can be replaced by $\overline{\rho}$, a mean reference density that is constant in space and time (adopting an incompressible baroclinic model).

An important consequence of the geostrophic relation is that the horizontal divergence of the geostrophic current vanishes :

$$\nabla_h . v_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \tag{3}$$

Thus if the flow is almost geostrophic it is almost horizontally non-divergent and we cannot reliably use the continuity equation to obtain the vertical velocity - instead we must use the vorticity equation (1).

However, because the flow is nearly geostrophic we can use the idea of a streamfunction even though we have 3 dimensional motion. We can think of each level, z, in the ocean as having its own streamfunction from which u_g and v_g may be computed:

$$v_g = \frac{\partial \psi}{\partial x}; u_g = -\frac{\partial \psi}{\partial y} \tag{4}$$

This does not completely specify ψ because only horizontal derivatives are involved. So we can add any arbitrary function of z to ψ and we shall choose this arbitrary function in such a way that $\frac{\partial \psi}{\partial z}$ has a useful meaning. To this end we write the pressure:

$$p = p_o(z) + p'(x, y, z, t) \tag{5}$$

where $p_o(z)$ is some sort of average value (average over x, y and t at each horizontal level) so that p' is relatively small at all levels.

Thus the p that appears in (2) is just $p'(x, y, z, t)$, and from (2) and (4) we see that:

$$\psi = \frac{p'}{\overline{\rho} f} \tag{6}$$

Furthermore we assume that the flow is in hydrostatic balance:

$$\begin{aligned} \frac{\partial p}{\partial z} &= \frac{dp_o}{dz} + \frac{\partial p'}{\partial z} = -g\rho \\ &= -g\left(\overline{\rho} + \rho_o(z)\right) - g\rho'(x, y, z, t) \end{aligned}$$

where we have similarly separated the density field out thus:

$$\rho = \overline{\rho} + \rho_o(z) + \rho'(x, y, z, t) \tag{7}$$

Now we choose $\overline{\rho} + \rho_o(z)$ to correspond to $p_o(z)$ and $\rho'(x, y, z, t)$ to correspond to $p'(x, y, z, t)$: i.e.

$$\frac{dp_o}{dz} = -g\left(\overline{\rho} + \rho_o\right); \frac{\partial p'}{\partial z} = -g\rho' \tag{8}$$

where ρ' is assumed small compared to ρ_o.

Thus, (6) and (8) imply that:

$$b' = -g\frac{\rho'}{\overline{\rho}} = f\frac{\partial \psi}{\partial z} \tag{9}$$

where b' is the buoyancy, with associated reference profile $b_o(z)$.

Equation (9) is nothing more than a statement of the thermal wind equation in the incompressible baroclinic model. Indeed taking $\mathbf{k} \times \nabla$ (9) (where $\mathbf{k}$ is a unit vector in the vertical) and using (4) we immediately obtain the thermal wind equation: $f\frac{\partial u}{\partial z} = -\frac{\partial b}{\partial y}$; $f\frac{\partial v}{\partial z} = \frac{\partial b}{\partial x}$.

Thus we see that $\frac{\partial \psi}{\partial x}$, $\frac{\partial \psi}{\partial y}$ have a meaning as a measure of geostrophic current but also $\frac{\partial \psi}{\partial z}$ has a meaning as a measure of buoyancy. It is this double nature of ψ which makes it very convenient in quasi-geostrophic theory.

We now go on to determine the equations that govern the evolution of ψ.

2.1 The quasi-geostrophic potential vorticity equation

We enter the geostrophic approximation in to Eq.(1) and write:

$$\frac{D_g}{Dt}\left(f + [curl\ \mathbf{v}_g]_z\right) = f\frac{\partial w}{\partial z} \tag{10}$$

where

$$[curl\ \mathbf{v}_g]_z = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \nabla_h^2 \psi$$

and

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_g.\nabla \tag{11}$$

Note that advection of vorticity in eq.(10) is by the horizontal geostrophic flow (we discuss conditions for the neglect of vertical advection below).

