Turbulence in the Atmosphere and Oceans

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Course description
The course will present the phenomena, theory, and modeling of turbulence in the Earth's oceans and atmosphere. The scope will range from centimeter to planetary scale motions. The regimes of turbulence will include homogeneous isotropic three dimensional turbulence, convection, boundary layer turbulence, internal waves, two dimensional turbulence, quasi-geostrophic turbulence, and planetary scale motions in the ocean and atmosphere. Prerequisites: the mathematics and physics required for admission to the graduate curriculum in the EAPS department, or consent of the instructor.

Course requirements
Class attendance and discussion, bi-weekly homework assignments.

Reference texts
Frisch, "Turbulence: the legacy of Kolmogorov"
Lesieur, "Turbulence in Fluids", 3rd revised edition
McComb, "The physics of turbulence"
Saffman, "Vortex dynamics"
Salmon, "Lectures on geophysical fluid dynamics"
Strogatz, "Nonlinear Dynamics and Chaos: With Applications in Physics, Biology, Chemistry, and Engineering"
Tennekes and Lumley, "A first course in Turbulence"

Chapter 12

Surface Quasi Geostrophy

The quasi-geostrophic (QG) theory introduced by Glenn describes the flow departures from solid body rotation in a rapidly rotating, stably stratified fluid. In its Boussinesq version, the flow evolves according to the coupled vorticity (ζ) and buoyancy (b) equations,

$$\partial_t \zeta = -J(\psi, \zeta) + f\partial_z w, \tag{12.1}$$
$$\partial_t b = -J(\psi, b) - N^2 w, \tag{12.2}$$
$$\zeta \equiv \nabla^2 \psi, \qquad b \equiv f\partial_z \psi. \tag{12.3}$$

Here ψ is the streamfunction for the horizontal geostrophic flow, $(u, v) = (-\psi_y, \psi_x)$, w is the vertical velocity, f is the constant vorticity due to the background rotation, while $N(z)$ is the buoyancy frequency of a reference state. The vorticity ζ is defined as in two-dimensional flows. Eliminating the vertical velocity, one obtains the pseudo-potential vorticity equation,

$$\partial_t q = -J(\psi, q), \qquad q \equiv \left(\partial_{xx} + \partial_{yy} + \partial_z S^{-2} \partial_z\right) \psi, \qquad S^2 \equiv (N/f)^2. \tag{12.4}$$

If we specialize to the case of constant N, the Prandtl ratio S can be subsumed into a rescaled vertical coordinate, Nz/f (in which case we retain the notation z for the rescaled coordinate), and the potential vorticity is now simply the three-dimensional Laplacian of the streamfunction. At flat lower boundaries, the condition of no normal flow is,

$$\partial_t b = -J(\psi, b), \qquad \text{at } z = 0. \tag{12.5}$$
$$\tag{12.6}$$

If a flat upper boundary is imposed, then (12.5) also holds at $z = H$.

The familiar special case of two-dimensional flow is obtaining by assuming that the streamfunction is independent of z. Charney suggested that a more geophysically

relevant limit is to retain vertical dependence in ψ and q assuming that the top and bottom boundaries are homogeneous, i.e. $b = 0$. Charney's model has become the de-facto standard tool for QG studies of oceans and atmospheres. A less familiar special case is that of surface quasi-gesotrophic (SQG) flow, in which it is assumed that $q = 0$, so that the interior equation is identically satisfied, and the flow is driven entirely by the surface b-distribution. If the surface is flat, if N^2 is a constant, and if there is no upper boundary, then the resulting equations are,

$$\partial_t b = -J(\psi, b), \qquad \text{at } z = 0, \tag{12.7}$$

$$b \equiv f\partial_z \psi, \tag{12.8}$$

$$q \equiv (\partial_{xx} + \partial_{yy} + \partial_{zz})\,\psi = \text{const}, \qquad \text{for } z > 0, \tag{12.9}$$

$$\psi \to 0, \qquad \text{as } z \to \infty. \tag{12.10}$$

One can easily generalize to the case of $q = q_0$ a non-zero constant. For example, in the presence of a uniform horizontal shear, with the total flow described by the streamfunction $1/2q_0y^2 + \psi$, one need only incorporate advection by the mean flow, $u = -q_0 y$, into the buoyancy equation.

