CHAPTER 5

OPEN-CHANNEL FLOW

1. INTRODUCTION

1 ***Open-channel flows*** are those that are not entirely included within rigid boundaries; a part of the flow is in contract with nothing at all, just empty space (Figure 5-1). The surface of the flow thus formed is called a ***free surface***, because that flow boundary is freely deformable, in contrast to the solid boundaries. The boundary conditions at the free surface of an open-channel flow are always that both the pressure and the shear stress are zero everywhere. But a flow can have a free surface but not be an open-channel flow. Closed-conduit flows that consist of two immiscible fluid phases of differing density in contact with each other along some bounding surface are not open-channel flows, because they are nowhere in contact with open space, but they do have a freely deformable boundary within them. Such flows are free-surface flows but not open-channel flows (Figure 5-2), although they are usually called ***stratified flows***, because the density difference between the two fluids gives rise to gravitational effects in the flow. On the other hand, open-channel flows are by their definition also free-surface flows.

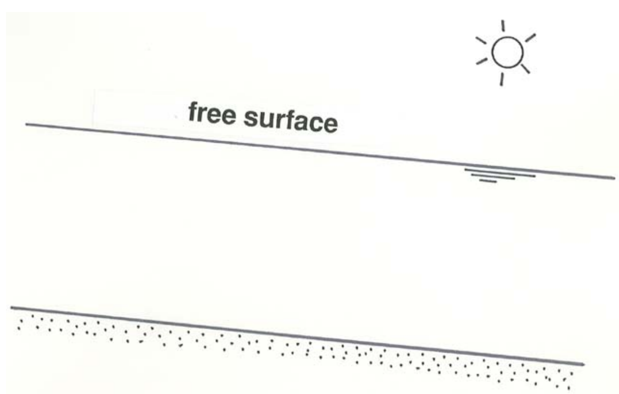

Figure 5-1. An open-channel flow.

2 In a narrow technical sense, flows of liquid at the Earth's surface, like ocean-surface currents or rivers, are not open-channel flows, because they are in contact with another fluid—the atmosphere—at a free surface within a two-phase fluid medium. But the contrast in density between water and air is so great that in studying Earth-surface liquid flows we usually ignore the presence of the overlying atmosphere.

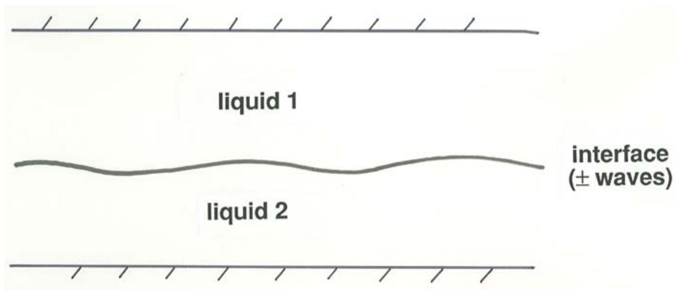

Figure 5-2. A free-surface flow that is not an open-channel flow.

3 All of the principles and techniques for dealing with velocity structure and boundary resistance that were developed for closed-conduit flows in earlier chapters hold as well for open-channel flows. In fact, much of the material in Chapter 4, on flow resistance and velocity structure, is about open-channel flows. But open-channel flows involve an important added element of complexity beyond what we have covered on laminar and turbulent flows in closed conduits: the presence of the free surface means that the geometry of the flow can change in the flow direction not just by being constrained to do so by virtue of the geometry of the boundaries but also by the behavior of the flow itself. This means that the acceleration of gravity can no longer be ignored by the expedient of subtracting out the hydrostatic pressure, as with closed-conduit flows, because the force of gravity helps to shape the free surface. So gravity must therefore be included as an additional independent variable in dealing with free-surface flows. You have already seen an example of this back in Chapter 1, when a sphere was towed underwater but near the free surface.

4 Also, under the right conditions gravity waves can be generated on the free surface, whether or not the fluid is flowing. When the deformable free surface is momentarily deformed in some small area by a deforming force of some sort—by the force of the wind, or by your agitating the water with your hand—the force of gravity acts to try to restore the free surface to its original planar condition. Provided that the viscosity of the liquid is not too high (have you ever tried to make waves in a vat of molasses?) this attempt at restoration of a deformed free surface leads to the propagation of gravity waves away from the region of surface disturbance.

5 This chapter is a selective presentation of some important topics in free-surface flow. I will defer consideration of the generation and propagation of gravity waves on the free surface of a standing or flowing body of liquid until Chapter 6, on oscillatory flow.

TWO PRACTICAL PROBLEMS

6 One of the interesting things about open-channel flow is the effect of gravity on the shape of the free surface relative to the solid boundary. Babbling brooks and white-water rivers clearly have complex free-surface geometries governed by bed relief, expansions and contractions of the channel, and, less obviously, upstream and downstream conditions. But all open-channel flow, even broad, majestic rivers like the Mississippi, or flows in laboratory channels we try to keep as nearly uniform as possible, are subject to such effects of gravity. To make this effect concrete, I will pose two questions at this point for you to think about. Both are of great practical importance to engineers dealing with open-channel flows.

7 Your intuition might have some trouble with the first question. Suppose that you set up a nice open-channel flow, in a wide rectangular channel, just for the sake of definiteness, with a planar bottom, which may or may not be sloping. Then, at a particular position down along the channel you introduce a smooth and gentle step in the channel bottom, either upward or downward (Figure 5-3). The question is: does the water surface rise or fall over the step, relative to its upstream level?

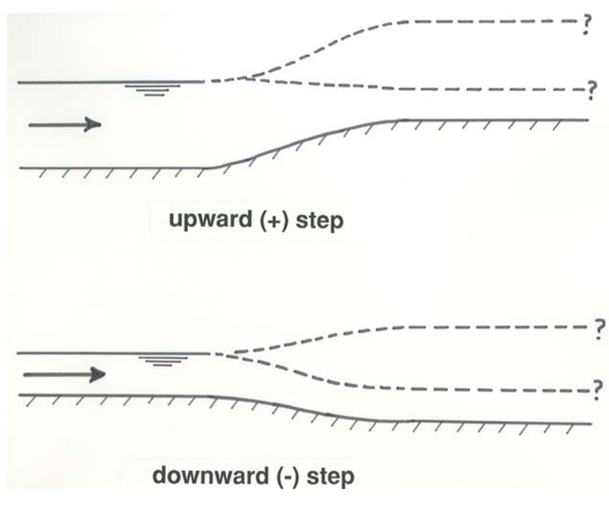

Figure 5-3. A positive step and a negative step in a channel bottom.

8 You will probably feel more comfortable with the second question. A river with a constant bottom slope is dammed at a certain point, so that the river has to merge somehow into a deep reservoir formed in the river valley (Figure

5-4). You can assume that far upstream in the channel the flow is very nearly uniform. That sloping water surface upstream has to pass continuously into the horizontal water surface of the reservoir, where the water velocity is negligible. What would the water-surface profile look like along a streamwise vertical cross section through the channel and the reservoir? Would it change very gradually, all the while sloping monotonically down toward the reservoir? Or would it continue unchanged all the way to the reservoir level, to meet the water surface in the reservoir by an abrupt change in water-surface slope?

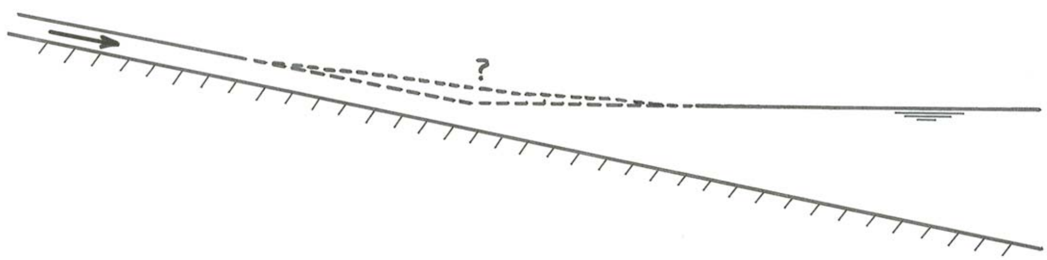

Figure 5-4. When a river enters a lake or reservoir, does the water surface in he river meet the water surface in the lake or reservoir in a smooth transition, or abruptly?