Now from the thermodynamic equation we write:

$$\frac{D_g b'}{Dt} + N^2 w = 0 \tag{12}$$

where we have assumed that $\frac{b'}{b_o} << 1$ and supposed that

$$N^2 = \frac{\partial b_o}{\partial z}, \tag{13}$$

the static stability, only depends on z. For the moment we have set buoyancy sources to zero on the rhs of eq(12).

We can eliminate w between (12) and (10) to give

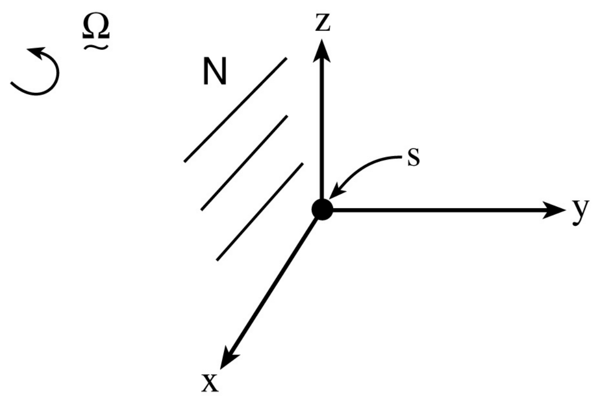

Figure 1: A point vortex of strength 's' in a rotating ($f = 2\Omega$), stratified (N) fluid.

$$\frac{D_g q}{Dt} = 0 \tag{14}$$

where (using equation 4 and 9):

$$q = \nabla_h^2 \psi + f + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{15}$$

is the 'quasi-geostrophic potential vorticity' (qgpv) and eq(14) is the qgpv equation. Note that the qgpv is conserved in *horizontal motion*, in the absence of sources and sinks.

The factor $\frac{f^2}{N^2}$ is a pure number which can obviously be interpreted as a scale factor; $\frac{N}{f}$ is of order 30 in the ocean, 100 in the atmosphere.

So, in analogy with our equation for 2-d barotropic flow, we may write our equation for 3 -dimensional baroclinic flow as:

$$\left(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\widetilde{\nabla}_3^2 \psi = 0 \tag{16}$$

$$\widetilde{\nabla}_3^2 = \nabla_h^2 + \frac{\partial^2}{\partial \widetilde{z}^2} \tag{17}$$

is a modified 3-d Laplacian operator in which (assuming f and N are constant):

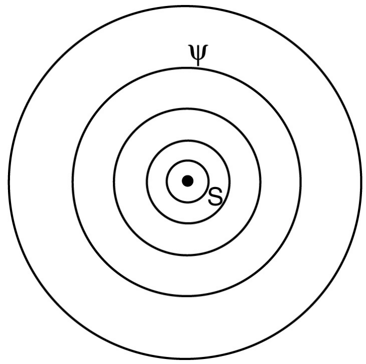

Figure 2: A point vortex of strength 's' induces a circulation Ψ given by eq., if f, N = constant, eq.(21).

$$\frac{\partial}{\partial \widetilde{z}} = \frac{f}{N}\frac{\partial}{\partial z}$$

thus stretching z, the geometric height by, typically, a large factor $\frac{N}{f} \times z = \widetilde{z}$.

3 The invertibility principle

We note that q and Ψ are related to one another through an Elliptic operator eq(17):

$$q = \widetilde{\nabla}_3^2 \psi \tag{18}$$

If the distribution of q is known, then, given suitable boundary conditions, it may be INVERTED for ψ - this is the INVERTIBILITY PRINCIPLE - it stems from the Elliptic nature of the $\widetilde{\nabla}_3^2$ operator:

$$\underbrace{\widetilde{\nabla}_3^{-2} q \Rightarrow \psi}_{inversion} \tag{19}$$

This elliptic property can be traced back to the assumption of balance - that the flow is geostrophically balanced in the horizontal and hydrostatically balanced in the vertical.

