Nonlinear waves

Theories of nonlinear waves have been developed for many systems; we shall treat two-dimensional internal gravity wave dynamics here.

$$\frac{\partial}{\partial t}\mathbf{u} + \boldsymbol{\zeta} \times \mathbf{u} = -\nabla(p + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}) + b\hat{\mathbf{z}}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b + wN^2 = 0$$

We assume that there is no variation in y and write the horizontal and vertical velocities in terms of a streamfunction

$$u = -\frac{\partial}{\partial z}\psi \quad , \quad w = \frac{\partial}{\partial x}\psi$$

Forming a horizontal vorticity equation ($\zeta = -\boldsymbol{\zeta}\cdot\hat{\mathbf{y}} = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \nabla^2\psi$) gives the set

$$\boxed{\begin{aligned} \frac{\partial}{\partial t}\zeta + J(\psi, \zeta) &= \frac{\partial b}{\partial x} \\ \frac{\partial}{\partial t}b + J(\psi, b + \int^z N^2(z')dz') &= 0 \end{aligned}}$$

We shall use $B(z) \equiv \int^z N^2$ for the basic stratification.

Steadily propagating waves

We begin by looking for steadily propagating nonlinear solutions. If we replace $\frac{\partial}{\partial t}$ by $-c\frac{\partial}{\partial x}$, we have

$$J(\psi + cz, \zeta) = \frac{\partial b}{\partial x}$$
$$J(\psi + cz, b + B(z)) = 0$$

The last equation implies

$$b + B(z) = \mathcal{B}(z + \frac{\psi}{c})$$

which has a (by no means unique) solution

$$b = B(z + \psi/c) - B(z)$$

For this solution, ψ and b go to zero smoothly and are related in the linear limit by

$$b \simeq B'(z)\frac{\psi}{c} = N^2(z)\frac{\psi}{c}$$

which is exactly what one gets for linear waves.

The vorticity equation now gives

$$\begin{aligned} J(\psi + cz, \nabla^2\psi) &= B'(z + \psi/c)\frac{1}{c}\frac{\partial\psi}{\partial x} \\ &= N^2(z + \psi/c)\frac{1}{c}J(\psi + cz, z) \\ &= J\left(\psi + cz, \frac{z}{c}N^2(z + \psi/c)\right) \end{aligned}$$

Therefore we have

$$\nabla^2\psi - \frac{z}{c}N^2(z + \psi/c) = \mathcal{Z}(z + \frac{\psi}{c})$$

or (taking the same kind of extension of the linear solution)

$$\nabla^2\psi = \frac{z}{c}N^2(z + \psi/c) - \frac{\psi + cz}{c^2}N^2(z + \psi/c)$$

Our final equation is

$$\boxed{\nabla^2\psi = -\frac{\psi}{c^2}N^2(z + \psi/c)}$$

When $N^2(z)$ is constant — uniform stratification — the wave equation is linear with solutions like

$$\psi = A\cos(kx)\sin(mz) \quad , \quad c^2 = N^2/(k^2 + m^2)$$

with $m = \pi/H$ for a domain height H bounded by horizontal surfaces. This is the standard internal gravity wave dispersion relationship. For this case, the linear waves are a solution to the nonlinear equations.

For a non-uniform stratification, if the wave amplitude is small enough, we can separate variables

$$\psi = \epsilon\Psi(x)F(z)$$

with

$$\frac{\partial^2}{\partial x^2}\Psi = -\gamma^2\Psi \quad , \quad \frac{\partial^2}{\partial z^2}F + \frac{N^2(z)}{c^2}F = -\gamma^2 F$$

The vertical structure equation gives a relationship between c and γ^2 while the horizontal equation just gives sinusoidal disturbances with wavenumber $k = \gamma$. Note that there will be a gravest long wave mode corresponding to the value $c = c_{lw}$ where γ^2 goes through zero.

$$\frac{\partial^2}{\partial z^2}F_{lw} + \frac{N^2(z)}{c_{lw}^2}F_{lw} = 0$$

Stokes' expansion

For small, but finite, amplitude waves, we can Taylor-expand the equation to get

$$\nabla^2\psi = -\frac{N^2}{c^2}\psi - \frac{1}{c^3}\frac{\partial N^2}{\partial z}\psi^2 - \frac{1}{2c^4}\frac{\partial^2 N^2}{\partial z^2}\psi^3 ...$$

This is like the oscillator equation with a nonlinear restoring force.

