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12.002 Physics and Chemistry of the Earth and Terrestrial Planets
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Cooling Planets

From the expression that we derived above for convective heat flow, we can calculate how fast a planet will cool due to convective circulation in its mantle. We already showed that conductive cooling can't penetrate much below several hundred kilometers depth in 4 Gy, so the main process by which planetary interiors cool has to be convection.

For mantle materials, viscosity, which shows up the Rayleigh number and the heat flow expression, is strongly dependent on temperature, and we have to take this into account when we compute the cooling. We can approximate the viscosity dependence on temperature (this is just a useful approximation in a form that will be mathematically tractable!) as:

$$\mu(T) = \mu_o e^{\gamma(T_o - T)}$$

where T is the temperature in the middle of the mantle. γ should probably be set such that a 100°C increase in temperature gives about an order of magnitude decrease in viscosity, so a value of .05 or so is reasonable. If there is a linear thermal gradient in the mantle, then T is also the average temperature of the mantle.

Next, we need to write an equation that relates the change in temperature of the mantle through time to the convective heat flow. Consider a vertical column of material with height d, surface area A and volume V. Then:

$$q(t) = -\left(\frac{1}{A}\right) \frac{\partial Q}{\partial t} + Q_r d = -\left(\frac{\rho C_p V}{A}\right) \frac{\partial T}{\partial t} + Q_r d$$

where Q_r is the heat production rate per unit volume and d is the thickness of the convecting layer. For simplicity we will assume that all the heat generated is added at the base of the convecting layer, although this is not in fact really correct and we would need to adjust for the internal generation of heat. Simplifying to remove the surface area and volume terms:

$$q(t) = -\rho C_p d \frac{\partial T}{\partial t} + Q_r d$$

Now set this equal to the expression for convective heat flow:

$$q(t) = \beta \frac{d^2 \rho^2 \Delta T^2 g \alpha C_p}{\mu(T)} = -\rho C_p d \frac{\partial T}{\partial t} + Q_r R$$

which is the same as:

$$\frac{\partial T}{\partial t} = -\beta \frac{d \rho \Delta T^2 g \alpha}{\mu(T)} + \frac{Q_r R}{\rho C_p d}$$

We will now substitute in the expression for $\mu(T)$, and so that we have an expression that is easy to solve, assume that ΔT is a constant.

(remember that ΔT is the superadiabatic temperature difference between the top and bottom of the mantle): For mantle materials, viscosity, which shows up the Rayleigh number and the heat flow expression, is strongly dependent on temperature, and we have to take this into account when we compute the cooling. We can approximate the viscosity dependence on temperature (this is just a useful approximation in a form that will be mathematically tractable!) as:

$$\mu(T) = \mu_o e^{-\gamma(T-T_o)}$$

γ should probably be such that a 100°C increase in temperature gives about an order of magnitude decrease in viscosity, so a value of .05 or so is reasonable. Then, substituting:

$$\frac{\partial T}{\partial t} = -\beta \frac{d \rho \Delta T^2 g \alpha}{\mu_o e^{\gamma(T_o-T)}} + \frac{Q_r R}{\rho C_p d}$$

Note that T is the average temperature of the mantle and ΔT is the superadiabatic temperature. If there is a linear thermal gradient in the mantle, then T is also the temperature halfway down through the mantle.

From this, we can calculate how fast a planetary mantle will cool by convection if we know the thickness of the mantle and the various other parameters. This differential equation is easily solved by rewriting and integrating in the steps below:

$$\rho C_p \mu_o e^{-\gamma(T-T_o)} \frac{\partial T}{\partial t} = -\beta d \rho^2 C_p \Delta T^2 g \alpha + Q_r (R/d) \mu_o e^{-\gamma(T-T_o)}$$

$$\left[\frac{d\rho C_p \mu_o e^{-\gamma(T-T_o)}}{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T-T_o)}} \right] \frac{\partial T}{\partial t} = -1$$

Integrate:

$$\left(\frac{\rho C_p}{\gamma Q_r} \right) \ln \left[\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T-T_o)} \right] = -t + \text{constant } t$$

Solve for the constant of integration by requiring $T=T_{in}$ at $t=0$:

$$\left(\frac{\rho C_p}{\gamma Q_r} \right) \ln \left[\frac{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T-T_o)}}{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T_{in}-T_o)}} \right] = -t$$

It might be more convenient to write this as a function of time rather than temperature, so rewriting (in many steps):

$$\left[\frac{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T-T_o)}}{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T_{in}-T_o)}} \right] = \exp \left(\frac{-t \gamma Q_r}{\rho C_p} \right)$$

$$\left(\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T-T_o)} \right) = \left(\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T_{in}-T_o)} \right) \exp \left(\frac{-t \gamma Q_r}{\rho C_p} \right)$$

$$\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - \left(\beta d^2 \rho^2 C_p \Delta T^2 g \alpha - Q_r R \mu_o e^{-\gamma(T_{in}-T_o)} \right) \exp \left(\frac{-t \gamma Q_r}{\rho C_p} \right) = Q_r R \mu_o e^{-\gamma(T-T_o)}$$