One can draw a direct and complete analogy between the electric potential induced by a distribution of charge, and the streamfunction induced by a potential vorticity field in quasi-geostrophic theory. Moreover, one can employ and exploit all the tricks that are commonly used in electrostatics (image charges, boundary sheets of charge etc), to problems of large-scale quasi-geostrophic dynamics.

Once ψ is known then horizontal differentiation yields the geostrophic currents, eq.(6), and vertical differentiation yields the buoyancy perturbation, eq.(9).

For example, let us compute the streamfunction induced by a point source of potential vorticity of strength 's', in a stratified, rotating fluid ($\frac{f}{N}$ assumed constant) of infinite extent – no boundaries. Of course we are computing the 'Green's function'

$$\widetilde{\nabla}_3^2\psi = s\delta(0) \tag{20}$$

In spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

and so

$$\psi = \text{constant} \times \frac{1}{r}.$$

where $r^2 = x^2 + y^2 + \widetilde{z}^2 = x^2 + y^2 + \frac{N^2}{f^2}z^2$ and N/f assumed constant.

One can deduce the 'constant' by integrating eq.(20) over a sphere; the rhs contributes 's', the lhs $\nabla\psi$ over the surface of a sphere: we find that

$$\psi = -\frac{s}{4\pi}\frac{1}{\left(x^2 + y^2 + \frac{N^2}{f^2}z^2\right)^{\frac{1}{2}}} \tag{21}$$

The form of this Green's function, although simple, is highly instructive. We note that:

- if $\frac{N}{f} >> 1$ (very strongly stratified), then the solution decays away rapidly in the vertical. But if $\frac{N}{f} << 1$, the influence of the q anomaly can be felt over great depths

- if $s > 0$, then we observe a cyclonic vortex with a 'pinching together' of the b surfaces: see fig.3.

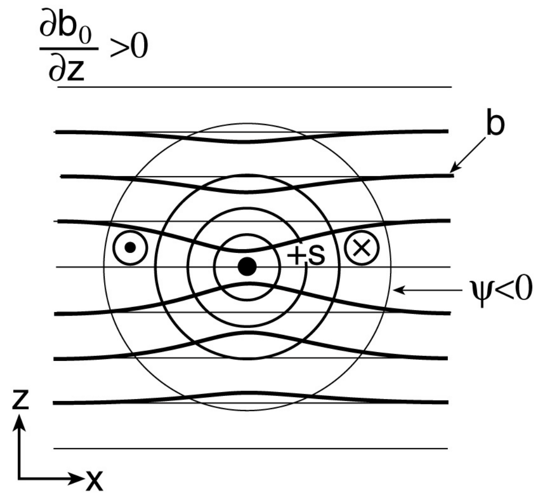

Figure 3: Cyclone

- if $s < 0$, then we observe an anticyclonic vortex with a 'pushing apart' of the b surfaces: see fig.4.

3.1 Boundary conditions

A distribution of q can only be inverted for ψ - Eq.19 - given appropriate boundary conditions. At lateral boundaries Dirichlet conditions on ψ are often appropriate i.e. ψ, rather than gradients of ψ, is typically specified at lateral boundaries. However, at upper and lower boundaries the buoyancy distribution is specified, providing, through (9), inhomogeneous Neumann conditions on ψ i.e. $\frac{\partial \psi}{\partial z}$ specified at upper and lower boundaries. A computational and (depending on one's point of view) conceptual simplification arises if, as we are liberty to do, the inhomogeneous Neumann condition on ψ are replaced by homogeneous ones thus.

Let us define a potential vorticity $\widehat{q}$ which is exactly equal to q in the interior of the fluid, except adjacent to the upper and lower boundaries. Next to these boundaries, just inside the fluid, we slip delta-function sheets of potential vorticity, q_{upper} and q_{lower} whose strengths are chosen to represent the buoyancy distributions on the boundary (see fig.5 below).