9 Before attacking these problems, you need a brief look at *uniform flow*, which is a useful reference for study of the nonuniformities introduced by the joint effect of gravity and the changing boundary geometry. Then I will have to expose you to more material on *flow energy*, because it turns out that this is the key to the problems posed above.

UNIFORM FLOW

10 ***Uniform flow*** serves as a good reference case from which to think about the effect of gravity on the free surface in an open-channel flow. Only if an open-channel flow can somehow be adjusted to be strictly ***uniform***, in the sense that the water surface is planar and the flow depth is the same at all cross sections along the flow (Figure 5-5), can the effect of gravity in shaping the flow be ignored.

11 Flows in the laboratory can be set up to be very nearly uniform, and outdoors flows like those in long canals are often also close to being uniform. But uniformity is an abstraction: real flows are never perfectly uniform, because, no matter how closely the conditions of flow are adjusted, there are always subtle free-surface effects that extend downstream from the source of the flow and upstream from the sink for the flow, or upstream and downstream from places where the channel geometry changes, like dams or bridge piers.

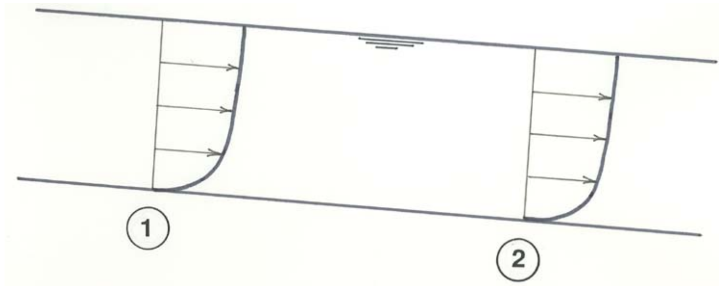

Figure 5-5. A uniform open-channel flow: the depth and the velocity profile is the same at all sections along the flow.

12 One kind of problem that is associated with uniform flow is what the channel slope will be if discharge Q, water depth d, and bed sediment size D are specified or imposed upon the flow. You can investigate this by building an open channel in your back yard, just nailed together out of wood, as if you were going to pan for gold. Try to make the channel several meters long and something like a meter wide, with a planar bottom and planar vertical sidewalls. Immerse the downstream end of the channel in one of those big above-ground swimming pools so many people have in their yards these days. (This is the key to imposing the flow depth on the channel upstream: the higher the water level in the swimming pool relative to the sediment bed in the channel, the deeper the flow in the channel.) Put a submersible pump in the pool to recirculate the water, and the transported sediment as well, to the upstream end of the channel at a given discharge Q. Lay a full bed of sand in the channel, thick enough so that the flow can redistribute it by erosion and deposition if it so desires, without exposing the channel bottom. Mount the upstream end of the channel on a scissors jack or the like, so that you can vary the slope of the channel.

13 It should seem obvious to you that for a given discharge, and an arbitrary channel-bottom slope you set at the beginning, the flow depth in the channel would vary from upstream to downstream: in general the flow in your channel is *nonuniform*, before the flow erodes sand from one end of the channel and deposits in at the other end in its desire to establish uniform flow. If that does not seem obvious to you, imagine that for a given discharge you first increased the channel slope; eventually you would have a condition in which the flow was relatively shallow at the upstream end and relatively deep at the downstream end (Figure 5-6A). On the other hand, if you decreased the channel slope to be very gentle, you would eventually have a condition in which the flow was relatively deep at the upstream end and relatively shallow at the downstream end (Figure 5-6B). Somewhere in between those two extreme conditions there would be a slope for which the flow was nearly uniform. The question then is: *what governs what the slope is for uniform flow*?

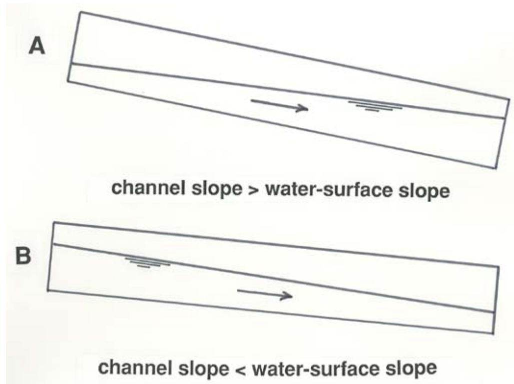

Figure 5-6. **A)** An open-channel flow for which the water-surface slope is less than the slope of the channel bottom. **B)** An open-channel flow for which the water-surface slope is greater than the slope of the channel bottom.

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14 The key to the answer lies in flow resistance, which was addressed at length in Chapter 4. But there we analyzed the dynamics of flow resistance after assuming that the flow had already been adjusted for uniformity. Now we are asking how we can predict what the slope will be for uniform flow. This an important engineering problem: if you have to design a drainage culvert or an irrigation channel, you want to make sure that the flow is not grossly non-uniform, or it might end up overflowing its banks either upstream or downstream, and make you vulnerable to lawsuits.

15 The problem is fairly straightforward. First of all, you have at your disposal the basic resistance equation for open-channel flow (Equation 4.11, repeated here):

$$\tau_o = \gamma d \sin\alpha \qquad (5.1)$$

You also have an empirical equation for bed shear stress τ_o in terms of a resistance coefficient, which could be the friction factor *f* or the Chézy coefficient *C* (Equation 4.18, repeated here in slightly rearranged form):

$$\tau_o = \frac{f}{8}\,\rho U^2 \qquad (5.2)$$

You also know the mean velocity *U*, because you have chosen *Q* yourself and you already know *d*, so by the relation $Q = Udb$ (where *b* is the known width of the channel) you can solve for *U*. Compute the mean-flow Reynolds number Re, go to a diagram like that in Figure 4-27 (that diagram was found for flow in a circular pipe, but it is known to give fairly good results for open-channel flow, provided that you use the hydraulic radius both for the channel flow and for the pipe flow) to find f and thus, by Equation 5.2, τ_o. Then, knowing τ_o, you can use Equation 5.1 to find the slope angle α. You could adjust the channel slope by use of your scissors jack—and you would *have* to do that if the channel bottom is rigid rather than mantled with loose sediment—but with the full bed of sediment, the flow eventually adjusts the slope to the condition of uniform flow by eroding sediment and one end and depositing sediment at the other end.

16 Now for a further aspect of uniform flow, one that is more relevant to natural open-channel flows on the Earth's surface. Excavate a very long, straight channel, ending at the brink of a large, deep, open pit into which the flow will fall freely, on a gently and uniformly sloping area of the land surface. A length of many kilometers would be good. Arrange to pass a discharge Q of your choosing down the channel. You can readily appreciate that if the channel is sufficiently long the flow in the channel will be close to being uniform, although you can assist the approach to uniformity by fiddling a bit with the flow at the downstream end, by installing a sluice gate or a porous weir to prevent the decrease in upstream depth as the flow falls out of the channel. You will also have to be prepared to feed in some bed sediment at the upstream end, to replenish what is transported down the channel and out the end, if the flow turns out to be strong enough to move some of the sediment. Otherwise, you would be modeling the long-term behavior of a real river, whereby the river gradually wears down the land area on which it flows, thus decreasing the slope of the land over the long term.

