12.002 Physics and Chemistry of the Earth and Terrestrial Planets
Fall 2008

Elastic Stress Strain Relations
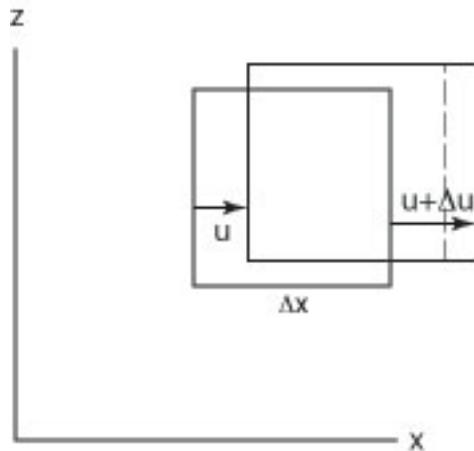
In elastic materials, stress is linearly related to strain, as opposed to viscous materials, where stress is related to strain rate.

What is strain?

Consider a cube with sides of lengths Δx, Δy, Δz (all equal). Suppose that the cube is translated by an amount (u,v,w) and at the same time stretched, only in the x-direction, by an amount Δu.

Normal strains are defined as change in length divided by the original length. So here, normal strain in the x-direction, written ε_{xx}, is:

$$\varepsilon_{xx} = \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$$

And similarly for the normal strain in the y-direction:

$$\varepsilon_{yy} = \frac{\Delta v}{\Delta y} = \frac{\partial v}{\partial y}$$

If a normal strain is positive, then the material is stretching (lengthening) in that direction; if negative, then the material is shortening in that direction.

Shear strains measure the extent to which a marker box, originally with right angle corners, is changed in shape so that its corners are no longer right angles. Consider the same box originally with sides (Δx, Δy, Δz) in length.
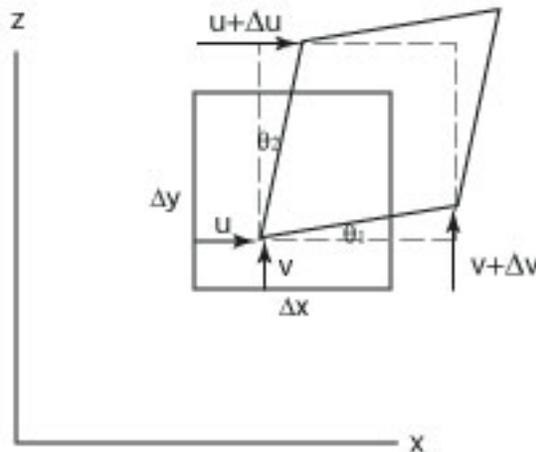
The shear strain, written $\varepsilon_{xy} = \varepsilon_{xy}$ is defined as the sum of the angles θ_1 and θ_2 divided by 2. Noting that for small strains, which we are considering here, $\theta_1 = \tan(\theta_1) = \Delta u/\Delta y$ and $\theta_2 = \tan(\theta_2) = \Delta v/\Delta x$, then:

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

There are of course comparable shear strains and normal strains that involve the z-dimension, but we will confine ourselves to two dimensions in the x-y plane.

You will remember from your physics course, Hooke's law for a spring, which says that:

F=-kx

In thinking about the earth, we will use the same relationship, except that we will consider force per unit area (stress) and strain rather than force and displacment. First we need to define normal and shear stresses:
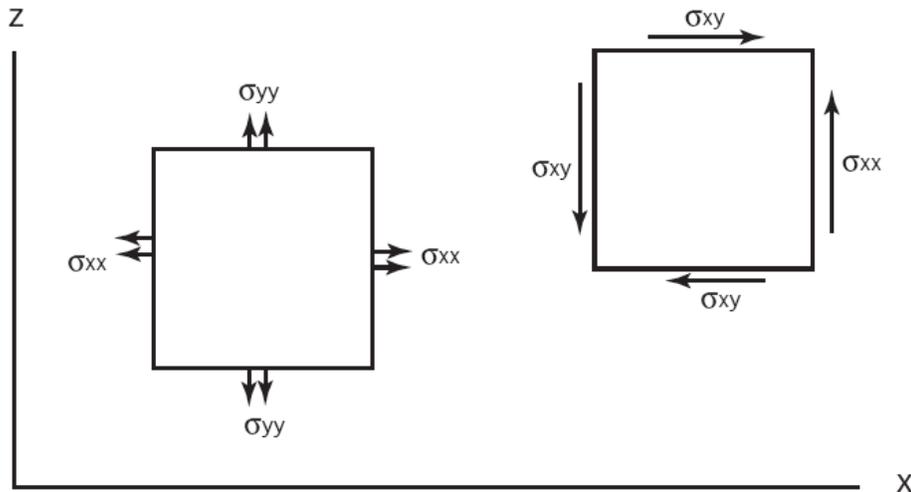
The diagram above shows the normal stresses and shear stresses acting on a small cube within a deforming medium. The normal stress σ_{xx} is defined as the stress (positive out) corresponding to a force in the x-direction acting on the plane normal to the x-axis, as shown. Similarly for the normal stress σ_{yy}. The shear stress, σ_{xy}, is defined as the stress corresponding to the force in the x-direction acting on a plane perpendicular to the y-axis, and vice versa for σ_{yx}. However, σ_{xy} and σ_{yz} must be or else the cube will be undergoing rotational acceleration, so we can write both shear stresses as σ_{xy}.

Now we can take advantage of Hooke's law, that relates force and displacement in an elastic material, except that we are going to use a two-dimensional version that relates stress and strain.

$$\sigma_{xx} = a_1 \varepsilon_{xx} + a_2 \varepsilon_{xy} + a_3 \varepsilon_{yy}$$

$$\sigma_{xy} = b_1 \varepsilon_{xx} + b_2 \varepsilon_{xy} + b_3 \varepsilon_{yy}$$

$$\sigma_{yy} = c_1 \varepsilon_{xx} + c_2 \varepsilon_{xy} + c_3 \varepsilon_{yy}$$

Although there a 9 constants of proportionality here, we can get rid of all but two of them by requiring the expression to hold true under rotation of the x-y axes – the expression wouldn't be useful if it held in only one direction, at least for an isotropic material. So, considering a rotation of axis by 90°, it is clear that the proportionality constant between σ_{xx} and ε_{xx} has to be the same

as that between σ_{yy} and ε_{yy}, etc. So right away we can simplify the expression a bit:

$$\sigma_{xx} = a_1\varepsilon_{xx} + a_2\varepsilon_{xy} + a_3\varepsilon_{yy}$$

$$\sigma_{xy} = b_1\varepsilon_{xx} + b_2\varepsilon_{xy} + b_3\varepsilon_{yy}$$

$$\sigma_{yy} = a_3\varepsilon_{xx} + c_2\varepsilon_{xy} + a_1\varepsilon_{yy}$$

It is easy to show, but a bit messy, that if we rotate our axes by an arbitrary angle θ, that the relation between stress and strain is identical under rotation only if most of the constants of proportionality are related to one another, with only two independent terms. It is common practice to call these two terms the Lame coefficients, written λ and μ, so that:

$$\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy}$$

$$\sigma_{xy} = \mu\varepsilon_{xy}$$

$$\sigma_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy}$$

μ is called the shear modulus because it relates shear stress to shear strain. λ doesn't have a particular name (that I know of) but it and μ together can be related to the bulk modulus (K). The bulk modulus relates the change in volume of a unit cube of material to the (negative) pressure applied to the cube of material. Of course, we have to do this in three dimensions. First we note that pressure is just the average normal stress applied to all the surfaces of the cube, so:

$$-P = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$$

Then we note that the volume increase of the unit cube can be related to the normal stresses by:

$$\Delta V = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz}) - 1 \approx \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

Finally, expanding our stress to strain expressions to three dimensions for normal stress/strain gives:

$$\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz}$$

$$\sigma_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz}$$

$$\sigma_{zz} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz}$$

Adding all three equations gives:

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = (3\lambda + 2\mu)(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$

or

$$-P = \Delta V(\lambda + 2\mu/3) = K\Delta V$$

so that the bulk modulus K is related to the lame coefficients by:

$$K = (\lambda + 2\mu/3)$$

The Seismic Wave Equation

Seismic waves travel by elastic deformation of the crust, mantle and core, on timescales that are very short (fractions of seconds) compared to the timescales of geologic strain (millions of years).