We expand

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 ...$$

$$c^{-2} = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2$$

and use the notation $S = N^2$ to get a sequence of problems

$$[\nabla^2 + \lambda_0 S]\psi_1 = 0$$

$$[\nabla^2 + \lambda_0 S]\psi_2 = -S\lambda_1\psi_1 - S'\lambda_0^{3/2}\psi_1^2$$

$$[\nabla^2 + \lambda_0 S]\psi_3 = -S\lambda_2\psi_1 - S\lambda_1\psi_2 - \frac{3}{2}S'\lambda_0^{1/2}\lambda_1\psi_1^2 - 2S'\lambda_0^{3/2}\psi_1\psi_2 - \frac{1}{2}S''\lambda_0^2\psi_1^3$$

The first equation gives

$$\psi_1 = \cos(kx)F(z) \quad , \quad \frac{\partial^2}{\partial z^2}F - k^2 F = -\lambda_0 S F$$

We apply solvability conditions to the second and third equations: if

$$[\nabla^2 + \lambda_0 S]\psi_n = R_n(x,z)$$

then

$$\frac{1}{H}\int_0^H dz \frac{k}{2\pi}\int_0^{2\pi/k} dx\, F(z)\cos(kx)R_n(x,z) = 0$$

or

$$\langle \psi_1 R_n(x,z) \rangle = 0$$

For the second equation, this implies $\lambda_1 = 0$ and therefore

$$\psi_2 = -\frac{1}{2}\lambda_0^{3/2}F_{20}(z) - \frac{1}{2}\lambda_0^{3/2}\cos(2kx)F_{22}(z)$$

with

$$[\frac{\partial^2}{\partial z^2} - n^2k^2 + \lambda_0 S]F_{2n} = S'F^2$$

Thus we see asymmetries in the shape of the wave. If we multiply the F_{22} equation above by F and integrate, we see that

$$-3k^2 \int F F_{22} = \int S' F^2$$

Thus for $S' > 0$, F_{22} will be negative and the $\cos(2kx)$ term will have a positive coefficient: streamfunction highs will be sharper and lows flatter. Also, the isotherms will have flat crests and sharp troughs. In addition, the ψ_2 field has a net mean flow contribution. We can get rid of this term if we choose by going back to the equation for the vorticity and writing it as

$$\nabla^2 \psi = \frac{z}{c} N^2(z + \psi/c) - \frac{\psi + cz}{c^2} N^2(z + \psi/c) + \mathcal{Z}_1(z + \psi/c)$$

with the last term order ϵ. The sequence of equations now has

$$\begin{aligned}
[\nabla^2 + \lambda_0 S]\psi_2 &= -S\lambda_1\psi_1 - S'\lambda_0^{3/2}\psi_1^2 + \mathcal{Z}_1(z) \\
[\nabla^2 + \lambda_0 S]\psi_3 &= -S\lambda_2\psi_1 - S\lambda_1\psi_2 - \frac{3}{2}S'\lambda_0^{1/2}\lambda_1\psi_1^2 \\
&\quad -2S'\lambda_0^{3/2}\psi_1\psi_2 - \frac{1}{2}S''\lambda_0^2\psi_1^3 + \mathcal{Z}_1'\lambda_0^{1/2}\psi_1 + \mathcal{Z}_2(z)
\end{aligned}$$

