3 Spatial Averaging

- 3-D equations of motion
- Scaling => simplifications
- Spatial averaging
- Shear dispersion
- Magnitudes/time scales of diffusion/dispersion
- Examples

Navier-Stokes Eqns

Conservative form momentum eqn; x-component only

$$\frac{\partial u}{\partial t}+\underbrace{\frac{\partial}{\partial x}(u^2)+\frac{\partial}{\partial y}(uv)+\frac{\partial}{\partial z}(uw)}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\underbrace{\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}$$

1 2 3 4 5

1 " storage" or local acceleration

2 advective acceleration

3 Coriolis acceleration [f = Coriolis param = 2ωsin(θ), θ = latitude]

4 pressure gradient

5 viscous stress

Turbulent Reynolds Eqns

$u = \bar{u} + u'$, etc $\quad (\bar{u} = \text{time average})$

Insert & time average (over bar)

$$\overline{\frac{\partial}{\partial x}\left[(\bar{u}+u')(\bar{u}+u')\right]} = \frac{\partial}{\partial x}\bar{u}^2 + \frac{\partial}{\partial x}\overline{u'^2}$$

e.g., x-component; term 2

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}$$

Continuity

$$\frac{\partial \bar{u}}{\partial t} + \underbrace{\frac{\partial}{\partial x}(\bar{u}^2) + \frac{\partial}{\partial y}(\bar{u}\bar{v}) + \frac{\partial}{\partial z}(\bar{u}\bar{w})} - f\bar{v}$$

x-momentum

1 2a 3

$$= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(\upsilon\frac{\partial \bar{u}}{\partial x} - \overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\upsilon\frac{\partial \bar{u}}{\partial y} - \overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\upsilon\frac{\partial \bar{u}}{\partial z} - \overline{u'w'}\right)$$

4 5 2b

Specific terms

Term 2a: could subtract $\overline{u}$ times continuity eqn

$$= \overline{u}\left[\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z}\right]$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial x} + \overline{w}\frac{\partial \overline{u}}{\partial x}$$

NC form of momentum eqn (term 2)

Terms 5 + 2b

x-comp of turbulent shear stress

$$\upsilon\frac{\partial \overline{u}}{\partial x} - \overline{u'^2} = \tau_{xx} \cong -\overline{u'^2} \cong 2\varepsilon_{xx}\frac{\partial \overline{u}}{\partial x}$$

$$\upsilon\frac{\partial \overline{u}}{\partial y} - \overline{u'v'} = \tau_{xy} \cong -\overline{u'v'} \cong \varepsilon_{xy}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)$$

$$\upsilon\frac{\partial \overline{u}}{\partial z} - \overline{u'w'} = \tau_{xz} \cong -\overline{u'w'} \cong \varepsilon_{xz}\left(\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{w}}{\partial x}\right)$$

ε_{xx}, ε_{xy}, ε_{xz} are turbulent (eddy) kinematic viscosities resulting from closure model (like E_{xx}, E_{xy}, E_{xz})

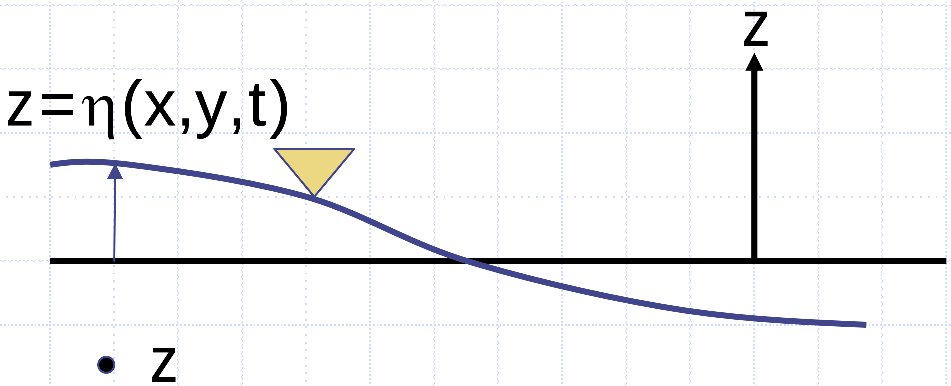
Specific terms, cont'd

Similar eqns for y and z except z has gravity.