17 The big question now is: what will the uniform flow depth be, given the imposed slope and discharge? Is the flow fast and shallow, or is it slow and deep? You have the same hydraulic relationships available as in the previous situation, but now their application is not as straightforward. Think about what you know and what you do not know. What is given is the slope angle α, the discharge Q, the bed sediment size D, and the channel width b The unknowns are the mean flow velocity U, the flow depth d, the resistance coefficient or friction factor f, and the boundary shear stress τ_o. You have four relationships available that involve these knowns and unknowns:

$\tau_o = \gamma d \sin\alpha$ (the basic resistance equation for uniform channel flow)

$Q = Udb$ (conservation of flow volume)

$\tau_o = (f/8)\rho U^2$ (the relationship between flow velocity and boundary shear stress)

$f = f(\mathrm{Re}, D/d)$ (the dependence of the friction coefficient on flow velocity, flow depth, and bed roughness)

These are the same equations used in the earlier situation. The difference is that you cannot proceed step by step to find the answer: you need to deal with them all at once. The problem is well posed (four unknowns, four equations), but you cannot obtain the solution analytically, in closed form; you need to find the solution by some iterative numerical technique. The important point here, though, is that there *is* a unique solution: for any given combination of channel slope, bed sediment, and water discharge, there is a certain flow depth, mean flow velocity, and boundary shear stress. And your intuition tells you, correctly in this case, that the uniform flow depth increases with water discharge and also with bed roughness: the greater the discharge, and the rougher the bed (meaning more resistance to flow), the greater the flow depth for uniform flow. It is clear also that the flow depth depends on the slope: the greater the slope, the smaller the flow depth.

ENERGY IN OPEN-CHANNEL FLOW

18 To address the two channel-transition problems posed earlier, we need to have a closer look at mechanical energy in an open-channel flow, and at how the partitioning of the various components of that mechanical energy, kinetic and potential, are changed at the transition in question.

19 I noted back in Chapter 3 that the Bernoulli equation is an expression of the work–energy theorem: the work done by the fluid pressure is equal to the change in kinetic energy of the flow. Remember that, in cases like this, if the change in kinetic energy is reversible a quantity called potential energy is defined as minus the work done, and then the sum of kinetic energy and potential energy, often called mechanical energy, is unchanged or conserved. Forces for which this is true, like the fluid pressure in this case, are said to be *conservative* forces. Gravity is a good example: a ball thrown upward gains potential energy on its way up at the same rate it loses kinetic energy, if the frictional resistance of the air is ignored. Frictional forces, on the other hand, degrade mechanical energy into thermal energy (more commonly called heat or heat energy).

20 Review the derivation of the Bernoulli equation in Chapter 3 and you will see that fluid pressure is a conservative force: in the absence of friction, the change in pressure potential energy per unit volume between two points 1 and 2 down a streamline, which is minus the work per unit volume $-(p_2 - p_1)$ by the fluid pressure, is equal to the change in kinetic energy per unit volume, $(\rho/2)(v_2^2 - v_1^2)$, so the two kinds of mechanical energy are interchangeable in this case also. It should therefore seem natural that when the fluid is in a gravity field a term for gravitational potential energy can be included in the Bernoulli equation as well. Because gravitational potential energy is mgh (where m is the mass of the body under consideration and h is the elevation relative to an arbitrary horizontal plane), the potential energy per unit volume is ρgh.

21 So in the expanded Bernoulli equation the mechanical energy per unit volume of fluid moving along a streamline, $v^2/2 + p + \rho gh$, is constant. This can be written a little more conveniently for our purposes as energy per unit weight of fluid E_w. Because weight equals volume multiplied by ρg,

$$E_W = \frac{v^2}{2g} + \frac{p}{\gamma} + h \tag{5.3}$$

Note that each term has the dimensions of length; E_w is called the ***total head***, and the terms on the right are called the ***velocity head***, the ***pressure head***, and the

elevation head, respectively. In a real fluid, friction degrades mechanical energy to heat as the fluid moves along a streamline. This decrease in mechanical energy from point to point, expressed per unit weight of fluid, is called the ***head loss***. If you add up all three terms on the right in Equation 5.3 the sum decreases downstream, no matter how the values of the individual terms change.

22 It would be nice to generalize Equation 5.3 so that it applies to an entire open-channel flow, not just to each streamline in it. The problem in doing this is that velocity, elevation, and pressure are not constant from point to point on a cross section. But if there are no strong fluid accelerations normal to the flow direction, pressure is close to being hydrostatically distributed: $p = \gamma(d\text{-}y)$. Then the sum of the elevation head and the pressure head can be written

$$\begin{aligned} h + \frac{p}{\gamma} &= h_o + y + \frac{p}{\gamma} \\ &= h_o + y + \frac{\gamma(d-y)}{\gamma} \\ &= h_o + d \end{aligned} \tag{5.4}$$

where h_o is the elevation of the channel bottom. Variations in pressure and elevation over the cross section are thus taken into account in Equation 5.3. Variation in velocity is still a problem, but in turbulent flows the velocity profile is so flat over most of the section that only a small correction need be made in order to replace v by the cross-sectional mean velocity U. Equation 5.3 can then be written between two cross sections 1 and 2 in a channel flow that varies only slowly downstream as

$$\begin{aligned} \text{head loss} &= (E_w)_2 - (E_w)_1 \\ &= \frac{U_2^2}{2g} + h_{o2} + d_2 - \left(\frac{U_1^2}{2g} + h_{o1} + d_1\right) \end{aligned} \tag{5.5}$$

A plot of E_w against downchannel position is called the ***energy grade line***, and the slope of this line (or, generally, curve) is the ***energy gradient*** or ***energy slope***.

23 In a *uniform* open-channel flow, for which both kinetic energy and potential energy are the same at every cross section but potential energy decreases downstream, the head loss is simply the rate of decrease of elevation head downstream, or in other words the slope of the water surface and bed surface, which is then also equal to the energy slope.

24 It is often useful to apply Equation 5.5 to an open-channel flow that varies rapidly enough that there is little head loss but slowly enough that the hydrostatic-pressure approximation is not too far wrong. Those conditions are not very restrictive: examples are a gentle rise or fall in the channel bed, as in the first "practical problem" posed earlier in this chapter (Figure 5-3) or a gentle expansion or contraction of the channel walls. The development in the rest of this section is meant to address such cases. Equation 5.5 becomes

$$\frac{U_2^2}{2g} + h_{o2} + d_2 = \frac{U_1^2}{2g} + h_{o1} + d_1 \tag{5.6}$$

25 A convenient quantity to substitute into Equation 5.6 is $d + U^2/2g$, called the ***specific head*** H_o:

$$H_o = d + \frac{U^2}{2g} \tag{5.7}$$

H_o, also called the ***specific energy***, is simply the head (i.e., flow energy per unit weight) relative to the channel bottom. Using H_o, Equation 5.6 becomes

$$H_{o2} + h_{o2} = H_{o1} + h_{o1} \tag{5.8}$$

or

$$H_{o2} = H_{o1} - (h_{o2} - h_{o1}) \tag{5.9}$$

26 Now look at a unit slice parallel to the flow direction in a two-dimensional flow. In other words, you do not have to worry about the sidewalls because they are far away relative to what is happening locally.) Discharge per unit width q is constant and equal to Ud. Substitution of $U = q/d$ into the definition for specific head eliminates U and provides a relation between d and H_o for each value of q:

$$H_o = \frac{q^2}{2gd^2} + d \tag{5.10}$$

The family of curves of H_o vs. d for various values of q is called the ***specific-head diagram*** or ***specific-energy diagram*** (Figure 5-7).

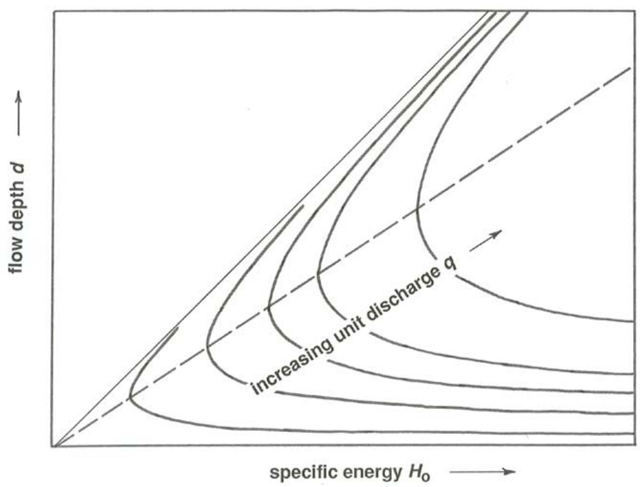

Figure 5-7. The specific-energy diagram. Each of the curves is for a given value of discharge per unit width, q.

