12.540 Principles of the Global Positioning System Lecture 03

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Review

- In last lecture we looked at conventional methods of measuring coordinates
- Triangulation, trilateration, and leveling
- Astronomic measurements using external bodies
- Gravity field enters in these determinations

Gravitational potential

- In spherical coordinates: need to solve

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\lambda^2}=0$$

- This is Laplace's equation in spherical coordinates

Solution to gravity potential

- The homogeneous form of this equation is a “classic” partial differential equation.
- In spherical coordinates solved by separation of variables, r=radius, λ=longitude and θ=co-latitude

$$V(r,\theta,\lambda) = R(r)g(\theta)h(\lambda)$$

Solution in spherical coordinates

- The radial dependence of form r^n or r^{-n} depending on whether inside or outside body. N is an integer
- Longitude dependence is $\sin(m\lambda)$ and $\cos(m\lambda)$ where m is an integer
- The colatitude dependence is more difficult to solve

Colatitude dependence

- Solution for colatitude function generates Legendre polynomials and associated functions.
- The polynomials occur when m=0 in λ dependence. t=cos(θ)

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$$

Legendre Functions

$$P_o(t) = 1$$

$$P_1(t) = t$$

$$P_2(t) = \frac{1}{2}(3t^2 - 1)$$

$$P_3(t) = \frac{1}{2}(5t^3 - 3t)$$

$$P_4(t) = \frac{1}{8}(35t^4 - 30t^2 + 3)$$

- Low order functions. Arbitrary n values are generated by recursive algorithms

Associated Legendre Functions

- The associated functions satisfy the following equation

$$P_{nm}(t) = (-1)^m (1-t^2)^{m/2} \frac{d^m}{dt^m} P_n(t)$$

- The formula for the polynomials, Rodriques' formula, can be substituted

Associated functions

$$P_{00}(t) = 1$$
$$P_{10}(t) = t$$
$$P_{11}(t) = -(1-t^2)^{1/2}$$
$$P_{20}(t) = \frac{1}{2}(3t^2 - 1)$$
$$P_{21}(t) = -3t(1-t^2)^{1/2}$$
$$P_{22}(t) = 3(1-t^2)$$

- Pnm(t): n is called degree; m is order
- m<=n. In some areas, m can be negative. In gravity formulations m=>0

http://mathworld.wolfram.com/LegendrePolynomial.html

Orthogonality conditions

- The Legendre polynomials and functions are orthogonal:

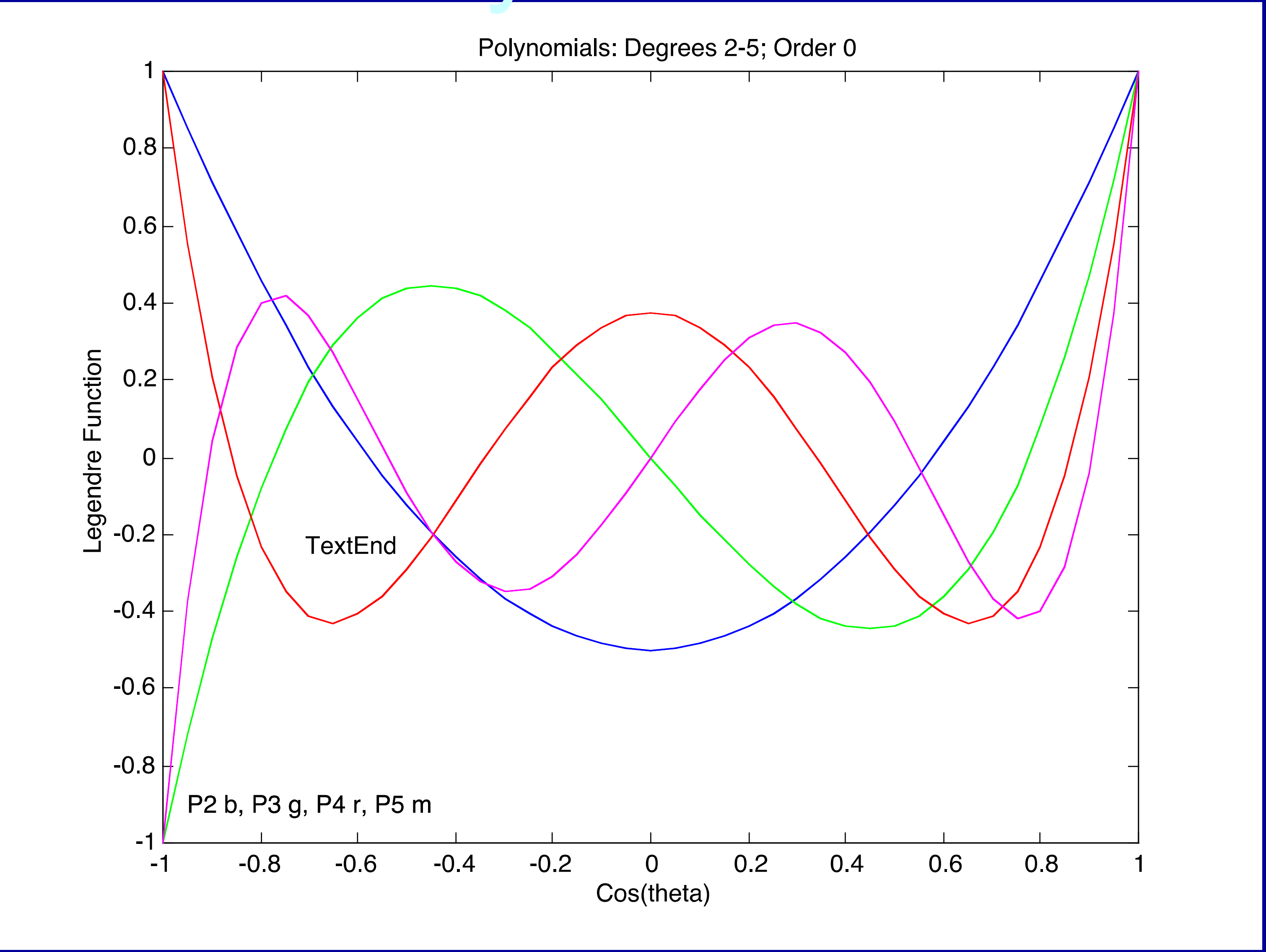
$$\int_{-1}^{1} P_{n'}(t) P_n(t)\,dt = \frac{2}{2n+1}\delta_{n'n}$$

$$\int_{-1}^{1} P_{n'm}(t) P_{nm}(t)\,dt = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n'n}$$