Thus:

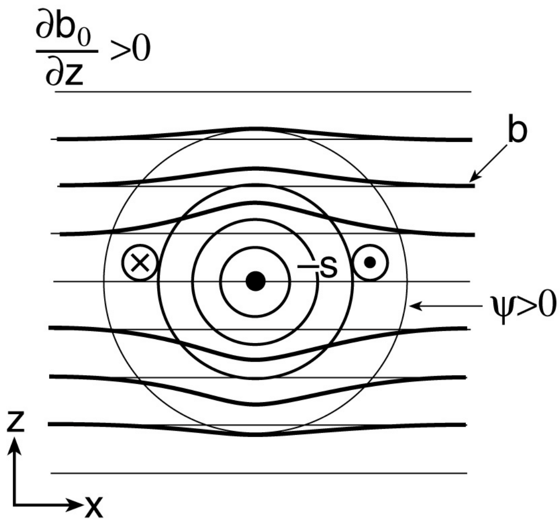

Figure 4: Anticyclone

$$\widehat{q} = q + q_{upper} + q_{lower} = \widetilde{\nabla}_3^2 \psi; \; \frac{\partial \psi}{\partial z} = 0 \text{ at top and bottom} \tag{22}$$

i.e. $\widehat{q}$ is inverted with homogeneous Neumann boundary conditions. The q_{upper}, q_{lower} are delta-function sheets of potential vorticity chosen to represent the buoyancy distributions on the upper and lower boundaries as follows:

$$q_{upper} = -\frac{f}{N^2} b\delta\,(top)\,; q_{lower} = \frac{f}{N^2} b\delta\,(bottom) \tag{23}$$

The equivalence of the two can be seen by integrating:

1. $\widehat{q}$, eq.(22) with homogeneous boundary conditions, over the body of the fluid, and invoking (23) and

2. q, eq.(18), over the body of the fluid with the inhomogeneous boundary conditions eq(9).

In both cases the result is the same.
From eq(23), we thus see that:

- a warm (cold) lower boundary is equivalent to a sheet of positive (negative) q adjacent to the boundary.

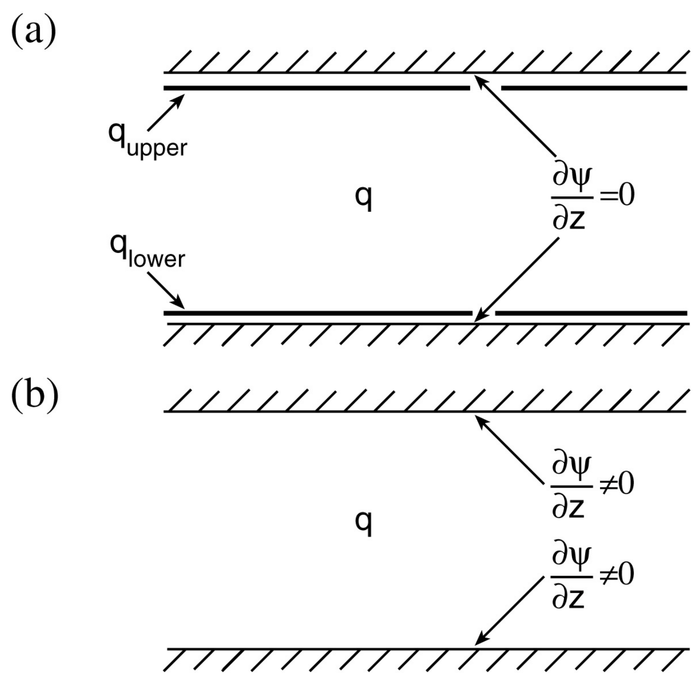

Figure 5: In (a) $\widehat{q} = q + q_{upper} + q_{lower}$ is inverted with homogeneous boundary conditions $\frac{\partial\psi}{\partial z} = 0$ where q_{upper}, q_{lower} are PV $\delta-$function sheets adjacent to the boundary but just interior to the fluid. In (b) q is inverted with inhomogeneous boundary conditions $\frac{\partial\psi}{\partial z} \neq 0$. If the sheets are chosen to have strengths given by eq(23), the interior solutions are identical.