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27 To illustrate the usefulness of the specific-head diagram, suppose that the flow approaching the step shown in Figure 5-3 is characterized by values of q, d, and H_o (i.e.: discharge per unit channel width; depth; and flow energy) that plot at point P_1 in Figure 5-8, on the upper part of the curve for the given q. Because the bottom rises by a positive distance $\Delta h = h_{o2} - h_{o1}$, by Equation 5.9 the specific head H_{o2} associated with the flow downstream of the transition lies a distance Δh to the left of H_{o1} along the H_o axis; P_2 is the corresponding point that represents the flow. Flow depth downstream of the step is therefore smaller by $(\Delta d)_P$ in Figure 5-8 than in the approaching flow, and by the relation $q = Ud$ the flow velocity is greater (Figure 5-9). Does that do damage to your intuition?

28 By virtue of the doubly branched form of the curves in Figure 5-7 there can also be an approaching flow, represented by point Q_1 on the lower part of the same curve, with exactly the same discharge and flow energy but with shallower depth and higher velocity. In this case the flow downstream of the transition, represented by the point Q_2 found by moving a distance Δh leftward along the H_o axis as before, has depth greater by $(\Delta d)_Q$ than the approaching flow, and smaller velocity (Figure 5-10). Open-channel hydraulicians speak of *upper and lower alternate depths*.

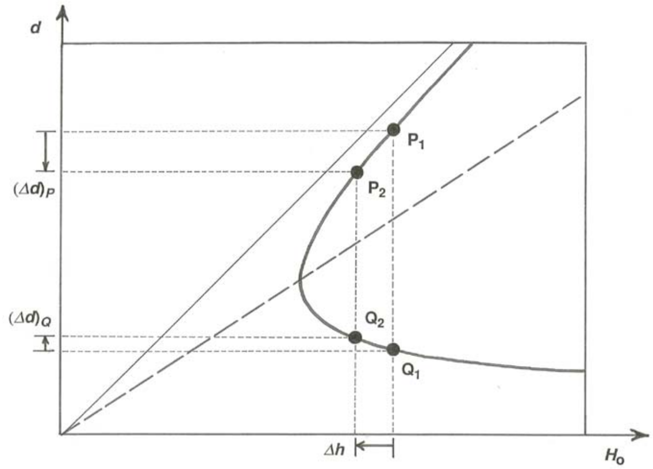

Figure 5-8. The specific-energy diagram for one particular value of q, to illustrate the effect of raising the channel bottom by a distance Δh. See text for explanation.

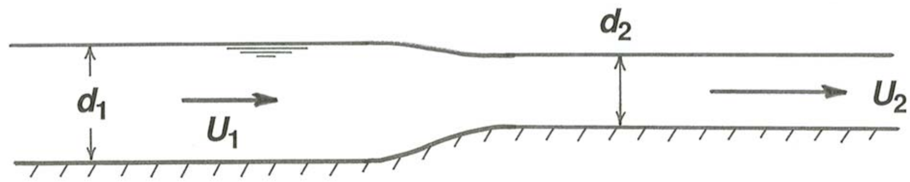

Figure 5-9. The effect of raising the channel bottom beneath a subcritical approaching flow. The depth decreases over the transition, and the mean flow velocity increases.

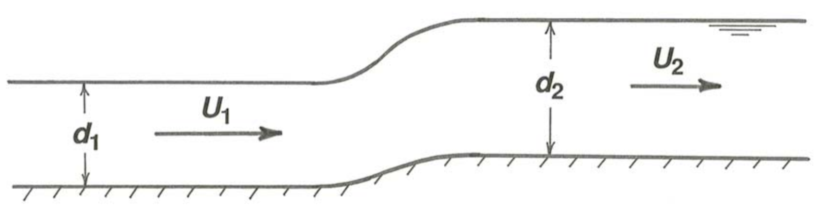

Figure 5-10. The effect of raising the channel bottom beneath a supercritical approaching flow. The depth increases over the transition, and the mean flow velocity decreases.

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29 For points at which the curves of d vs. H_0 have vertical tangents, depth and velocity do not change in the transition. Flows corresponding to these points are called ***critical flows***. The equation for such points is found in two steps. First, differentiate the function in Equation 5.10 to find $dH_0/d(d)$, set this derivative equal to zero, and solve for q as a function of d. The result is

$$q_c^2 = gd_c^3 \qquad (5.11)$$

where the subscript c indicates that the equation is for the critical condition of vertical tangency. Then substitute this expression for q_c^2 into Equation 5.10 to obtain

$$H_{oc} = \frac{3}{2}\, d_c \qquad (5.12)$$

again with the subscript c denoting critical flow.

The locus of points in the specific-head diagram for which the flow is critical is thus a straight line with a slope of 2/3. It is shown in Figure 5-7 as a dashed line extending upward and to the right from the origin. Flows corresponding to points above the line are ***subcritical*** (deeper depths and lower velocities), and flows corresponding to points below the line are ***supercritical*** (shallower depths and higher velocities).

30 Thus, to every combination of discharge per unit width q and flow energy (represented by H_0) there correspond two different possible flow states, with different depth and velocity given by the two intersections of the curve of d vs. H_0 for that q and the vertical line associated with that H_0. In some kinds of transitions along the channel, the flow is forced all the way from one of these states to the other, thereby passing through the critical state during the transition. Any flow, whatever its origin and therefore whatever its depth and discharge, falls at some point on one of the curves in the specific-head diagram, and is therefore either supercritical or subcritical (or critical). The behavior of that flow in a transition is radically different depending on whether the flow is subcritical or supercritical. This difference in behavior is fundamentally a consequence of the requirement of conservation of flow energy expressed by Equation 5.6, together with the conservation-of-mass requirement that

$$q = \frac{U_1}{d_1} = \frac{U_2}{d_2} \qquad (5.14)$$

For example, in the transition examined above, the variables U_1, d_1, h_{o1}, and h_{o2} are all given, and Equations 5.6 and 5.12 then specify exactly which combination of U_2 and d_2 must hold.

31 It happens that the condition for critical flow corresponds to a mean-flow Froude number $U/(gd)^{1/2}$ of unity. To verify this, simply substitute Equation 5.11, the condition for critical flow, into Equation 5.7, the definition for H_0, to obtain a relation between U and d for critical flow: $U^2 = gd$, or Fr = 1.

Subcritical flows are characterized by Froude numbers less than one, and supercritical flows are characterized by Froude numbers greater than one.

32 You will see in Chapter 6, on oscillatory flow, that the speed *c* of a gravity wave in shallow water is $(gd)^{1/2}$, where *d* is the water depth. If you substitute this wave speed *c* for the denominator $(gd)^{1/2}$ in the definition of the Froude number, you see that for a Froude number equal to one the mean flow velocity is equal to the speed of surface waves. A water-surface wave that is moving in the upstream direction appears to an observer on the channel bank to be standing still. This means that if the Froude number of the flow is greater than one, wavelike disturbances cannot propagate upstream: the flow coming from upstream cannot know what is in store for it at positions downstream. In subcritical flow, on the other hand, the upstream flow *can* be influenced, commonly for long distances, by conditions downstream.

33 That last point is well illustrated by one final consideration of the upward step shown in Figure 5-3. As the step height is gradually increased, the corresponding point on the upper branch of the specific head diagram moves leftward and downward from point P toward the point of vertical tangent, C. The farther along the curve the point shifts, the greater is the decrease in flow depth over the step. But there is a limit to this effect: the specific energy cannot decrease beyond that corresponding to the point C of vertical tangent, because the flow has to stay on the q = constant curve. So what happens as the step is raised even further? The flow over the step remains critical and the depth upstream of the step increases. Instead of having no effect on the upstream flow, as was the case for lower steps, the step now acts as a dam: its effect is felt far upstream.