This kind of ambiguity is characteristic of nonlinear waves — there can be many different solutions depending on the choice of functionals. Without some further specification, there will be aspects which are undetermined. One such consideration is elimination of the mean zonal flow at second order; another might be requiring that the volume between two isotherms be the same as in the case when the buoyancy is just $B(z)$. Since the streamlines are isotherms, this comes back to the same requirement: we want $\int dx\, \psi = 0$. Therefore we choose

$$\mathcal{Z}_1 = \frac{1}{2}\lambda_0^2 S' F^2$$

and ψ_2 is just

$$\psi_2 = -\frac{1}{2}\lambda_0^{3/2}\cos(2kx) F_{22}(z)$$

For the third equation, we have

$$\lambda_2\langle S\psi_1^2\rangle + 2\lambda_0^{3/2}\langle \psi_1^2\psi_2 S'\rangle + \frac{1}{2}\lambda_0^2\langle\psi_1^4 S''\rangle - \lambda_0^{1/2}\langle \mathcal{Z}_1'\psi_1^2\rangle = 0$$

Substituting for the last term, doing the averages of the cosines, and integrating by parts a few times gives

$$\lambda_2\langle F^2 S\rangle + \frac{3}{2}\lambda_0^{3/2}\langle F^2 F_{22} S'\rangle + \frac{1}{8}\lambda_0^2\langle F^4 S''\rangle = 0$$

This determines the corrections to the phase speed:

$$c = \lambda^{-1/2} \quad \Rightarrow \quad c \simeq \lambda_0^{-1/2} - \frac{1}{2}\lambda_0^{-3/2}\lambda_2$$

For the case $S' > 0$, $S'' = 0$ (buoyancy increasing quadratically with height), we have

$$c_2 = \frac{3}{4}\frac{\langle F^2 F_{22} S'\rangle}{\langle F^2 S\rangle} < 0$$

The wave moves more slowly, which makes sense since it extends downward more into the weakly stratified zone.

Solitary waves

We can also find long wave solutions limit which are spatially isolated. Since such solutions decay to zero, the original choices for $\mathcal{B}(Z) = B(Z)$ and $\mathcal{Z}(Z) = -ZN^2(Z)/c$ are necessary; the values are set along a streamline by the fields flowing in from $+\infty$ where the streamlines correspond to the background state. We now analyze

$$\nabla^2\psi = -\frac{\psi}{c^2}N^2(z + \psi/c)$$

under the assumption that $u/c \sim \epsilon$ and $L \sim \epsilon^{-1/2}H$ so that $\frac{\partial^2}{\partial x^2} \sim \epsilon$ compared to $\frac{\partial^2}{\partial z^2}$. We then have the simpler sequence

$$[\frac{\partial^2}{\partial z^2} + \lambda_0 S]\psi_1 = 0$$
$$[\frac{\partial^2}{\partial z^2} + \lambda_0 S]\psi_2 = -\frac{\partial^2}{\partial x^2}\psi_1 - \lambda_1 S\psi_1 - \lambda_0^{3/2}S'\psi_1^2$$

The lowest order equation now constrains only the vertical structure to be the long wave structure function $\psi_1 = \Psi(x)F_{lw}$ and the eigenvalue to be $\lambda_0 = c_{lw}^{-2}$. The solvability condition for the next order problem is

$$\langle F_{lw}^2\rangle\frac{\partial^2}{\partial x^2}\Psi + \lambda_1\langle F_{lw}^2 S\rangle\Psi + \lambda_0^{3/2}\langle F_{lw}^3 S'\rangle\Psi^2 = 0$$

To have decaying solutions at infinity, we must have $\lambda_1 < 0$; since $c = c_{lw} - \frac{1}{2}\lambda_1 c_{lw}^3$, this implies that solitary waves move **faster** than the longest linear wave. This makes sense from the linear dispersion relation: if

$$c = \frac{N}{\sqrt{k^2 + m^2}} \simeq \frac{N}{m}(1 - \frac{1}{2}\frac{k^2}{m^2})$$

and $k^2 < 0$ representing exponential decay (at the front) or growth (at the back), the phase speed will be increased rather than decreased from the long wave value N/m. In addition, to turn the rising wave on the left to a decaying wave on the right, we must have $\frac{\partial^2}{\partial x^2}\Psi \propto -\Psi$ in the center; this will occur when Ψ is positive. Solitary waves are single-signed; for $S' > 0$, we find only waves with the isotherms depressed.