The QG and SQG equations are complementary description of stratified rotating flows. One can always divide the total flow at any instant into a part induced by the surface b-distribution and a part induced by the interior q-distribution. The QG approximation has attracted more attention after the seminal work of Charney. However there are meteorological and oceanographic problems for which the SQG approximation is thought to be more appropriate. Examples are the evolution of temperature anomalies at the tropopause (Juckes, JAS, 1994) and density anomalies at the ocean surface (LaCasce and Mahadevan, JMR, 2006).

12.1 2D turbulence versus QG turbulence

Some of the distinctions between SQG and 2D flows are immediately evident from the form of the equations. In two-dimensional flow, the streamfunction induced by a point vortex, $\zeta = \delta(\boldsymbol{x}')$, in an unbounded domain is $\psi(x) = -(2\pi)^{-1}\text{In}(|\boldsymbol{x} - \boldsymbol{x}'|)$. In SQG, $b = \delta(\boldsymbol{x})$ results in the flow $\psi(x) = -(2\pi|\boldsymbol{x} - \boldsymbol{x}'|)^{-1}$. The circumferential velocities around the vortex are proportional to r^{-1} for 2D flow and r^{-2} for SQG flow, r being the distance from the vortex centre.

The more singular SQG Green's function has several important consequences. Nearby point vortices rotate about each other more rapidly than in the twodimensional case; in consequence, a greater ambient strain is required to pull them apart, since the rapid rotation averages out the effects of the strain. Conversely, distant eddies are less tightly bound to each other than in two-dimensional flow. Taken together, a

greater tendency to form localized vortex assemblages is implied. Since the flow dies away from a point vortex as r^{-2} rather than r^{-1} in SQG, the aggregate effect of distant eddies on the local velocity field is more limited. SQG is qualitatively characterized by the preponderance of spatially local rather than long range interactions.

In terms of spectral amplitudes, if $\psi = \Re\left(\hat{\psi}_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$ in two-dimensional flow, then $\zeta = \Re\left(\hat{\zeta}_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$ with,

$$\hat{\zeta}(\boldsymbol{k},t) = -|\boldsymbol{k}|^2\hat{\psi}(\boldsymbol{k},t). \tag{12.11}$$

Since the vertical structure of a sinusoidal disturbance in SQG theory is $e^{-|\boldsymbol{k}|z}$, the analogous relation is,

$$\hat{b}(\boldsymbol{k},t) = -|\boldsymbol{k}|\hat{\psi}(\boldsymbol{k},t). \tag{12.12}$$

In both models, the flow can be thought of as determined by a smoothing operator acting on the conserved scalar, but in SQG there is less smoothing. This implies that large-scale strain will play a relatively smaller role in the advection of small-scale features in SQG, resulting in a cascade of variance to small scales that is more local in wavenumber. Held et al. (JFM, 1994) discuss the implications of the locality of interactions for atmospheric and oceanic flows.

12.2 Conserved properties

In the first part of the lecture we consider SQG flows with a rigid lower boundary and no upper boundary. These SQG equations preserve energy,

$$E \equiv \frac{1}{2}\langle|\nabla\psi|^2 + \psi_z^2\rangle = -\frac{1}{2}\overline{\psi b}^{z=0} - \frac{1}{2}\langle\psi q\rangle.$$

where $\langle\cdot\rangle$ denotes an average over the full 3D domain and the overbar a 2D average along the lower boundary. For zero potential vorticity flows, the conservation of energy becomes,

$$E = -\frac{1}{2}\overline{\psi b}^{z=0}.$$

Furthermore the SQG equations conserve the buoyancy variance along the lower boundary,

$$\Theta = \frac{1}{2}\overline{b^2}^{z=0}$$

Indeed, they conserve the average of any function of buoyancy, not just the square, so that we have to worry about whether or not the energy and buoyancy variance tell the whole story.