34 You might be wondering at this point how the flow condition represented by the alternate point on the specific-head diagram can be attained. To see how that might happen, suppose that the geometry of the step in Figure 5-3 is changed a bit: after the crest of the step is reached, the channel bottom falls smoothly again to its original height. If now for the approaching subcritical flow with given q the step height is raised to the point where the flow over the step has just attained the critical condition, represented by point C on the curve for the given q in Figure 5-11, the passage of the flow downstream to the original level is manifested on the specific-head diagram as a shift from point P to point Q vertically below the original point P but on the lower (supercritical) limb of the q curve. The flow is now at the same elevation, and has the same energy (i.e., the same channel-bottom elevation and the same specific head), but it is now flowing at a greatly different combination of depth and velocity, corresponding to supercritical flow (Figure 5-12). What is happening, physically, in contrast to graphically, is that the critical flow at the crest of the step accelerates down the lee side of the step, to attain a supercritical velocity (and, by virtue of conservation of mass, a shallower depth). If the step is raised even beyond what is needed to attain the critical condition, then the flow upstream is dammed, and its depth increases, forcing point P upward to the right along the curve for the given q in the specific-energy diagram.

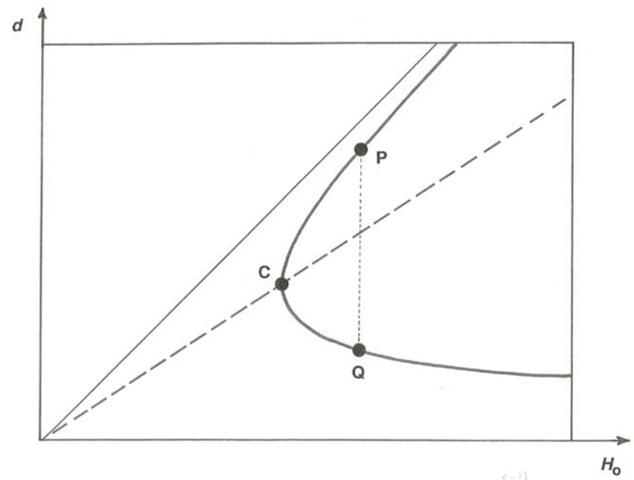

Figure 5-11. The specific-energy diagram for one particular value of *q*, to illustrate the effect of raising and then lowering the channel bottom to force the flow to pass from subcritical to supercritical.

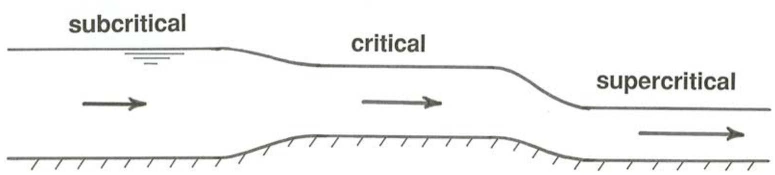

Figure 5-12. The behavior of the flow over a rise and then a fall of the channel bottom, when an approaching subcritical flow is forced to the critical condition by raising the step by a sufficiently large increment.

35 A final comment is that the supercritical flow downstream of the step does not stay supercritical for a very great distance, unless the slope of the bottom downstream of the step becomes much steeper. If the bottom retains its gentle slope, a hydraulic jump is likely to be formed at some point downstream, with the consequence that the flow reverts to its original subcritical condition; see the following section.

THE HYDRAULIC JUMP

36 We still have not milked the positive-step example, as arranged in Figure 5-12, for all the insight it affords. We made the implicit assumption that the flow coming from upstream had a combination of depth and velocity corresponding to the given *q* that was the outcome of the particular gentle channel slope that exists for a long distance upstream; see the earlier section on uniform flow. The combination of slope, discharge per unit width, and bed roughness was such as to provide subcritical flow at that *d* and *U*. We should expect that the flow would like to settle back to that same subcritical condition, somewhere far downstream of the step. But you have just seen that for a sufficiently high step—just high enough for the flow to attain the condition of critical flow, but not so high as to change the upstream flow—the flow for some distance downstream of the step is supercritical. How, then, does the flow pass from being supercritical, just downstream of the step, to subcritical far downstream? The answer is that commonly in situations like this the change from supercritical to subcritical is abrupt, in the form of what is called a ***hydraulic jump***, rather than gradual.

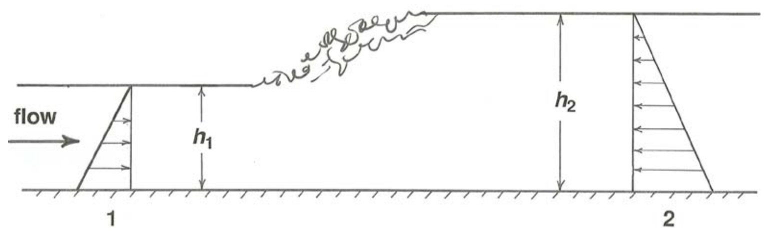

Figure 5-13. The hydraulic jump. The distribution of hydrostatic pressure is shown at section 1, upstream of the jump, and at section 2, downstream of the jump.

37 Hydraulic jumps are a striking feature of open-channel flow. You have all seen them, if only in your kitchen sink. You turn the faucet on full force, and the descending jet impinges on the bottom of the sink to form a thin, fast-moving sheet of water, with supercritical depth and velocity, that spreads out in all directions. But at a certain radius from the point of impact of the jet, which depends on the force of the down-flowing jet, the flow jumps up to a deeper and slower flow as it moves toward the drain. The jump is in the form of a steep and nearly stationary front accompanied by strong turbulence (Figure 5-13). Another situation in which a hydraulic jump commonly forms is downstream of a change from a relatively steep channel slope, with which supercritical flow is associated, to a relatively gentle channel slope, over which a uniform flow would be subcritical. If the change in slope is sufficiently rapid, the transition from supercritical flow to subcritical flow is in the form of a hydraulic jump rather than a smooth change in depth and velocity.

38 The nature of the hydraulic jump cannot be accounted for by use of the energy equation, because there is a substantial dissipation of energy owing to the turbulence associated with the jump; we need to appeal instead to conservation of momentum.

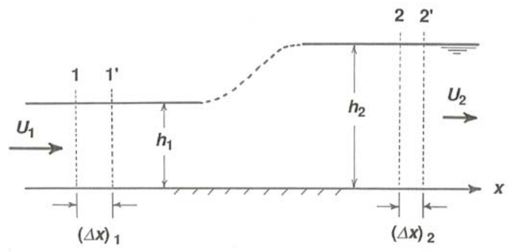

Figure 5-14. Definition sketch for deriving the moment diagram for flow through a hydraulic jump. The block of fluid contained between sections 1 and 2 at a given time is located between sections 1' and 2' a short time Δt later.

39 Figure 5-13 is a cross-section view of the flow from upstream of the hydraulic jump to downstream of it. Look at a block of the flow bounded by imaginary vertical planes at cross sections 1 and 2. The distributions of hydrostatic pressure forces are shown on the upstream and downstream faces of the block. You would have to locate section 2 quite a distance downstream of the jump, because it takes a long distance for the downstream flow to become organized. In the absence of any submerged obstacle to the flow between sections 1 and 2, the only streamwise forces on the fluid in the block are the pressure forces on the upstream and downstream faces; the hydraulic jump itself exerts no force on the flow. To see the effect of these forces, we need to do some momentum bookkeeping for use in Newton's second law, $F = ma$. For that purpose, look at Figure 5-14, a slight redrawing of Figure 5-13.