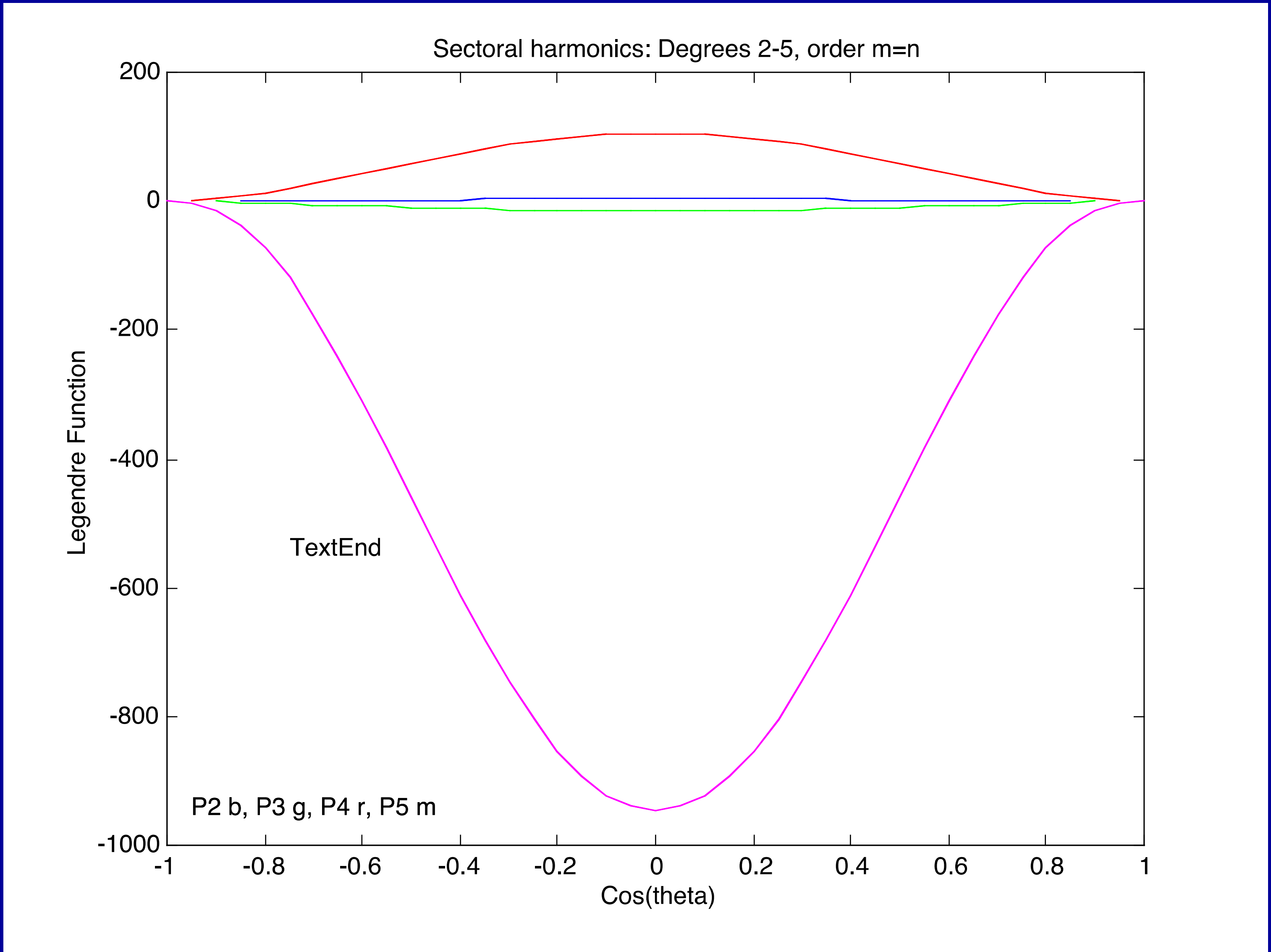
Examples from Matlab

- Matlab/Harmonics.m is a small matlab program to plots the associated functions and polynomials
- Uses Matlab function: Legendre