We can write an explicit solution to

$$A\frac{\partial^2}{\partial x^2}\Psi - c_1\Psi + B\Psi^2 = 0$$

with

$$A = \frac{c_0^3\langle F_{lw}^2\rangle}{2\langle F_{lw}^2 S\rangle} \quad , \quad B = \frac{\langle F_{lw}^3 S'\rangle}{2\langle F_{lw}^2 S\rangle}$$

in the form of the hyperbolic secant squared solitary wave

$$\Psi = \frac{6A}{BL^2}\operatorname{sech}^2(x/L) \quad , \quad c_1 = \frac{4A}{L^2}$$

Kortweg-deVries equation

We can examine the time-dependent behavior of nonlinear disturbances in the limit where dispersion and nonlinearity are weak, but comparable, effects. We again use multiple scale expansions with ϵ as a "marker"

$$\left(\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T}\right)\zeta + \epsilon J(\psi,\zeta) - \frac{\partial}{\partial x}b = 0$$
$$\left(\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T}\right)b + \epsilon J(\psi,b) + S\frac{\partial}{\partial x}\psi = 0$$
$$\zeta = \frac{\partial^2}{\partial z^2}\psi + \epsilon\frac{\partial^2}{\partial x^2}\psi$$

To think of these as nondimensional, scale $x \sim L$, $z \sim H$, $\psi \sim UH$, $\zeta \sim U/H$, $c \sim N_0 H$, $t \sim L/N_0 H$, and $b \sim N_0 U$. Then S is the nondimensional function N^2/N_0^2 and $\epsilon = U/N_0 H = H^2/L^2$. The last equality expresses the relationship between U and L so that nonlinearity and dispersion are similar in strength.

The expansion gives

$$\frac{\partial}{\partial t}\zeta_1 - \frac{\partial}{\partial x}b_1 = 0$$
$$\frac{\partial}{\partial t}b_1 + S\frac{\partial}{\partial x}\psi_1 = 0$$
$$\zeta_1 = \frac{\partial^2}{\partial z^2}\psi_1$$

and

$$\frac{\partial}{\partial t}\zeta_2 - \frac{\partial}{\partial x}b_2 + \frac{\partial}{\partial T}\zeta_1 + J(\psi_1,\zeta_1) = 0$$
$$\frac{\partial}{\partial t}b_2 + S\frac{\partial}{\partial x}\psi_2 + \frac{\partial}{\partial T}b_1 + J(\psi_1,b_1) = 0$$
$$\zeta_2 = \frac{\partial^2}{\partial z^2}\psi_2 + \frac{\partial^2}{\partial x^2}\psi_1$$

At lowest order, we have disturbances moving at the long wave speed c_{lw} with vertical structure F_{lw}:

$$\psi_1 = \Psi(x - c_0 t, T)F(z) \quad , \quad \zeta_1 = -\frac{SF}{c_0^2}\Psi \quad , \quad b_1 = \frac{SF}{c_0}\Psi$$

(we'll drop the lw subscript henceforth). At the next order, we have

$$\frac{\partial}{\partial t}\frac{\partial^2 \psi_2}{\partial z^2} - \frac{\partial}{\partial x}b_2 = c_0 F\frac{\partial^3}{\partial x^3}\Psi + \frac{SF}{c_0^2}\frac{\partial}{\partial T}\Psi + \Psi\frac{\partial \Psi}{\partial x}\frac{F^2 S'}{c_0^2}$$

$$\frac{\partial}{\partial t}b_2 + S\frac{\partial}{\partial x}\psi_2 = -\frac{SF}{c_0}\frac{\partial}{\partial T}\Psi - \Psi\frac{\partial \Psi}{\partial x}\frac{F^2 S'}{c_0}$$