40 In a short time interval Δt, the block of fluid moves downstream from positions 1 and 2 to positions 1' and 2'. In that time it has lost momentum equal to that of the fluid that was between sections 1 and 1'. That momentum, written per unit flow width (remember that the channel is of the same width from upstream to downstream of the hydraulic jump) is $[\rho d_1(\Delta x)_1]U_1$, where U_1 is the mean velocity at section 1. This can be expressed slightly differently, keeping in mind that $U_1 = (\Delta x)_1/\Delta t$ and $q = Ud$, as $\rho d_1 U_1^2 \Delta t$, or $\rho q U_1 \Delta t$. This can be written in still another form by eliminating U_1 by use of the relationship $q = Ud$ again: $(q^2\rho/d_1)\Delta t$. Likewise, during Δt the fluid block has gained momentum equal to that of the fluid that has moved in to occupy the volume between sections 2 and 2': $(q^2\rho/d_2)\Delta t$. The change in momentum as the fluid block moves from position 1–2 to position 1'–2' is then

$$\left(\frac{q^2\rho}{h_1}\right)\Delta t - \left(\frac{q^2\rho}{h_2}\right)\Delta t \qquad (5.15)$$

or

$$\left(\frac{q^2\rho}{h_1} - \frac{q^2\rho}{h_2}\right)\Delta t \qquad (5.16)$$

The time rate of change of momentum of the fluid block is then obtained by dividing by the time interval Δt:

$$\frac{q^2\rho}{h_1} - \frac{q^2\rho}{h_2} \qquad (5.17)$$

41 By Newton's second law, we can set this rate of change of momentum equal to the net streamwise force on the fluid block, F_1 (acting in the downstream direction) minus F_2 (acting in the upstream direction). The linear distribution of hydrostatic pressure forces on the upstream and downstream faces of the fluid block make it easy to find the resultant forces F_1 and F_2:

$$F_1 = \int_0^{h_1} \rho g y\, dy = \frac{1}{2}\rho g h_1^{\,2} \qquad (5.18)$$

and likewise $F_2 = (1/2)\rho g d_2^{\,2}$. The net force on the fluid block is then

$$F1 - F2 = \frac{\rho g h_1^{\,2}}{2} - \frac{\rho g h_2^{\,2}}{2} \qquad (5.19)$$

Finally, setting this net force equal to the rate of change of momentum,

$$\frac{q^2\rho}{h_1} - \frac{q^2\rho}{h_2} = \frac{\rho g h_1^{\,2}}{2} - \frac{\rho g h_2^{\,2}}{2} \qquad (5.20)$$

We can massage this a bit to put it into a form that is more convenient for our purposes by rearranging and dividing through by ρg:

$$\left(\frac{q^2}{g h_1} + \frac{h_1^{\,2}}{2}\right) - \left(\frac{q^2}{g h_2} + \frac{h_2^{\,2}}{2}\right) = 0 \qquad (5.21)$$

What is commonly done is to define a quantity

$$M = \frac{q}{gd} + \frac{d^2}{2} \tag{5.22}$$

called the ***momentum function***. Then Equation 5.21 boils down to $M_1 - M_2 = 0$, which says that the momentum function does not change through the transition, provided that no streamwise forces other than the hydrostatic pressure forces (like resistance forces exerted by obstacles in the channel bottom) act on the fluid block.

42 Just as with the specific energy in an earlier section, we can plot a useful graph of the momentum function M against the flow depth d (Figure 5-15). And just as with the specific-energy diagram (Figure 5-7), you can verify the shape of the curve in Figure 5-15 by assuming a value for q, choosing some values for d, and computing the corresponding values of M; in this case, however, there is no unrealistic limb of the function below the $d = 0$ axis. There is a family of curves, of the general shape shown in Figure 5-15, one for each value of discharge per unit width q. As with the specific-energy diagram, all points on the upper limb of each curve, above the point of vertical tangent, represent supercritical flow, and all points on the lower limb, below the point of vertical tangent, represent subcritical flow.

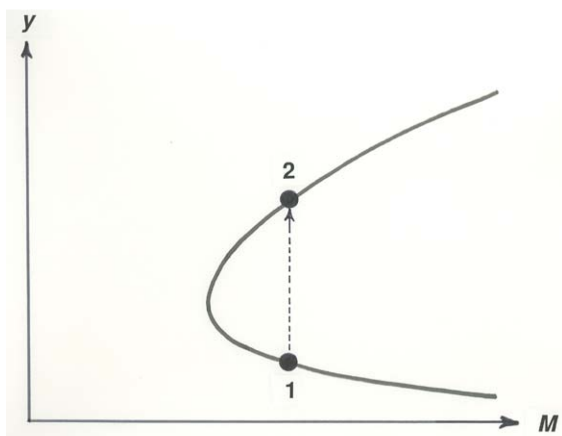

Figure 5-15. The momentum diagram: a plot of the momentum function M vs. flow depth y, shown for one of a family of curves for values of discharge per unit width, q.

43 Now we have the tools to predict the height of the hydraulic jump. We start at point 1 on the lower, supercritical limb of the curve in Figure 5-15, and jump up to point 2, at the same value of M but on the upper, subcritical limb, corresponding to the deeper, subcritical flow downstream of the hydraulic jump. You can see that the closer to the critical condition the upstream supercritical flow is, the smaller is the height of the hydraulic jump to subcritical flow, represented by the vertical distance between the respective points of intersection of the M = constant vertical line with the two limbs of the curve in Figure 5-15.

44 (Just as the shapes of the curves in the family of curves with q as the parameter in Figure 5-15 differ from the shapes of the corresponding curves in Figure 5-7, the specific-head diagram, so do the equations for the condition of critical flow—but that need not concern us here. You yourself can take the one further step, the same as for the specific-head diagram, to find the shape of the curve for critical flows in Figure 5-14, the momentum-function diagram.)

45 Finally, one incidental note is in order. The subcritical flow downstream of the jump, which emerges from the considerations above, is not exactly of the same depth and velocity as the subcritical uniform flow that is ultimately attained far downstream of the step; there is some slow further adjustment to that condition.

HYDRAULIC REGIMES OF OPEN-CHANNEL FLOW

46 Now that you know about supercritical vs. subcritical flow as well as about laminar vs. turbulent flow, various phenomena of open-channel flow can be drawn together into a single graph, to give you an idea of the wide range of hydraulic regimes of flow that can exist. Figure 5-16 is a graph of mean flow depth against mean flow velocity for steady uniform open-channel flow in a wide rectangular channel. If bed roughness is present, its height is assumed to be a small fraction of the flow depth. Both depth and velocity span several orders of magnitude, a far greater range than is found in the sediment-transporting flows encountered in natural flow environments.

47 It is easy to plot curves in Figure 5-16 corresponding to Fr = 1, for the transition between subcritical flow and supercritical flow, and to Re = 500 (Re based on flow depth), for the transition between laminar flow and turbulent flow. In a log–log plot like Figure 5-16 both of these conditions plot as straight lines; the line for Fr = 1 slopes upward to the right, and the line for Re = 500 slopes downward to the right. These two lines partition the graph into four sectors: *turbulent subcritical* in the upper left (the most common in natural open-channel flows), *turbulent supercritical* in the upper right, *laminar subcritical* in the lower left, and *laminar supercritical* in the lower right.

48 The usefulness of a graph like Figure 5-16 is that it helps to put into perspective the wide range of open-channel flows. The flow regimes shown in Figure 5-15 are just extensions of the concept of flow regimes introduced in the discussion of flow around a sphere in Chapter 3.