There is a remarkable confusion in the literature on the appropriate definition of energy for SQG flows. The discussion hinges on a misunderstanding about the difference between surface averaged energy,

$$E_s = \overline{\frac{1}{2}\langle|\nabla\psi|^2 + b^2}$$

and volume averaged energy,

$$E = \frac{1}{2}\langle|\nabla\psi|^2 + b^2\rangle.$$

For a flow with a rigid lower lid, it is straightforward to prove,

$$\int E(K)\mathrm{d}K\mathrm{d}z = \int E_s(K)e^{-2Kz}\mathrm{d}K\mathrm{d}z = \frac{1}{2}\int K^{-1}E_s(K)\mathrm{d}K.$$

the two spectra have different units and, more importantly, different slopes.

12.3 SQG turbulence

Blumen (1978) has presented the Kolmogorov-Kraichnan scaling arguments for the spectral shapes expected in the SQG turbulent inertial ranges, and these have been compared with numerical simulations by Pierrehumbert, Held and Swanson (1994).

The power spectra $E(K)$ and $\Theta(K)$ are defined so that,

$$E = \int_0^\infty E(K)\mathrm{d}K, \qquad \Theta = \int_0^\infty \Theta(K)\mathrm{d}K \tag{12.13}$$

with $\Theta(K) = KE(K) = \mathcal{U}(K)$, where $\mathcal{U}(K)$ is the power spectrum of the velocity field. The spectral fluxes are defined by $\partial_t E = -\partial_K F_E$ and $\partial_t \Theta = -\partial_K F_\Theta$. In equilibrium, both F_E and F_Θ must be constant.

The dimensions of F_E/F_Θ are L. The fundamental tenet of the Kolmogorov-Kraichnan scale analysis is that the only available length scale is the local eddy scale K^{-1}. Since the fluxes must be independent of K, only one of F_E and F_Θ can be nonzero. Nonzero F_E yields the energy cascading spectrum; since the dimensions of E are L^2T^{-2} and the dimensions of F_Θ are L^3T^{-3}, dimensional analysis implies,

$$E(K) = C_E(F_E)^{2/3}K^{-3}, \qquad \Theta(K) = C_\Theta(F_E)^{2/3}K^{-1}. \tag{12.14}$$

Following similar reasoning, the spectrum in the variance cascading range is,

$$E(K) = C_E(F_\Theta)^{2/3}K^{-8/3}, \qquad \Theta(K) = C_\Theta(F_\Theta)^{2/3}K^{-5/3}. \tag{12.15}$$

Following Kraichnan, the strain rate due to eddies with scale $1/K$ is $\sqrt{\int K^2 \mathcal{U}(K) \mathrm{d}K} = \sqrt{\int K \Theta(K) \mathrm{d}K}$. Substituting (12.15) for the spectrum, we find that the enstrophy cascade is dominated by local strain in SQG. In the direct enstrophy cascade range of QG, the K^{-1} local scaling spectrum is the same as the passive scalar spectrum predicted for the strongly nonlocal case in which straining is dominated by large eddies with a fixed timescale. Thus, the spectral behavior shades continuously over to the nonlocal scaling. This is because in QG the conserved quantity q is coincidentally the straining rate.

It is important to notice that the energy spectrum E refers to the 3D volume integral of potential plus kinetic energies. However the kinetic and potential energies at the lower boundaries both scale like the buoyancy variance spectrum Θ. In SQG the energy spectrum in the interior differs from that at the surface, because each normal mode of the system decays at a different rate away from the boundary so that the surface and interior spectra can be quite different.