We multiply both equations by F and average to find

$$-\frac{1}{c_0^2}\frac{\partial}{\partial t}\langle SF\psi_2\rangle - \frac{\partial}{\partial x}\langle Fb_2\rangle = c_0\langle F^2\rangle\frac{\partial^3 \Psi}{\partial x^3} + \frac{\langle F^2 S\rangle}{c_0^2}\frac{\partial \Psi}{\partial T} + \frac{\langle F^3 S'\rangle}{c_0^2}\Psi\frac{\partial \Psi}{\partial x}$$

$$\frac{\partial}{\partial t}\langle Fb_2\rangle + \frac{\partial}{\partial x}\langle SF\psi_2\rangle = -\frac{\langle SF^2\rangle}{c_0}\frac{\partial \Psi}{\partial T} - \frac{\langle F^3 S'\rangle}{c_0}\Psi\frac{\partial \psi}{\partial x}$$

Multiplying the first equation by $-c_0$ and adding gives

$$\left(\frac{\partial}{\partial t} + c_0\frac{\partial}{\partial x}\right)\left(\frac{\langle SF\psi_2\rangle}{c_0} + \langle Fb_2\rangle\right) = -\frac{2\langle F^2 S\rangle}{c_0}\frac{\partial \Psi}{\partial T} - c_0^2\langle F^2\rangle\frac{\partial^3 \Psi}{\partial x^3} - \frac{2\langle F^3 S'\rangle}{c_0}\Psi\frac{\partial \Psi}{\partial x}$$

The solvability condition gives the KdV equation:

$$\frac{2\langle F^2 S\rangle}{c_0}\frac{\partial \Psi}{\partial T} + c_0^2\langle F^2\rangle\frac{\partial^3 \Psi}{\partial x^3} + \frac{2\langle F^3 S'\rangle}{c_0}\Psi\frac{\partial \Psi}{\partial x} = 0$$

or

$$\boxed{\frac{\partial \Psi}{\partial T} + A\frac{\partial^3 \Psi}{\partial x^3} + B\frac{\partial \Psi^2}{\partial x} = 0}$$

with A and B defined as before.

Characteristics of the KdV eqn: The KdV eqn. has solitary wave solutions

$$\Psi = \frac{6A}{BL^2}\text{sech}^2\left(\frac{x - c_1 T}{L^2}\right) \quad , \quad c_1 = \frac{4A}{L^2}$$

as we found before. Note that if $\mathcal{A} = 6A/BL^2$, the speed is proportional to the amplitude $c_1 = 2B\mathcal{A}/3$ and the width varies inversely $L = (6A/B)^{1/2}\mathcal{A}^{-1/2}$.

We can set up two waves to "collide" and find that they pass through (with a phase shift). `Demos, Page 7: Collision` *<two waves>* *<summary>*

SOLITONS: An initial disturbance breaks up into a train of solitary waves ranked in amplitude plus a weak dispersive field. This is **not** a numerical artifact or accident associated with a particular shape of initial conditions; "inverse scattering theory" gives an exact solution to the KdV problem treating the initial condition as a potential. The solitons correspond to the bound states of the potential and the dispersive waves map to the scattering modes.

`Demos, Page 7: kdv solns` *<dispersive amp=0.1>* *<one soliton amp=1>* *<two amp=3>* *<three amp=7>*

INTERNAL WAVES: As a warning, though, we cannot be sure that similar "particle-like" properties apply when the waves have different vertical modes or propagate in different directions.

Group solitary waves

Nonlinearity can also act on wave groups,where it can create solitary wave packets and lead to modulational instability — the breakdown of a regular wave into groups. The math here is more complicated, unfortunately. We again expand the wave into

$$\psi = \epsilon\psi_1(x, z, t, X, T, \tau) + \epsilon^2\psi_2 + \epsilon^3\psi_3 + ...$$

etc., with $X = \epsilon x$, $T = \epsilon t$, $\tau = \epsilon^2 t$. The sequence of problems becomes