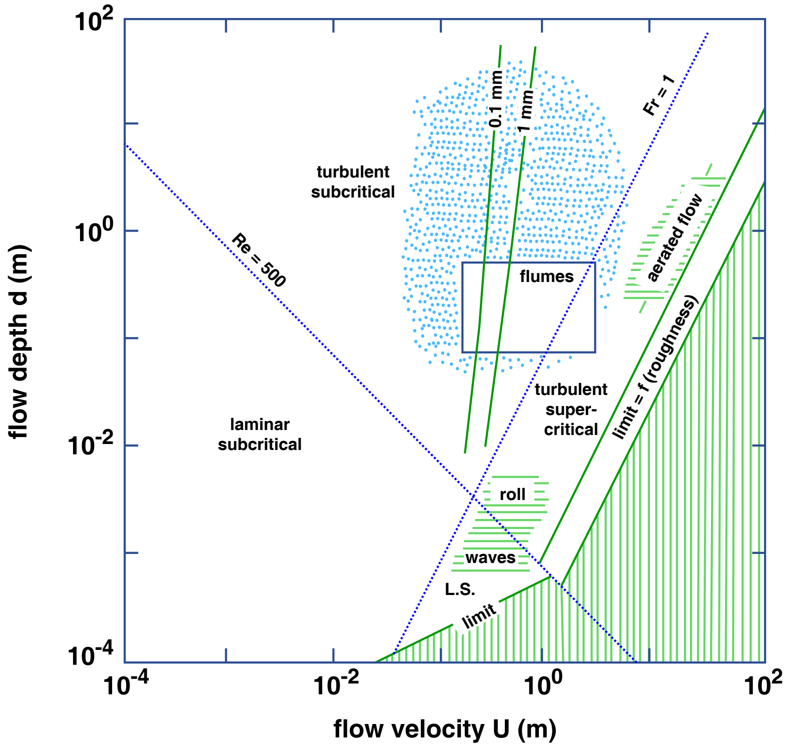

Figure by MIT OpenCourseWare.

Figure 5-16. Hydraulic regimes of open-channel flow in a graph of mean flow depth vs. mean flow velocity. See text for explanation of curves.

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49 In the lower part of Figure 5-16 are two curves (one for laminar flow and the other for turbulent flow) sloping upward to the right, below which steady, uniform open-channel flows cannot exist. These curves are defined by the condition that channel slope approaches the vertical, giving the greatest gravitational driving force possible. It is easy to get an exact solution for the curve that expresses this condition for laminar flow, by integrating Equation 4.17 to find the mean velocity U as a function of flow depth, fluid properties γ and μ, and channel slope angle ϕ, and then taking $\sin\phi = 1$, giving $U = d^2\gamma/3\mu$. This plots as a straight line in Figure 5-16. A sheet of rainwater running down a soapy windowpane is an example of flows represented by this line.

50 It is not as easy to obtain the limiting curve for turbulent flows, because we have to work with a resistance diagram like that in Figure 4-31. The curve shown in Figure 5-16 was drawn in an approximate way by obtaining the friction factor f from the smooth-flow curve in Figure 4-27 and using that value together with $\tau_o = \gamma d \sin\alpha$ in Equation 4.11. Figure 4-27 is for circular pipes, but it should be roughly applicable to open-channel flow provided that the pipe diameter is appropriately replaced. Four times the flow depth was used in place of pipe diameter in computing the Reynolds number in Figure 4-27, because as noted in Chapter 4 the hydraulic radius of a circular pipe is one-fourth the pipe diameter, whereas the hydraulic radius of a very wide open-channel flow is just about equal to the flow depth. For rough flows the limiting curve in Figure 5-15 would be displaced upward somewhat, because the friction factor is greater for a given Reynolds number.

GRADUALLY VARIED FLOW

51 Nonuniform flows for which the changes in depth and velocity are so abrupt that radial accelerations distort the vertical distribution of fluid pressure from the hydrostatic condition are called ***rapidly varied flows***. Flow over a sharp-crested dam or weir, and flow under a sluice gate, are good examples. Such flows are difficult to deal with analytically, and I will not pursue them here, although they are important in many engineering applications.

52 Nonuniform open-channel flows for which the changes in depth and velocity are slow enough in the downstream direction that the vertical distribution of fluid pressure from the free surface to the bottom is not much different from hydrostatic are called ***gradually varied flows***. An example is the flow transition over a gentle step, introduced at the beginning of this chapter (Fig. 5-3). It is completed in a sufficiently short distance that loss of flow energy by friction can be neglected, but the fluid accelerations are still sufficiently small that the vertical distribution of fluid pressure is close to being hydrostatic. In most gradually varied flows, however, the change takes place over a distance sufficiently great that we cannot assume zero energy loss due to bottom friction. The second channel-transition example posed at the beginning of this chapter falls into that category.

53 To see what happens to the elevation of the water surface through a transition over such a long distance that bottom friction cannot be neglected, we need to start with the equation for the total head at a cross section of the flow and differentiate each term with respect to distance in the flow direction. (In what follows, I am going to write *y* instead of *d* for the flow depth.) Start with Equation 5.3, written using the elevation of the channel bottom h_o (cf. Equation 5.4):

$$E_w = \frac{U^2}{2g} + y + h_o \tag{5.22}$$

Differentiate Equation 5.15 with respect to the flow direction *x*:

$$\frac{dE_w}{dx} = \frac{\mathrm{d}(U^2/2g)}{dx} + \frac{dy}{dx} + \frac{dh_o}{dx} \tag{5.23}$$

The term on the left side of Equation 5.23 is the rate of change in total energy in the downstream direction. This is always negative, because energy is inevitably lost by friction. Think in terms of the downward slope of the line formed by plotting E_w as a function of downstream distance. This slope, denoted by S_e, is what was called the ***energy slope***, or the ***energy gradient***, or the ***slope of the energy line*** earlier in this chapter. By convention, such a negative slope is considered to be positive S_e, so we replace dE_w/dx in Equation 5.23 by $-S_e$.

54 Friction loss in nonuniform flow is not well studied, but to get somewhere just in a qualitative way we can assume that the friction loss in slightly to moderately nonuniform flow is not greatly different from what it would be in uniform flow—and we have already dealt with that satisfactorily in Chapter 4. Remember the Chézy coefficient I introduced back then? According to Equation 4.20, repeated here as Equation 5.24,

$$U = C(y\sin\alpha)^{1/2} \tag{5.24}$$

Assuming that $\tan\alpha \approx \sin\alpha$, which is a very good approximation for the small angles we are dealing with here, and keeping in mind that the slope $\tan\alpha$ is just S_e, and solving for S_e,

$$S_e = \frac{U^2}{C^2 y} \tag{5.25}$$

This can be written a little more usefully by getting rid of U by use of the relation $q = Uy$; remember that the discharge per unit channel width q (which is constant along the channel) is related to the mean velocity U by this relation. Then Equation 5.25 can be written

$$S_e = \frac{q^2}{C^2 y^3} \tag{5.26}$$

55 Now for some manipulation of the right side of Equation 5.23. The first term on the right can be massaged in the following way to put it into a more useful form. In what follows, again keep in mind that the discharge per unit channel width q is related to the mean velocity U and the flow depth y by the equation $q = Uy$.

$$\begin{aligned} \frac{d}{dx}\left(\frac{U^2}{2g}\right) &= \frac{d}{dx}\left(\frac{q^2}{2gy^2}\right) \\ &= \frac{q^2}{2g}\,\frac{d}{dx}\,\left(\frac{1}{y^2}\right) \\ &= \frac{q^2}{2g}\left(\frac{-2}{y^3}\right)\,\frac{dy}{dx} \\ &= -\,\frac{q^2}{gy^3}\,\frac{dy}{dx} \end{aligned} \tag{5.27}$$

56 For later convenience, one more thing needs to be done with this result. Go back to Equation 5.11, which gives the relationship between q and y that holds when the flow is critical, and write the depth as y instead of d (just a matter of notation, as explained above), and substitute that equation into Equation 5.27. What you get is

$$\frac{d}{dx}\left(\frac{U^2}{2g}\right) = -\frac{{y_c}^3}{y^3}\frac{dy}{dx} \tag{5.28}$$

With regard to the second term on the right in Equation 5.23, we do not have to do anything further with it, because it just represents the rate of change of flow depth in the downstream direction.