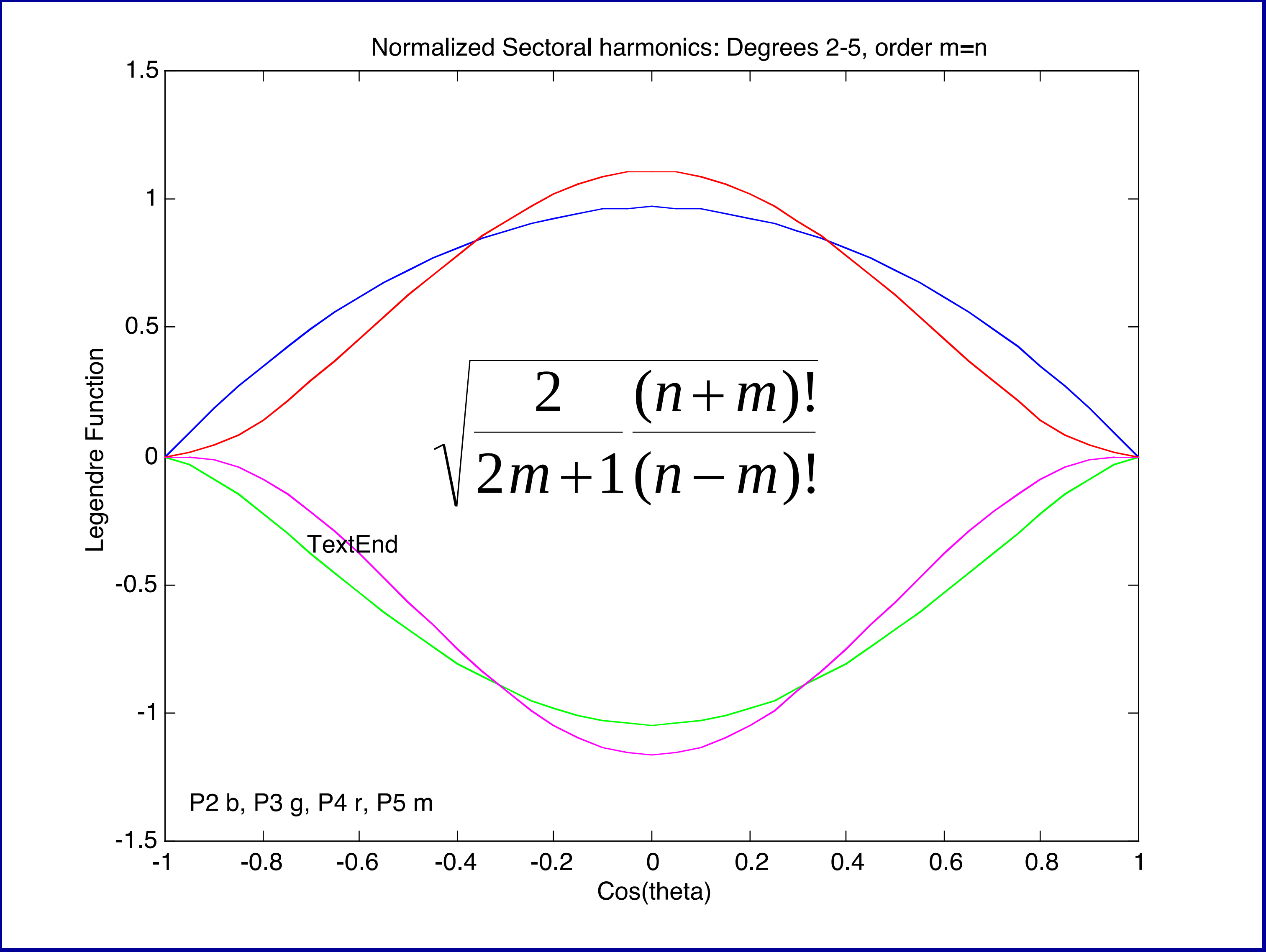
Polynomials



“Sectoral Harmonics” m=n



Normalized



$$\sqrt{\frac{2}{2m+1}\frac{(n+m)!}{(n-m)!}}$$

Surface harmonics

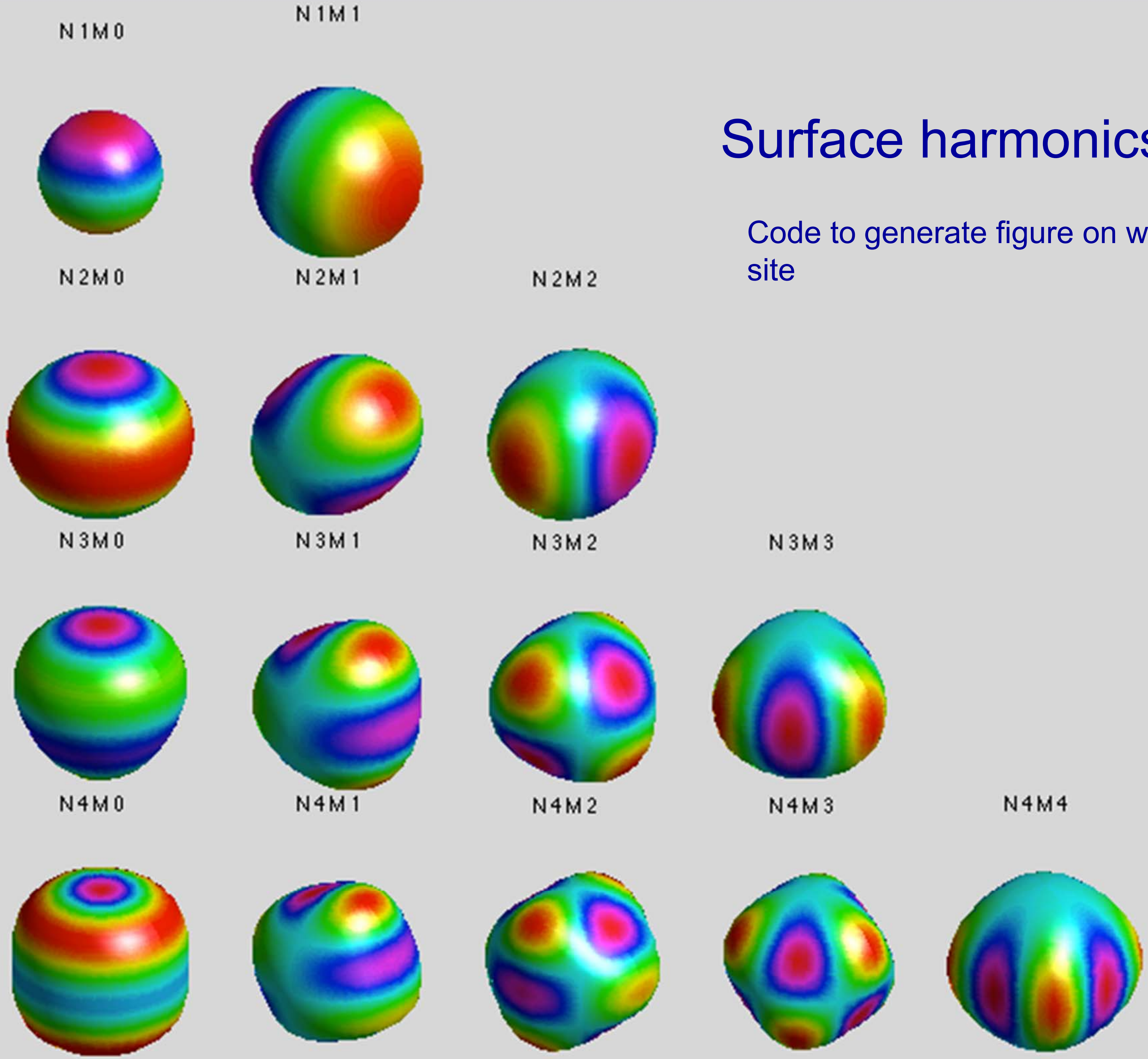
- To represent field on surface of sphere; surface harmonics are often used