$$\begin{aligned} \frac{\partial}{\partial t}\zeta_1 - \frac{\partial}{\partial x}b_1 &= 0 \\ \frac{\partial}{\partial t}b_1 + N^2\frac{\partial}{\partial x}\psi_1 &= 0 \\ \nabla^2\psi_1 &= \zeta_1 \end{aligned}$$

with $N^2 = const$ for simplicity and to avoid the Stokes' wave kind of nonlinearity. These give the basic wave solution

$$\begin{aligned} \psi_1 &= A(X, T, \tau)\phi + c.c. \\ \zeta_1 &= -K^2A(X, T, \tau)\phi + c.c. \\ b_1 &= NKA(X, T, \tau)\phi + c.c. \end{aligned}$$

with $\phi = \exp(\imath kx - \imath c_0 kt)\sin(mz)$, $K = \sqrt{k^2 + m^2}$, and $c_0 = N/K$.

At order ϵ^2, we have

$$\begin{aligned} \frac{\partial}{\partial t}\zeta_2 + \frac{\partial}{\partial T}\zeta_1 - \frac{\partial}{\partial x}b_2 - \frac{\partial}{\partial X}b_1 + J(\psi_1, \zeta_1) &= 0 \\ \frac{\partial}{\partial t}b_2 + \frac{\partial}{\partial T}b_1 + N^2\frac{\partial}{\partial x}\psi_2 + N^2\frac{\partial}{\partial X}\psi_1 + J(\psi_1, b_1) &= 0 \\ \nabla^2\psi_2 + 2\frac{\partial^2}{\partial x\partial X}\psi_1 &= \zeta_2 \end{aligned}$$

We note that $J(\psi_1, \zeta_1) = 0$ and $J(\psi_1, b_1) = 0$. The solvability condition comes from looking at the part of the second order solution which is proportional to ϕ; taking the second equation minus c_0 times the first gives

$$(c_0^2\nabla^2 + N^2)\frac{\partial \psi_2}{\partial x} = -2NK\frac{\partial}{\partial T}\psi_1 - 2N^2(1-\frac{k^2}{K^2})\frac{\partial}{\partial X}\psi_1$$

Therefore we require

$$\frac{\partial}{\partial T}\psi_1 + c_g\frac{\partial}{\partial X}\psi_1 = 0 \quad , \quad c_g = \frac{N}{K}(1-\frac{k^2}{K^2})$$

and

$$A(X,T,\tau) = A(X - c_gT, \tau)$$

In addition to the forced solution, we can add a free solution at order ϵ^2 so that

$$\psi_2 = P(X - c_gT, T, \tau)f^{(\psi)}(z)$$

$$b_2 = B(X - c_gT, T, \tau)f^{(b)}(z) - \frac{N}{K}\frac{\partial^2}{\partial x \partial X}\psi_1$$

At order ϵ^3 (sigh!) we find

$$\frac{\partial}{\partial t}\zeta_3 + \frac{\partial}{\partial T}\zeta_2 + \frac{\partial}{\partial \tau}\zeta_1 - \frac{\partial}{\partial x}b_3 - \frac{\partial}{\partial X}b_2 + J(\psi_1, \zeta_2) + J(\psi_2, \zeta_1) + \frac{\partial(\psi_1, \zeta_1)}{\partial(X, y)} = 0$$

$$\frac{\partial}{\partial t}b_3 + \frac{\partial}{\partial T}b_2 + \frac{\partial}{\partial \tau}b_1 + N^2\frac{\partial}{\partial x}\psi_3 + N^2\frac{\partial}{\partial X}\psi_2 + J(\psi_1, b_2) + J(\psi_2, b_1) + \frac{\partial(\psi_1, b_1)}{\partial(X, y)} = 0$$

$$\nabla^2\psi_3 + 2\frac{\partial^2}{\partial x \partial X}\psi_2 + \frac{\partial^2}{\partial X^2}\psi_1 = \zeta_3$$