12.3.1 Fjörtöft's argument

Fjörtöft's (1953) argument for the direction of the energy flux in 2D turbulent flows can also be applied to the SQG flow problem with minor modifications. Suppose we have unit energy at a wavenumber K such that,

$$\Theta(K) = KE(K),$$

and we wish to transfer it elsewhere through inviscid interactions. Let a fraction α_1 go to larger scales $(K/2)$ and α_2 to smaller scales $(2K)$. Then our energy and variance pictures look like,

Wavenumber	$K/2$	K	$2K$
Init. energy	0	1	0
Init. variance	0	K	0
Final energy	α_1	$1-\alpha_1-\alpha_2$	α_2
Final variance	$K\alpha_1/2$	$K(1-\alpha_1-\alpha_2)$	$K2\alpha_2$

If we conserve both energy and variance by this interaction (i.e., we're in an inertial range), we find $\alpha_1 = 2\alpha_2$ so that,

Wavenumber	$K/2$	K	$2K$
Init. energy	0	1	0
Init. enstrophy	0	K	0
Final energy	$2\alpha_2$	$1-3\alpha_2$	α_2
Final enstrophy	$K\alpha_2$	$K(1-3\alpha_2)$	$K2\alpha_2$

More energy is transferred to large scales and more variance to small scales. In SQG the baricenter of the energy remains at wavenumber K while the baricenter of the buoyancy variance shifts to $K(1+1.5\alpha_2)$. In the energy cascade to larger scale, the vertical scale can increase – the flow becomes more barotropic.

12.4 Spectral transfers

A study of triad interactions illustrates the dynamics behind the direct and inverse cascades of SQG turbulence. Let us transform the streamfunction to wavenumber space,

$$\psi = \hat{\psi}(k,\ell,t)\exp(\imath kx + \imath\ell y - Kz), \tag{12.16}$$
$$b = \hat{b}(k,\ell,t)\exp(\imath kx + \imath\ell y - Kz). \tag{12.17}$$

We introduce a shorthand $\psi_j = \hat{\psi}(k,\ell,t)$ so that each different subscript j corresponds to a different set of $\{k,\ell\}$ values. The streamfunction is related to the buoyancy variance by,

$$b_j = (k_j^2+\ell_j^2)^{1/2}\psi_j = -K_j\psi_j.$$

Now we can project out the equation for the amplitude of one mode by multiplying the equation by $\exp(-i\boldsymbol{k}_2\cdot\boldsymbol{x})$ and surface averaging,

$$K_2\partial_t\psi_2 = \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3=0}\sum (k_1\ell_3 - k_3\ell_1)K_3\psi_1^*\psi_3^*.$$

These equations are identical to those obtained for the QG problem except for the replacement of K_2^2 and K_3^2 with K_2 and K_3 as a result of the different relationship between the two conserved quantities: energy and enstrophy in QG, energy and buoyancy variance in SQG. Following the approach described for the QG problem, the equations can be rewritten in the form,

$$K_2\partial_t\psi_2 = \frac{1}{2}\sum (k_1\ell_2 - k_2\ell_1)(K_1-K_3)\psi_1^*\psi_3^*$$

with the definition $\boldsymbol{k}_3 = -\boldsymbol{k}_1 - \boldsymbol{k}_2$.

Let us look at one triad of wavenumbers $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3$ and choose the labelling such that $K_1 < K_2 < K_3$. The dynamics of this triad is given by,

$$K_1\partial_t\psi_1 = (k_1\ell_2 - k_2\ell_1)(K_3-K_2)\psi_3^*\psi_2^* \tag{12.18}$$
$$K_2\partial_t\psi_2 = (k_1\ell_2 - k_2\ell_1)(K_1-K_3)\psi_1^*\psi_3^* \tag{12.19}$$
$$K_3\partial_t\psi_3 = (k_1\ell_2 - k_2\ell_1)(K_2-K_1)\psi_2^*\psi_1^* \tag{12.20}$$

