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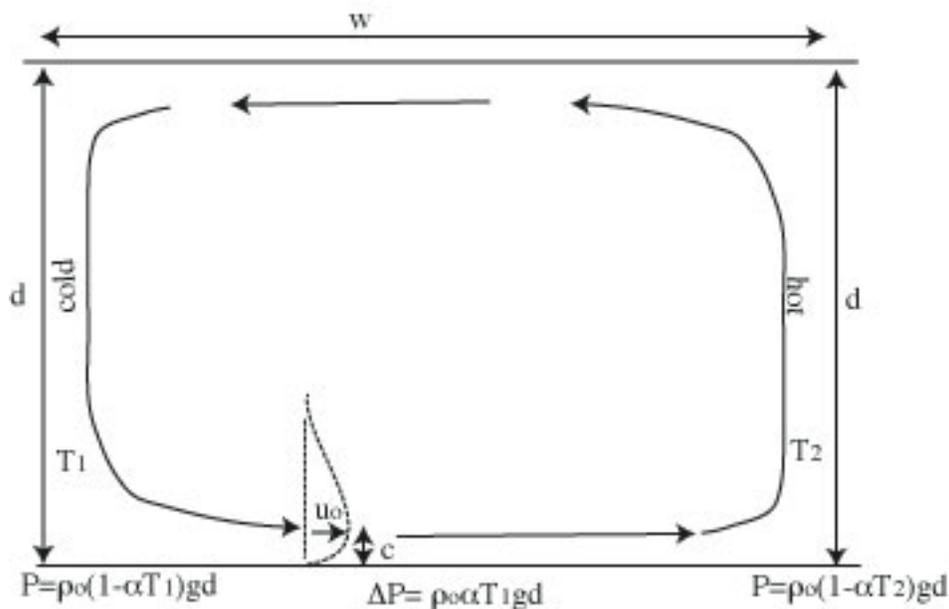
12.002 Physics and Chemistry of the Earth and Terrestrial Planets
Fall 2008

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Convective Heat Flow

In order to compute convective heat flow, averaged over the surface extent of a convecting cell, we need to calculate the difference in the heat content of the upwelling limb of the convective cell and the downwelling limb of the convective cell. We need to compute the rate at which heat is being transferred upwards and then divide by the surface area of the convective cell.

First we will calculate the rate at which a convective cell circulates. Note that all of these calculations are a bit approximate, assume that there is no significant conductive heat transfer, and don't really treat the details of the convection geometry. We are going to ignore the adiabatic gradient, and just assume that the temperature gradients that we are talking about are those in excess of the adiabat.



Consider a convective cell with an upwelling limb at temperature T_1 and a downwelling limb at temperature T_2 . The density of the upwelling and downwelling limbs are then:

$$\rho_1 = \rho_0 (1 - \alpha T_1) \quad \rho_2 = \rho_0 (1 - \alpha T_2)$$

The pressure within each limb as a function of depth is just:

$$P_1 = \rho_o (1 - \alpha T_1) g z \quad P_2 = \rho_o (1 - \alpha T_2) g z$$

Integrating over z to find the force (per unit length into the diagram) gives:

$$F_1/\text{length} = \rho_o (1 - \alpha T_1) g d^2/2 \quad F_2/\text{length} = \rho_o (1 - \alpha T_2) g d^2/2$$

The net horizontal force on the sides of the cell is then just:

$$\Delta F/\text{length} = (\rho_o \alpha \Delta T) g d^2/2$$

Because inertial terms for flow in planetary mantles (but not liquid cores!) are negligible (this will be a homework problem too), the total force on the convecting cell, or any piece of it, must be zero. Therefore this horizontal force must be balanced by shear forces applied to the top and bottom of the cell.

Suppose that the flow along the base of the cell reaches a maximum velocity of u_o at a height c above the base of the cell. Then the shear stress on the base of the cell is approximately:

$$\sigma_{xz} = - \mu u_o / c$$

The horizontal material flux within this basal channel will also scale with both u_o and c , so:

$$\text{flux}/\text{length} = \beta' c u_o$$

where β' is some dimensionless constant that describes the shape of the flow. Multiplying this by the width of the cell, w , gives the total force per unit length on the base of the cell:

$$\Delta F/\text{length} = - \mu w u_o / c$$

If there is a fixed lithosphere on top, we need to do this for the top of cell as well, so the value should be more or less doubled). Setting the total force on the cell equal to zero gives:

$$(\rho_o \alpha \Delta T) g d^2/2 - \mu w u_o / c = 0$$

and solving for u_o gives:

$$u_o = (\rho_o \alpha \Delta T) g (cd^2/w) / 2\mu$$

The amount of heat, Q_2 , carried upwards in the upwelling limb in an amount of time Δt (and per unit length into the diagram) is just the flux times the heat per unit volume:

$$Q_2/\text{length} = (\beta' c u_o) (T_2 \rho C_p) \Delta t$$

and similarly the amount of heat carried downward in the down-welling limb is:

$$Q_1/\text{length} = (\beta' c u_o) (T_1 \rho C_p) \Delta t$$

Thus the net heat (per unit length) carried upward in a time Δt is just the difference between the two quantities. The average heat flow out of the surface of the cell is equal to this difference divided by Δt and divided by the surface area of the cell (or width of the cell, since we have computed Q per unit length into the diagram).

$$q_{av} = (\beta' c u_o) (T_2 - T_1) (\rho C_p) / w$$

substituting in for u_o we find:

$$q_{av} = (\beta' c^2 / 2w^2) (d^2 \rho^2 \Delta T^2 g \alpha C_p / \mu)$$

The first term in parentheses is dimensionless and reflects the geometry of the flow, both in terms of the cell width, channel depth, whether there is a fixed lithosphere on top of the cell and a number of other geometrical factors that we have glossed over. If we write this dimensionless number as β , then:

$$q_{convective} = \beta (d^2 \rho^2 \Delta T^2 g \alpha C_p / \mu)$$

(Because β incorporates the term $(c/w)^2$, it will typically be much less than one. We will give a default value to use for doing the homework problems.)

Remember that ΔT is the temperature difference between the top and bottom of the cell corrected for the adiabatic gradient (also called the “super-adiabatic temperature”). We will leave it to the homework to show that, ignoring the adiabat, the ratio between the purely convective heat flow and the purely conductive heat flow computed for the same layer is the Rayleigh number times a dimensionless constant. This is another way of thinking about the Rayleigh number, although not a very conventional one.

Cooling Planets

From the expression that we derived above for convective heat flow, we can calculate how fast a planet will cool due to convective circulation in its mantle. We already showed that conductive cooling can't penetrate much below several hundred kilometers depth in 4 Gy, so the main process by which planetary interiors cool has to be convection.

For mantle materials, viscosity, which shows up the Rayleigh number and the heat flow expression, is strongly dependent on temperature, and we have to take this into account when we compute the cooling. We can approximate the viscosity dependence on temperature (this is just a useful approximation in a form that will be mathematically tractable!) as:

$$\mu(T) = \mu_0 e^{\gamma(T_0 - T)}$$

where T is the temperature in the middle of the mantle. γ should probably be set such that a 100°C increase in temperature gives about an order of magnitude decrease in viscosity, so a value of .05 or so is reasonable. If there is a linear thermal gradient in the mantle, then T is also the average temperature of the mantle.

Next, we need to write an equation that relates the change in temperature of the mantle through time to the convective heat flow. Consider a vertical column of material with height d , surface area A and volume V . Then:

$$q(t) = -\left(\frac{1}{A}\right) \frac{\partial Q}{\partial t} + Q_r d = -\left(\frac{\rho C_p V}{A}\right) \frac{\partial T}{\partial t} + Q_r d$$

where Q_r is the heat production rate per unit volume and d is the thickness of the convecting layer. For simplicity we will assume that all the heat generated is added at the base of the convecting layer, although this is not in

fact really correct and we would need to adjust for the internal generation of heat. Simplifying to remove the surface area and volume terms:

$$q(t) = -\rho C_p d \frac{\partial T}{\partial t} + Q_r d$$

Now set this equal to the expression for convective heat flow:

$$q(t) = \beta \frac{d^2 \rho^2 \Delta T^2 g \alpha C_p}{\mu(T)} = -\rho C_p d \frac{\partial T}{\partial t} + Q_r d$$

which is the same as:

$$\frac{\partial T}{\partial t} = -\beta \frac{d\rho \Delta T^2 g \alpha}{\mu(T)} + \frac{Q_r}{\rho C_p}$$

We will now substitute in the expression for $\mu(T)$, and so that we have an expression that is easy to solve, assume that ΔT is a constant (remember that ΔT is the superadiabatic temperature difference between the top and bottom of the mantle):

$$\frac{\partial T}{\partial t} = -\beta \frac{d\rho \Delta T^2 g \alpha}{\mu_o e^{\gamma(T_o - T)}} + \frac{Q_r}{\rho C_p}$$

From this, we can calculate how fast a planetary mantle will cool by convection if we know the thickness of the mantle and the various other parameters.