We can collect the terms which are independent of $x - c_0t$

$$(\frac{\partial P}{\partial T} - c_g\frac{\partial P}{\partial X})\frac{\partial^2 f^{(\psi)}}{\partial z^2} - \frac{\partial B}{\partial X}f^{(b)} + 2J(\psi_1, \frac{\partial^2}{\partial x \partial X}\psi_1) = 0$$

$$(\frac{\partial B}{\partial T} - c_g\frac{\partial B}{\partial X})f^{(b)} + N^2\frac{\partial P}{\partial X}f^{(\psi)} - \frac{N}{K}J(\psi_1, \frac{\partial^2}{\partial x \partial X}\psi_1) = 0$$

We use

$$J(\psi_1, \frac{\partial^2}{\partial x \partial X}\psi_1) = k^2 m\frac{\partial |A|^2}{\partial X}\sin 2mz$$

to find $f^{(\psi)} = f^{(b)} = \sin 2mz$ and

$$-4m^2\frac{\partial P}{\partial T} + 4m^2 c_g\frac{\partial P}{\partial X} - \frac{\partial B}{\partial X} + 2k^2 m\frac{\partial |A|^2}{\partial X} = 0$$

$$\frac{\partial B}{\partial T} - c_g\frac{\partial B}{\partial X} + N^2\frac{\partial P}{\partial X} - \frac{N}{K}k^2 m\frac{\partial |A|^2}{\partial X} = 0$$

These determine the corrections to the mean stratification and flow in the region of the wave packet.

Collecting the terms which are proportional to ϕ gives

$$2KN\frac{\partial A}{\partial \tau} + \imath\frac{kN^2}{K^2}(2 - 3\frac{k^2}{K^2})\frac{\partial^2 A}{\partial X^2} + 2\imath kmN\frac{\mathbf{k}^2 - m^2}{K}PA - \imath kmBA = 0$$

which determines the packet amplitude.

Solitary Waves: For these, we look at the steady solutions for the mean corrections

$$P = \frac{k^2 K^3 m(k^2+3m^2)}{N(K^6-4m^6)}|A|^2$$
$$B = \frac{2k^2 K^2 m(K^4+m^4)}{(K^6-4m^6)}|A|^2$$

and

$$\frac{\partial A}{\partial \tau} + \imath\alpha\frac{\partial^2 A}{\partial X^2} + \imath\beta|A|^2 A = 0$$

the nonlinear Schrödinger equation, with

$$\alpha = \frac{kN}{K^3}\left(1-\frac{3k^2}{2K^2}\right) \quad , \quad \beta = -\frac{k^2m^2K(3K[k^2-m^2]^2+k^5+2k^3m^2+3km^4)}{N(K^6-4m^6)}$$

Letting

$$A = |A|\exp(-\imath k c_1 \tau)$$

gives

$$-c_1 k|A| + \alpha\frac{\partial^2|A|}{\partial X^2} + \beta|A|^3 = 0$$

which has solutions

$$|A| = A_0 \mathrm{sech}(X/L) \quad , \quad c_1 = \frac{A_0^2\beta}{2k} \quad , \quad L^2 = \frac{2\alpha}{\beta A_0^2}$$

Again, the nonlinear group travels faster with a speed dependent on the amplitude and a width decreasing as A_0 increases.

Modulational instability: If we begin with an infinite regular wave $A = 1+\delta A$, we can look at the perturbation by envelope variability. The P and B equations will show rapid oscillations around

$$P = \frac{k^2 K^3 m(k^2+3m^2)}{N(K^6-4m^6)}(\delta A+\delta A^*)$$
$$B = \frac{2k^2 K^2 m(K^4+m^4)}{(K^6-4m^6)}(\delta A+\delta A^*)$$

and we'll ignore the high frequency variations. Then

$$\frac{\partial \delta A}{\partial \tau} + \imath\alpha\frac{\partial^2 \delta A}{\partial X^2} + \imath\beta(\delta A+\delta A^*) = 0$$

This gives

$$\delta A \sim e^{\sigma\tau} \quad , \quad \sigma^2 = k^2(2\alpha\beta - k^2)$$

This can be positive near $k < 0.766m$ where β becomes large.