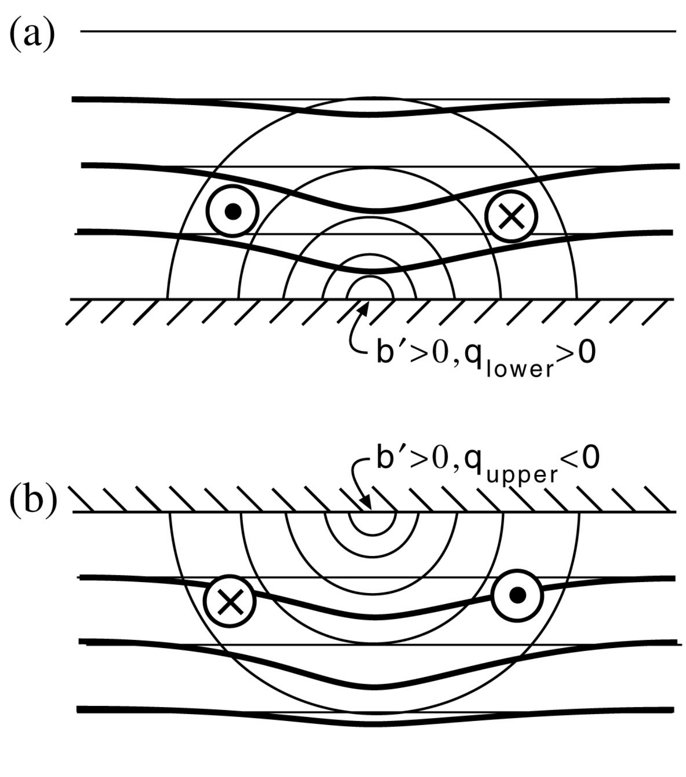

Figure 6: (a) warm lower boundary induces cyclonic circulation aloft (b) warm upper boundary induces anticyclonic circulation below.

- a warm (cold) upper boundary is equivalent to a sheet of negative (positive) q adjacent to the boundary.

The use of potential vorticity sheets in this way to represent boundary buoyancy distributions - often called ‘Bretherton’ PVsheets, enables one to think about surface buoyancy distributions and interior q distributions in one conceptual framework, often leading to deeper understanding. They will also help us interpret layer/level quasi-geostrophic models discussed later.

4 The momentum equations in the quasi-geostrophic approximation

What is the momentum equation that is consistent with the vorticity equation, Eq(1). If the Rossby number is small the horizontal momentum equations reduce to a statement of geostrophic balance. But a better approximation to the true horizontal velocity is obtained by substituting the geostrophic current in to the acceleration terms (which are already Rossby number smaller than the Coriolis terms). The resulting momentum equation may be written:

$$\frac{D_g \mathbf{v}_g}{Dt} + f\mathbf{k} \times \mathbf{v}_h + \frac{1}{\overline{\rho}} \nabla p = 0 \tag{24}$$

where $\overline{\rho}$ is a constant density and $\frac{D_g}{Dt}$ is given by eq(11) and $\mathbf{v}_h = \mathbf{v}_g + \mathbf{v}_{ag}$.

Note that in the quasi-geostrophic momentum equations, advection by the vertical velocity is neglected (see below).

Taking the $k \bullet \nabla \times$ of (24) and noting that :

$$\nabla_h . \mathbf{v}_{ag} + \frac{\partial w}{\partial z} = 0$$

we obtain eq.(10), the quasi-geostrophic vorticity equation.

5 Conditions for the validity of the quasi-geostrophic equations

Let k be a typical horizontal wavenumber, $2\pi/k$ is then a typical wavelength; and let

H be a scale of vertical variation, so that:

$$\left|\nabla_h^2\right| \sim k^2; \quad \frac{\partial}{\partial z} \sim \frac{1}{H}$$

We suppose that:

V, W are typical horizontal and vertical currents respectively

Δb – a measure of the horizontal variation in b

$$N^2 \text{ is the static stability}$$

Now we can estimate the magnitude of the above quantities from data, but they are not independent of one-another. We will thus make use of dynamical and kinematic constraints to link them together.