57 The last term in Equation 5.23 represents the slope of the channel bottom (remember that h_o was defined as the elevation of the channel bottom), and because in the realm of channel flows the downward slope is arbitrarily defined as positive that last term can just be written $-S_o$, where S_o is the slope of the channel bottom. S_o can be written in a form that you will see is useful: think about the hypothetical uniform flow that could pass down the given bottom slope (which, remember, in reality has a nonuniform flow at some different depth passing over it). The depth of this hypothetical uniform flow over any given bottom slope is called the ***normal depth***, y_n. Just as with S_e, you can express S_o in terms of the Chézy equation by using this normal depth y_n:

$$S_o = \frac{q^2}{C^2 {y_n}^3} \tag{5.29}$$

58 So now, upon substitution of all these reworked forms of the various terms into Equation 5.23, and then bringing the terms with dy/dx to the left and the other two to the right, the equation reads as follows:

$$\frac{dy}{dx}\left(1 - \frac{{y_c}^3}{y^3}\right) = \frac{q^2}{C^2 {y_n}^3} - \frac{q^2}{C^2 y^3} \tag{5.30}$$

Rewrite the second term on the right in the form

$$\frac{q^2}{C^2\left(\frac{y^3}{{y_n}^3}\right){y_n}^3}$$

and apply to this term the expression for S_o given in Equation 5.29 to obtain

$$S_o\left(\frac{{y_n}^3}{y^3}\right)$$

and substitute that result into Equation 5.30, replacing the last term back with S_o also. Finally, solve for dy/dx to get the grand finale:

$$\frac{dy}{dx} = S_o \frac{1 - \left(\frac{y_n}{y}\right)^3}{1 - \left(\frac{y_c}{y}\right)^3} \qquad (5.24)$$

59 To get you back on the ground after that tortuous (also torturous?) exercise in manipulation (see Figure 5-17 for a summary "road map"), what Equation 5.31 does is give, for a flow that is slowly changing its depth in the downstream direction, the rate of change of depth with downstream distance, as a function of (1) the bottom slope S_o, (2) the critical depth y_c associated with the given discharge (that is, the flow depth you would see if a flow with that discharge per unit width were in the form of critical flow—which it is not), and (3) the normal depth y_n associated with the given discharge (that is, the flow depth you would see if a flow with that bottom slope and that discharge per unit width were uniform—which it is not). The only thing that stands in the way of perfection is the assumption we made that the friction loss in nonuniform flow at a given depth and discharge is the same as would be seen in the corresponding uniform flow at the same depth and discharge.

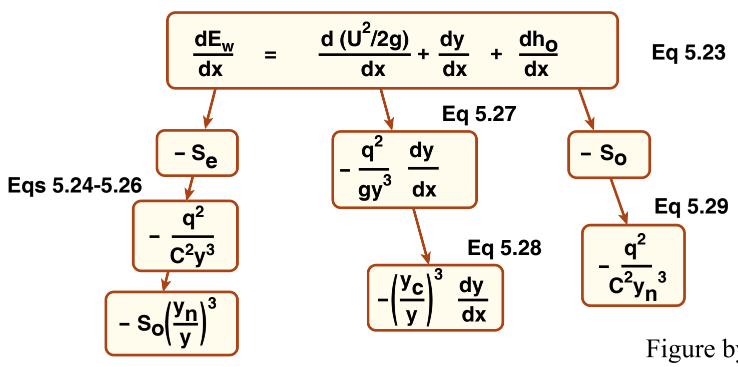

Figure by MIT OpenCourseWare.

Figure 5-17. A "road map" to aid in following the analysis of backwater curves in the text.

60 People do numerical integrations of Equation 5.31 to get approximate but reasonable water-surface profiles in real gradually varied flows. But what is also commonly done is just to use Equation 5.31 as a qualitative guide to the profile shape to be expected. We will do a little of that here, so that we can finally address the problem of what the water surface looks like as the river runs into the deep reservoir—and that is just one of the many important problems that can be attacked by this approach.

61 What you need to think about is the sign of *dy/dx* on the left side of Equation 5.31, because if *dy*/*dx* is positive then the flow depth increases downstream, and if *dy*/*dx* is negative then the flow depth decreases downstream—and this is just the information we need in order to keep track of what the water surface does relative to the channel bottom.

62 The derivative *dy*/*dx* is *positive* (meaning that the depth increases downstream) if in Equation 5.31 both the numerator and the denominator are positive or if both the numerator and the denominator are negative. Conversely, *dy*/*dx* is *negative*, and the depth decreases downstream, if the numerator and the denominator have different sign.

63 Another thing we can do is think about the conditions under which (1) *dy/dx becomes zero*, meaning that the flow approaches the uniform condition, or (2) *dy/dx approaches infinity*, meaning that the water surface gets steeper and steeper (obviously, something has to happen before it gets to be vertical!), or (3) *dy/dx becomes equal to* S_o, meaning that the water surface approaches horizontality.

64 Suppose that our river flow is subcritical, as is usually the case for large rivers, meaning that the depth is greater than critical and the velocity is less than critical. We can express this by the condition $y > y_c$. So the *denominator* in the fraction in the right side of Equation 5.31, which in the following I will call *F*, is always *less than one*. With regard to the *numerator*, you know already that whatever the actual shape of the water-surface profile, the depth must ultimately increase when the reservoir is reached, so $y > y_n$ as well. Also, because we said that the approaching river is subcritical, you know that $y_n > y_c$. You can easily convince yourself that these three inequalities guarantee that the fraction *F* must be positive and less than one, so *dy*/*dx* is positive and less than S_o, meaning that the depth gradually increases downstream.

65 As *y* gets larger and larger in the process, both the numerator and the denominator of *F* go to one, meaning that *dy*/*dx* goes to S_o, which if you think about it a little bit is the same as saying that *the water surface itself becomes horizontal*. So our conclusion is that the water-surface profile is as shown in Figure 5-18A: it is asymptotic to the uniform-flow profile upstream, and to the horizontal water surface of the reservoir downstream. This kind of curve is called a ***backwater curve***, for reasons I suppose are obvious.

66 To carry this analysis just a little further, what is the effect of assuming that the river upstream is flowing at conditions closer to being critical? You can see by inspection of the fraction *F* that as $y_n \rightarrow y_c$, *F* itself stays closer and closer to one for $y > y_n$, meaning that the transition from the almost uniform flow upstream to the horizontal reservoir level downstream is sharper and sharper (that is, it takes place over a shorter and shorter distance), until, for critical flow upstream, the river meets the reservoir at a sharp angle (Figure 5-18B)!

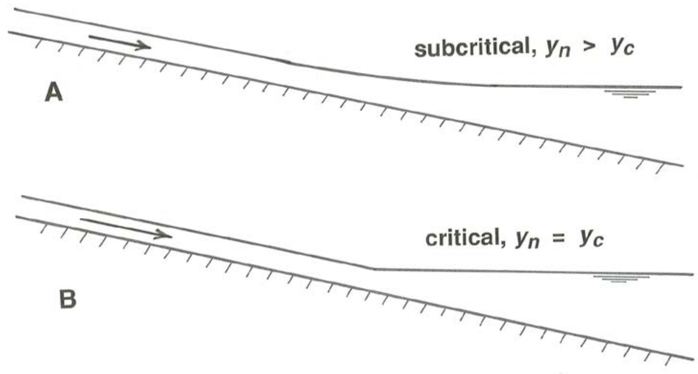

Figure 5-18. Qualitative water-surface profiles when a river in **A)** subcritical flow and **B)** supercritical flow enters a lake or a reservoir.

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67 For big rivers flowing well below the critical condition, the backwater effect is felt not just for kilometers but for tens of kilometers upstream, and the superelevation of the actual water surface above the hypothetical point of intersection between the uniform flow and the reservoir level can be many meters. You can imagine the importance of being able to predict the magnitude of this superelevation at all points upstream, when you are worrying about how many homes and farms and businesses you are going to be flooding when you build that dam.

68 I have just scratched the surface of the business of analyzing backwater effects. There are many qualitatively different kinds of backwater curves, depending upon whether the approaching flow is subcritical, critical, or supercritical, and upon whether (1) $y > y_n$ and $y > y_c$, (2) y is between y_n and y_c, or (3) $y < y_n$ and $y < y_c$.