$$Y_{nm}(\theta,\lambda)=\sqrt{\frac{2m+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_{nm}(\theta)e^{im\lambda}$$

- Be cautious of normalization. This is only one of many normalizations
- Complex notation simple way of writing cos(mλ) and sin(mλ)

Surface harmonics

Code to generate figure on web site



Zonal ---- Tesserals ----------------------------Sectorial

Gravitational potential

- The gravitational potential is given by:

$$V = \iiint \frac{G\rho}{r} dV$$

- Where ρ is density,
- G is Gravitational constant 6.6732×10^{-11} $m^3 kg^{-1} s^{-2}$ ($N\ m^2 kg^{-2}$)
- r is distance
- The gradient of the potential is the gravitational acceleration

Spherical Harmonic Expansion

- The Gravitational potential can be written as a series expansion

$$V = -\frac{GM}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n}P_{nm}(\cos\theta)[C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)]$$

- Cnm and Snm are called Stokes coefficients

Stokes coefficients

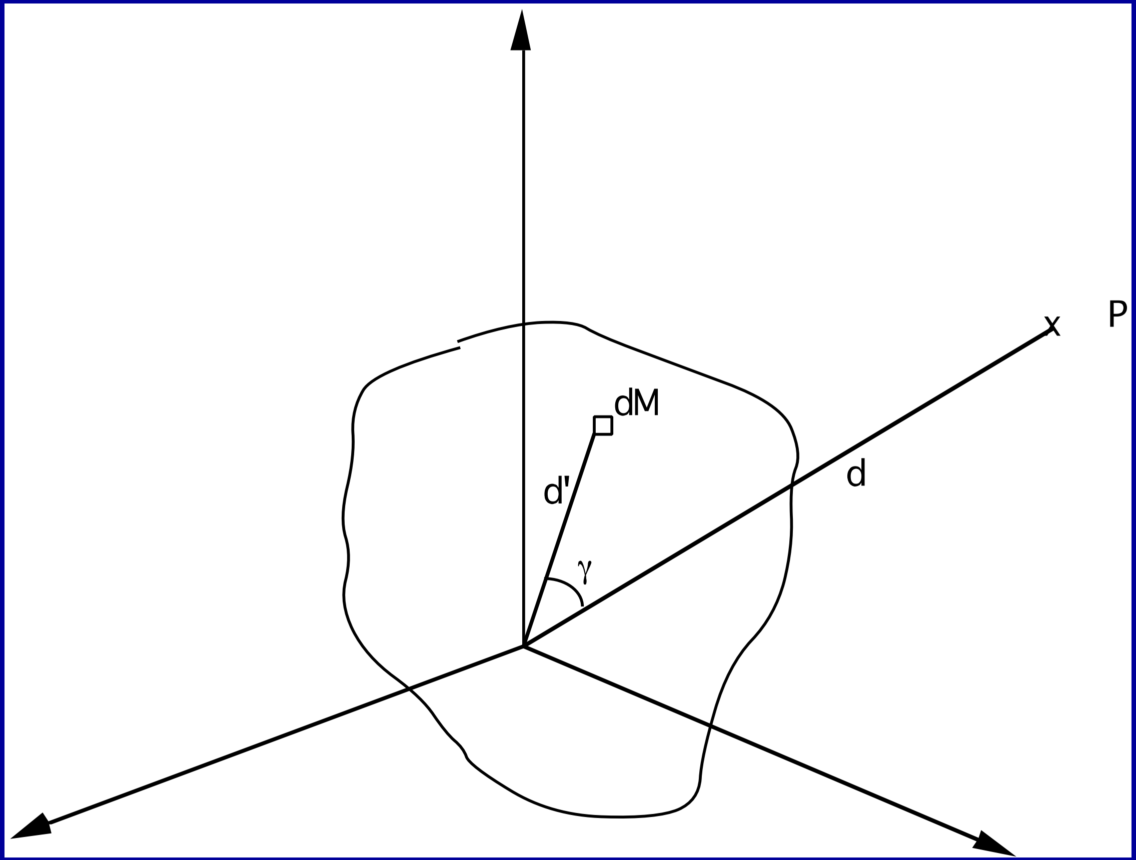
- The Cnm and Snm for the Earth's potential field can be obtained in a variety of ways.
- One fundamental way is that 1/r expands as:

$$\frac{1}{r} = \sum_{n=0}^{\infty} \frac{d'^n}{d^{n+1}} P_n(\cos\gamma)$$

- Where *d'* is the distance to *dM* and *d* is the distance to the external point, γ is the angle between the two vectors (figure next slide)

1/r expansion

- Pn(cosγ) can be expanded in associated functions as function of θ,λ



Computing Stoke coefficients

- Substituting the expression for 1/r and converting γ to co-latitude and longitude dependence yields:

$$P_n(\gamma) = \frac{4\pi}{2n+1} \sum_{m=0}^{n} Y_{nm}^{*}(\theta', \lambda') Y_{nm}(\theta, \lambda)$$

$$V = \iiint \frac{GdM}{r} = 4\pi \iiint \frac{dM}{2n+1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{d'^{n}}{d^{n+1}} Y_{nm}^{*}(\theta', \lambda') Y_{nm}(\theta, \lambda)$$

The integral and summation can be reversed yielding integrals for the Cnm and Snm Stokes coefficients.

Low degree Stokes coefficients

- By substituting into the previous equation we obtain:

$$C_{10} = GM \iiint z' dM \quad C_{11} = GM \iiint x' dM$$
$$S_{11} = GM \iiint y' dM$$

$$C_{20} = \frac{GM}{2} \iiint 2z^2 - x^2 - y^2 dM$$
$$C_{21} = GM \iiint xz dM \quad S_{21} = GM \iiint yz dM$$
$$C_{22} = \frac{GM}{4} \iiint x^2 - y^2 dM \quad S_{22} = \frac{GM}{2} \iiint xy dM$$

Moments of Inertia

- Equation for moments of inertia are:

$$I = \begin{bmatrix} \iiint y^2 + z^2 dM & \iiint xydM & \iiint xzdM \\ \iiint xydM & \iiint z^2 + x^2 dM & \iiint yzdM \\ \iiint xzdM & \iiint yzdM & \iiint x^2 + y^2 dM \end{bmatrix}$$

- The diagonal elements in increasing magnitude are often labeled A B and C with A and B very close in value (sometimes simply A and C are used)

Relationship between moments of inertia and Stokes coefficients

- With a little bit of algebra it is easy to show that:

$$C_{20} = GM(\frac{A+B}{2} - C)$$

$$C_{22} = \frac{1}{4}GM(B-A)$$

$$S_{22} = \frac{1}{2}GMI_{12}$$

C_{21} S_{21} are related to I_{13} and I_{23}

Spherical harmonics

- The Stokes coefficients can be written as volume integrals
- $C_{00} = 1$ if mass is correct
- C_{10}, C_{11}, $S_{11} = 0$ if origin at center of mass
- C_{21} and $S_{21} = 0$ if Z-axis along maximum moment of inertia

Global coordinate systems

- If the gravity field is expanded in spherical harmonics then the coordinate system can be realized by adopting a frame in which certain Stokes coefficients are zero.
- What about before gravity field was well known?

Summary

- Examined the spherical harmonic expansion of the Earth's potential field.
- Low order harmonic coefficients set the coordinate.
  - Degree 1 = 0, Center of mass system;
  - Degree 2 give moments of inertia and the orientation can be set from the directions of the maximum (and minimum) moments of inertia. (Again these coefficients are computed in one frame and the coefficients tell us how to transform into frame with specific definition.) Not actually done in practice.
- Next we look in more detail into how coordinate systems are actually realized.