Since $\frac{D}{Dt_h} \sim kV$, the criterion for geostrophic balance of the horizontal current is that:

$$R_o = \frac{kV}{f} << 1 \tag{25}$$

assuming that the vertical advection of momentum is negligible.

The criterion for the neglect of vertical advection (which is equivalent to the criterion for the neglect of $\frac{\partial w}{\partial z}$ relative to $\frac{\partial u}{\partial x}$or $\frac{\partial v}{\partial y}$ in the continuity equation) is:

$$\frac{w\frac{\partial}{\partial z}}{\mathbf{v}.\nabla_h} \sim \frac{\partial w/\partial z}{\partial u/\partial x} \sim \frac{W/H}{kV} << 1 \tag{26}$$

So what are the appropriate scales to choose for W?

From the buoyancy equation (12):

$$\frac{D}{Dt_h}b \sim N^2 w \Longrightarrow W \sim \frac{kV\Delta b}{N^2} \tag{27}$$

and eq(26) becomes:

$$\frac{w\frac{\partial}{\partial z}}{\mathbf{v}.\nabla_h} \sim \frac{W}{kVH} \sim \frac{\Delta b}{N^2 H} << 1 \tag{28}$$

Eq.(28) says that horizontal variations in b must be much smaller than vertical variations in b for vertical advection to be neglected.

Thus the quasi-geostrophic equations assume two things:

$$R_o = \frac{kV}{f} << 1$$

and

$$\frac{\Delta b}{N^2 H} << 1$$

We now want to show that conditions (28) both flow essentially from the largeness of the Richardson number.

To restrict the variety of possible systems further we make use of the vorticity equation. If the governing vorticity equation is (1). Thus:

$$\frac{D}{Dt}\left[curl\ \mathbf{v}\right]_z \ \sim k^2V^2 \text{ and } f\frac{\partial w}{\partial z} \ \sim \frac{fW}{H}$$

For a W scale we use, from eq(27):

$$W \ \sim \frac{kV\Delta b}{N^2} \sim \ \frac{fV^2}{N^2H}$$

where we have exploited the observed smallness of R_o on the large-scale and hydrostatic balance, and scaled Δb as suggested by thermal wind balance:

$$\Delta b \sim \ \frac{fV}{kH} \tag{29}$$

The terms in the vorticity equation are thus of comparable magnitude when:

$$k^2H^2 \sim \frac{f^2}{N^2}$$

a balance which can be expressed in terms of R_i and R_o thus:

$$R_iR_o^2 \ \sim 1 \tag{30}$$

where

$$R_i = \frac{N^2H^2}{V^2} \tag{31}$$

is the Richardson number. Thus if $R_i >> 1$, then according to eq(30):

$$R_o \ \sim \frac{1}{\sqrt{R_i}} << 1$$

which is (25).

Furthermore, using (29),

$$\frac{\Delta b}{N^2H} = \frac{1}{R_iR_o} = \frac{1}{\sqrt{R_i}} << 1$$

which is criterion (28).

Thus we see that:

the approximations inherent in the quasi-geostrophic equations flow essentially from the largeness of the Richardson number.

The Richardson number takes on a value of ~ 100 in large-scale atmospheric flows and $\sim 10^4$ in the ocean on the large scale.

Finally, note that (30) implies that:

$$R_i R_o^2 = \frac{N^2 H^2}{V^2} \frac{k^2 V^2}{f^2} \sim 1$$

and so

$$k \sim \frac{f}{NH} = \frac{1}{L_\rho}$$

the inverse of the Rossby radius of deformation. The vorticity balance (1) thus holds on the deformation radius, L_ρ, typically 1000 km in the atmosphere and 30 km in the ocean.