This triad conserves energy and buoyancy variance internally,

$$\partial_t\sum K_j|\psi_j|^2 = \sum E_j = 0$$

$$\partial_t \sum K_j^2 |\psi_j|^2 = \sum K_j E_j = 0$$

From the triad equations, we also have,

$$\partial_t E_1 = -\frac{K_3 - K_2}{K_3 - K_1} \partial_t E_2 \tag{12.21}$$

$$\partial_t E_3 = -\frac{K_2 - K_1}{K_3 - K_1} \partial_t E_2. \tag{12.22}$$

Energy leaving component 2 will transfer into both 1 and 3; when it does so, the triads with $K_3 - K_2 > K_2 - K_1$ put more energy into the larger scale mode than the smaller scale one, and put more buoyancy variance into the smaller scale mode than the larger scale one.

Following Merilees and Warn (JAS, 1975), who studied triads interactions in the QG problem, the relative magnitude of the energy flows from or to the middle wave, represented by K_2, will be considered for the SQG problem. We can write,

$$\frac{\partial_t E_1}{\partial_t E_3} = \frac{K_3 - K_2}{K_2 - K_1} = \frac{\sqrt{1 + r^2 + 2r\cos\phi} - 1}{1 - r}, \tag{12.23}$$

$$\frac{\partial_t \Theta_1}{\partial_t \Theta_3} = \frac{K_1}{K_3} \frac{K_3 - K_2}{K_2 - K_1} = \frac{1 - r}{\sqrt{1 + r^2 + 2r\cos\phi}} \frac{\partial_t E_1}{\partial_t E_3}, \tag{12.24}$$

where ϕ denotes the angle between the wavenumbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, and $r = K_1/K_2$.

The region in (ϕ, r) space where the vector $\boldsymbol{k}_1$ may terminate is shown in Fig.1. The conditions $K_1 < K_2$ and $\partial_t E_1/\partial_t E_3 \geq 0$ provide the respective boundaries of this region, i.e. $r = 1$ and $\cos\phi = -r/2$. The energy and enstrophy exchange diagram for QG is very similar to Fig.1, although the relative magnitudes of the regions where $\partial_t E_1/\partial_t E_3 \geq 1$ and $\partial_t Z_1/\partial_t Z_3 \geq 1$ are slightly different. In the present case about 61% of the interactions lead to a larger exchange of depth-integrated energy with low wavenumbers, $\partial_t E_1/\partial_t E_3 \geq 1$. More available potential energy on the boundaries, i.e. boundary buoyancy variance, is sent to high wavenumbers in 57% of the interactions, $\partial_t \Theta_1/\partial_t \Theta_3 < 1$.

12.5 The effect of a rigid upper lid

A rigid upper lid changes the properties of SQG turbulence, because it introduces a vertical scale in the problem,

$$\partial_t b = -J(\psi, b), \qquad \text{at } z = 0, H, \tag{12.25}$$

$$b \equiv f\partial_z \psi, \tag{12.26}$$

$$q \equiv (\partial_{xx} + \partial_{yy} + \partial_{zz})\,\psi, \qquad \text{for } 0 < z < H. \tag{12.27}$$

A remarkable property of the finite-depth SQG problem is that it transitions between quasi-two-dimensional barotropic flow at large scales and baroclinic three-dimensional flow at small scales.