We can derive the seismic wave equation from the stress-strain relationship in an elastic material, with Lame coefficients μ and λ. This can be done in the general 3D case, and is by real seismologists, but here we will just derive the wave equation for simple plane P and S waves. Let's start with P-waves, and consider a wave traveling in the x-direction. The wave train consists of a energy packet that travels at velocity v_p; within the energy packet, the elastic material vibrates back and forth in the x-direction. For a plane wave, there is no variation in displacement or velocity (or anything else) in the y- and z- directions. Therefore we can consider all derivatives with respect to y and z to be zero.

The basic physics boils down to F=ma. Remember that on geological timescales, acceleration and momentum terms in any force balance are effectively zero (outside of the core), but for very short-term deformation, the acceleration is important (otherwise we could not have waves). Let's start with F=ma, but write this in terms of the forces acting on a cube of material with dimensions Δx, Δy and Δz. Thus:

$$\left[\sigma_{xx}(x + \Delta x)\Delta y\Delta z\right] - \left[\sigma_{xx}(x)\Delta y\Delta z\right] = (\rho\Delta x\Delta y\Delta z)\left(\frac{\partial^2 u}{\partial t^2}\right)$$

or, dividing through by Δx, Δy and Δz and taking the limit as Δx becomes small:

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho\left(\frac{\partial^2 u}{\partial t^2}\right)$$

Now let's replace σ_{xx} by the normal strains using the equations derived in the preceding section:

$$\frac{\partial}{\partial x}\left[(\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy}\right] = \rho\left(\frac{\partial^2 u}{\partial t^2}\right)$$

We also know how to relate ε_{xx} and ε_{yy} to derivatives of displacement, from the preceding section. Making that substitution gives:

$$\frac{\partial}{\partial x}\left[(\lambda + 2\mu)\left(\frac{\partial u}{\partial x}\right) + \lambda\left(\frac{\partial v}{\partial y}\right)\right] = \rho\left(\frac{\partial^2 u}{\partial t^2}\right)$$

This can be further simplified by noting that μ and λ are constants and remembering that for this particular type of plane wave, all derivatives with respect to y are zero. So:

$$(\lambda + 2\mu)\left(\frac{\partial^2 u}{\partial x^2}\right) = \rho\left(\frac{\partial^2 u}{\partial t^2}\right)$$

Finally, we can divide to get:

$$\left[\frac{(\lambda + 2\mu)}{\rho}\right]\left(\frac{\partial^2 u}{\partial x^2}\right) = \left(\frac{\partial^2 u}{\partial t^2}\right)$$

This is the wave equation for a plane P-wave traveling in the x-direction. It should remind you of the equation of motion for a harmonic oscillator. From this, we can easily find the velocity with which the wave propagates through the material; remember that this is the group velocity and is not the

same as the rate of movement of the material at any one point. Suppose that we have a (non-dispersive) wave traveling at velocity v_p. Then the form of the wave will $u(x-v_p t)$ where u is some function that describes the displacement due to the wave. Substituting this into the equation above, we find:

$$\left[\frac{(\lambda + 2\mu)}{\rho}\right]u''(x - v_p t) = v_p^2 u''(x - v_p t)$$

or:

$$v_p = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$$

Using essentially the same procedure, it is easy to show (you get to do it for homework) that the velocity of a planar S-wave is:

$$v_s = \sqrt{\frac{\mu}{\rho}}$$

Although we have derived the wave equation only for simple planar waves, the P- and S- wave velocities that we have obtained are valid for all seismic waves. You will sometimes see the P-wave velocity written in terms of the bulk modulus, K, instead of the Lame coefficient λ:

$$v_p = \sqrt{\frac{(K + 4\mu/3)}{\rho}}$$