If we let T_{in} be sufficiently large, then the exponential term on the left hand side is very small and can be dropped:

$$(\beta d^2 \rho^2 C_p \Delta T^2 g \alpha) \left[1 - \exp\left(\frac{-t\gamma Q_r}{\rho C_p}\right) \right] = Q_r R \mu_o e^{-\gamma(T-T_o)}$$

$$\ln\left\{\left(\frac{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha}{Q_r R \mu_o}\right) \left[1 - \exp\left(\frac{-t\gamma Q_r}{\rho C_p}\right) \right]\right\} = -\gamma(T - T_o)$$

$$\ln\left(\frac{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha}{Q_r R \mu_o}\right) + \ln\left[1 - \exp\left(\frac{-t\gamma Q_r}{\rho C_p}\right) \right] = -\gamma(T - T_o)$$

or finally, in “convenient” form:

$$\ln\left(\frac{Q_r R \mu_o}{\beta d^2 \rho^2 C_p \Delta T^2 g \alpha}\right) - \ln\left[1 - \exp\left(\frac{-t\gamma Q_r}{\rho C_p}\right) \right] = \gamma(T - T_o)$$

To get a first estimate of what this looks like, let’s assume we are looking at planets with the same density as the earth but a different radius. In this case g will simply scale with the planetary radius. ρ and α , density and coefficient of thermal expansion, are independent of planet size:

$$g = 10m/s^2 * R/R_e$$

$$Q_R = .01\mu W/m^3 = 10^{-8} W/m^3$$

$$\kappa = 10^{-6} m^2/s$$

$$\rho = 4000kg/m^3$$

$$C_p = 1260J/kgK$$

$$K = 5W/mK$$

$$\alpha = 10^{-5} K^{-1}$$

$$\beta = .01$$

$$\gamma = .05$$

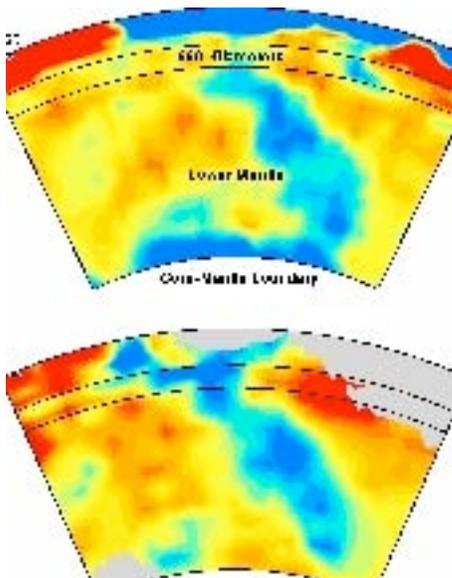
$$\mu_o = 10^{22} Pa \cdot s$$

$$T_o = 1300^\circ C$$

$$\Delta T = 10^\circ C$$

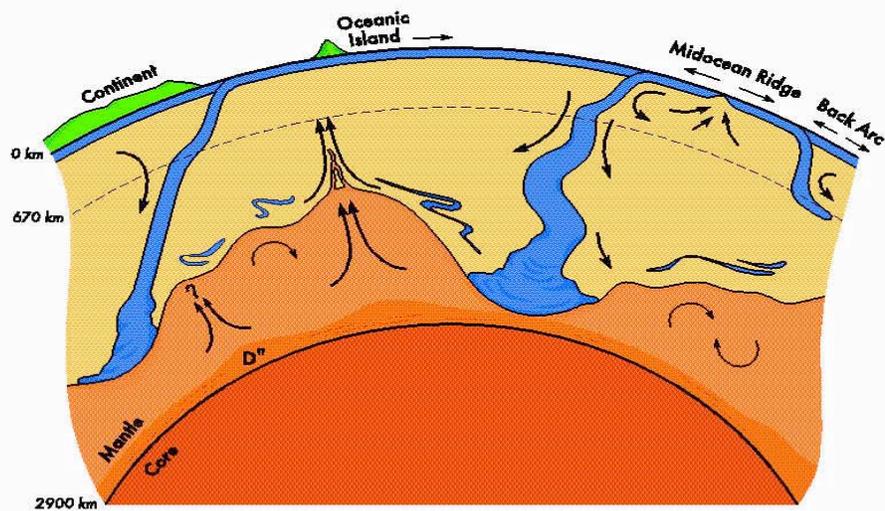
We can choose any value of d , the thickness of the convecting layer, provided that it is less than R .

Descending slab in the Earth – part of the convection system:



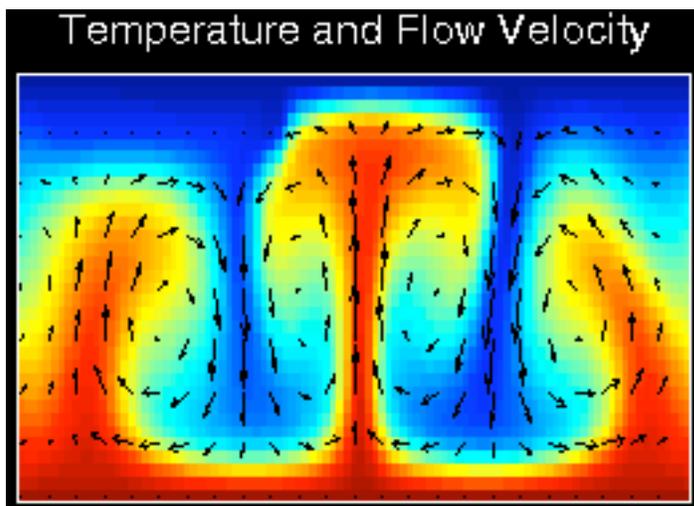
Courtesy of Prof. Robert van der Hilst at MIT. Used with permission.

For all planets:



<http://www-geology.ucdavis.edu/~kellogg/mantle.jpeg>

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