The solution to the SQG problem with a bottom rigid lid shows that as the horizontal scales get larger (or K gets smaller), the penetration depth of the buoyancy anomalies increases, with aspect ratio given by the Prandtl ratio, $S = N/f$. At large enough scale, the penetration will reach deep into the interior flow all the way to the upper lid. The effect of an upper limit in the penetration of buoyancy anomalies is best described in terms of the solutions of the SQG problem,

$$\hat{\psi}(\boldsymbol{k}, z) = \frac{\cosh[S(z+H)K]}{SK\sinh(SHK)}\hat{b}(\boldsymbol{k}, 0), \tag{12.28}$$

which at the upper surface becomes,

$$\hat{\psi}(\boldsymbol{k}, 0) = (SK)^{-1}\tanh(SHK)\hat{b}(\boldsymbol{k}, 0). \tag{12.29}$$

The remarkable property of this finite-depth SQG model results from the properties of the hyperbolic tangent in the inversion. At large scales, or $K \ll (SH)^{-1}$, the buoyancy is related to the streamfunction like $\hat{b}(\boldsymbol{k}, 0) \sim S^2HK^2\hat{\psi}(\boldsymbol{k}, 0)$, while at small scales, or $K \gg (SH)^{-1}$, the inversion is approximately $\hat{b}(\boldsymbol{k}, 0) \sim SK\hat{\psi}(\boldsymbol{k}, 0)$. Thus the relation at the surface of streamfunction to advected quantity (buoyancy) transitions from a 2D-like inversion at large scales, to an SQG-like inversion at small scales, with the transition occurring at the wavenumber

$$K_t \equiv (SH)^{-1} = \frac{f}{NH}. \tag{12.30}$$

The transition scale is the deformation radius.

Tulloch and Smith (2006) have recently suggested that the transition between 2D and SQG spectral slopes in finite depth fluids can be seen in measurements of atmospheric spectra. The horizontal spectra of atmospheric wind and temperature at the tropopause have a steep -3 slope at synoptic scales, but transition to $-5/3$ at wavelengths of order $500-1000$ km. The basic idea is that temperature perturbations generated at the planetary scale excite a direct cascade of energy with a slope of -3 at large scales, $-5/3$ at small scales and a transition near horizontal wavenumber $K_t = f/NH$, where f is the Coriolis parameter. Ballpark atmospheric estimates for N, f and H give a transition wavenumber near the one observed. Numerical simulations also support the expected behavior.

12.6 The Eady problem

Introducing an environmental horizontal buoyancy gradient in SQG, analogous to the β-effect in two-dimensional flow, results in,

$$\partial_t b = -J(\psi, b + \Lambda y), \qquad \text{at } z = 0, H. \tag{12.31}$$

The constant Λ can equivalently be thought of as due to a background vertical shear in the x-component of the flow, $f u_z = -\Lambda$. As this contributes nothing to the interior potential vorticity, the interior equation is unaltered. This system now supports linear waves with the dispersion relation,

$$\omega = -\Lambda k/K, \tag{12.32}$$

where $\psi = \Re\left(\hat{\psi} e^{i(kx+\ell y-\omega t)}\right)$. (This should be contrasted with the familiar Rossby wave dispersion relation, $\omega = -\beta k/K^2$.) These are edge waves that decay away from the surface as e^{-Kz}. The interaction between two such waves, one at the surface and another at the tropopause, gives rise to baroclinic instability in Eadys (1949) classic model of that process.

In the Eady problem perturbations develop as a result of a baroclinic instability of the basic state. the most unstable wavenumber $(k, \ell) = (2.4f/NH, 0)$ can be considered as the injection scale. One might therefore expect a 2D inverse cascade of energy toward larger scales and an SQG direct cascade of buoyancy variance toward smaller scales. The corresponding spectral slopes are expected to be,

Variable	Inverse cascade range	Direct cascade range
Buoyancy	$K^{-5/3}$	$K^{-5/3}$
Energy	$K^{-5/3}$	$K^{-8/3}$

The direct numerical simulations shown in the webpage do not seem to reproduce the transition between the direct and inverse cascade regimes. however more needs to be done to properly explore the parameter space of the problem.

Held et al. (1994) discuss in detail the properties of the direct and inverse cascades in Eady turbulence. In the direct cascade of buoyancy variance to small scales they show that strains generates filaments of high vorticity. This vorticity field is of interest when one considers the geostrophic momentum (GM) equations (Hoskins 1975). In this extension of quasigeostrophic theory, one approximates the momentum by the momentum of the geostrophic flow, but advects it with the full, geostrophic plus ageostrophic, flow. It turns out that the GM equations can be solved by transforming to *geostrophic coordinates* in which coordinate system the equations simply reduce to quasigeostrophy. Therefore, one can take a quasi-geostrophic solution, such as those shown in the webapge, and transform it into a solution of the GM equations. The Jacobian of the transformation is essentially $1-\zeta/f$, where ζ is the vorticity. When the Rossby number ζ/f reaches unity, GM predicts the formation of a frontal singularity.

The implication is that one can anticipate a filigree of microfrontal singularities in homogeneous turbulent simulations of the geostrophic momentum equations. The possibility that the quasi-geostrophic equations themselves would also form such a pattern of discontinuities has been raised by Constantin et al. (1994). But GM predicts that the flow will develop a singularity well before the SQG equations do. The main point is that SQG can be expected to reproduce well the eddy stirring of buoyancy at the boundaries, but not necessarily the final frontal collapse.

The inverse energy cascade in SQG appears to have much in common with that in the two-dimensional case. held et al. (1994) show snapshots from the free evolution of an SQG flow with an initial white noise temperature field. In the movie of Eady turbulence there is a transfer of energy toward larger scales during the spinup phase. Vortices form as the cascade proceeds, more or less as in two dimensional flow (e.g. McWilliams 1984), and the evolution can be thought of as the movement of the vortices in the flow field induced by other vortices, with occasional intense encounters. There is considerable pairing of vortices and the sporadic formation of larger groups, but we have not yet attempted to determine whether there is a greater tendency for the formation of assemblages than in two-dimensions, as suggested by the discussion at the beginning of this chapter. A qualitative difference hinted at by the movie is that vortex encounters are more violent than in two-dimensional flow: rather than merger accompanied by the formation of relatively passive filaments, encounters such as that seen on the left edge of the domain are almost invariably accompanied by the formation of small satellite vortices, through the filamentary instabilities described above. The formation of these satellite vortices should modify the evolution of the vortex size probability distribution in important ways.

Rhines (1975) has discussed the way in which the inverse energy cascade in two-dimensional flow is halted by the presence of an environmental vorticity gradient, the beta-effect (see also Vallis and Maltrud 1992). In the two-dimensional case, within the $K^{-5/3}$ inverse energy cascade range the characteristic inverse timescale, or advective frequency, of an eddy with wavenumber is,

$$\omega_{ad} = \int K^2 E(K) \mathrm{d}K \propto K^{2/3}$$

Comparing with the Rossby wave dispersion relation, $\omega_R = -\beta k/K^2$, one sees that wave dispersion will eventually dominate, except along the $k = 0$ axis. The transition is a fairly sharp one: $\omega_R/\omega_{ad} \propto K^{-5/3}$ for $l = 0$. In the SQG case, the surface flow and buoyancy are predicted to have a $K^{-5/3}$ spectral shape as well. The edge wave frequency is $\omega_E = -\Lambda k/K$, giving the ratio $\omega_E/\omega_{ad} \propto K^{-2/3}$. Thus, we still expect a transition between turbulent and wavelike behaviour, but a more gradual one, with increasing scale. Numerical experiments in which the inverse energy cascade is arrested with an environmental temperature gradient have yet to be performed.

Further reading

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Held, I., R. Pierrehumbert, S. Garner, K. Swanson, *J. Fluid Mech.*, **282**, 1-20, 1995.

Pierrehumbert, R. I. Held, K. Swanson, *Chaos, Solitons and Fractals*, **4**, 1111-1116, 1994.

Tulloch, R. and K. S. Smith, A New Theory for the Atmospheric Energy Spectrum: Depth-Limited Temperature Anomalies at the Tropopause, *Proc. Nat. Acad. Sci.*, **103**, 14690-14694, 2006.