For nearly horizontal flow (w ~ 0), z-mom => hydrostatic

$$0 = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} - g$$

4 6



$$\overline{p} = p_a + \int_z^{\eta} \rho g dz$$

Use in x and y eqns

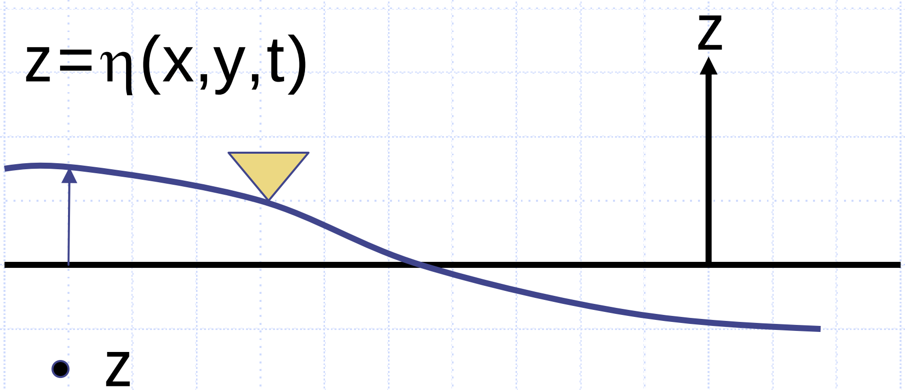
Specific terms, cont'd

Term 4

$$\frac{\partial \overline{p}}{\partial x} = \frac{\partial}{\partial x}\left[p_a + \int_z^{\eta} \rho g dz \right]$$

$$= \frac{\partial p_a}{\partial x} + \frac{\partial \eta}{\partial x}\rho_s g + \int_z^{\eta} \frac{\partial \rho}{\partial x} g dz$$

4a 4b 4c



4a atmospheric pressure gradient (often negligible)

4b barotropic pressure gradient (barotropic => $\rho = \rho_s$ = const)

4c baroclinic pressure gradient (baroclinic => density gradients; often negligible)

Simplification

Neglect ε_{xx}; $\varepsilon_{xy} = \varepsilon_h$; $\varepsilon_{zx} = \varepsilon_z$

Neglect all pressure terms except barotropic

And drop over bars

$$\underbrace{\frac{\partial u}{\partial t}}_{1}+\underbrace{\frac{\partial}{\partial x}(u^2)+\frac{\partial}{\partial y}(uv)+\frac{\partial}{\partial z}(uw)}_{2a}-\underbrace{fv}_{3}=\underbrace{-g\frac{\partial \eta}{\partial x}}_{4b}+\underbrace{\frac{\partial}{\partial y}\left(\varepsilon_h\frac{\partial u}{\partial y}\right)}_{2b1}+\underbrace{\frac{\partial}{\partial z}\left(\varepsilon_z\frac{\partial u}{\partial z}\right)}_{2b2}$$

Contrast with mass transport eqn

$$\underbrace{\frac{\partial \bar{c}}{\partial t}}_{1}+\underbrace{\frac{\partial}{\partial x}(uc)+\frac{\partial}{\partial y}(vc)+\frac{\partial}{\partial z}(wc)}_{2a}=\underbrace{\frac{\partial}{\partial x}\left(E_x\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial y}\left(E_y\frac{\partial c}{\partial y}\right)}_{2b1}+\underbrace{\frac{\partial}{\partial z}\left(E_z\frac{\partial c}{\partial z}\right)}_{2b2}\underbrace{\pm\sum r}_{7}$$

Pressure gradient (4) in momentum eqn => viscosity (2b) not always important to balance advection (2a); depends on shear (separation). For mass transport, diffusivity (2b) always needed to balance advection (2a)

Comments

- 3D models include continuity + three components of momentum eq (z may be hydrostatic approx) + n mass transport eqns
- Above are primitive eqs (u, v, w); sometimes different form, but physics should be same
- Sometimes further simplifications
- Spatial averaging => reduced dimensions

Further possible simplifications

Neglect terms 2a and 2b1

$$\frac{\partial u}{\partial t} + - fv = -g\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial z}\left(\varepsilon_z \frac{\partial u}{\partial z}\right)$$

Linear shallow water wave eqn

Also neglect term 1

$$-\ fv = -g\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial z}\left(\varepsilon_z \frac{\partial u}{\partial z}\right)$$

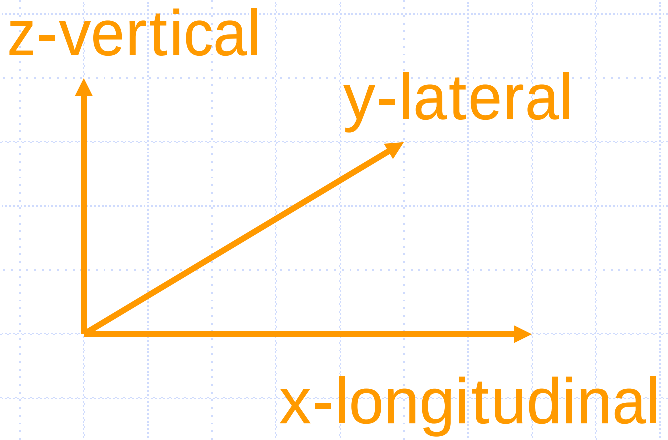
Steady Ekman flow

Also neglect term 2b2

$$-\ fv = -g\frac{\partial \eta}{\partial x}$$

Geostrophic flow

Spatial averaging



3-D equations

$\phi(x,y,z,t)$ ocean

2-D vertical average

$\phi(x,y,t)$ shallow coastal; estuary

2-D lateral average

$\phi(x,z,t)$ long reservoir; deep estuary/fjord

1-D vertical & lateral average

$\phi(x, t)$ river; narrow/shallow estuary

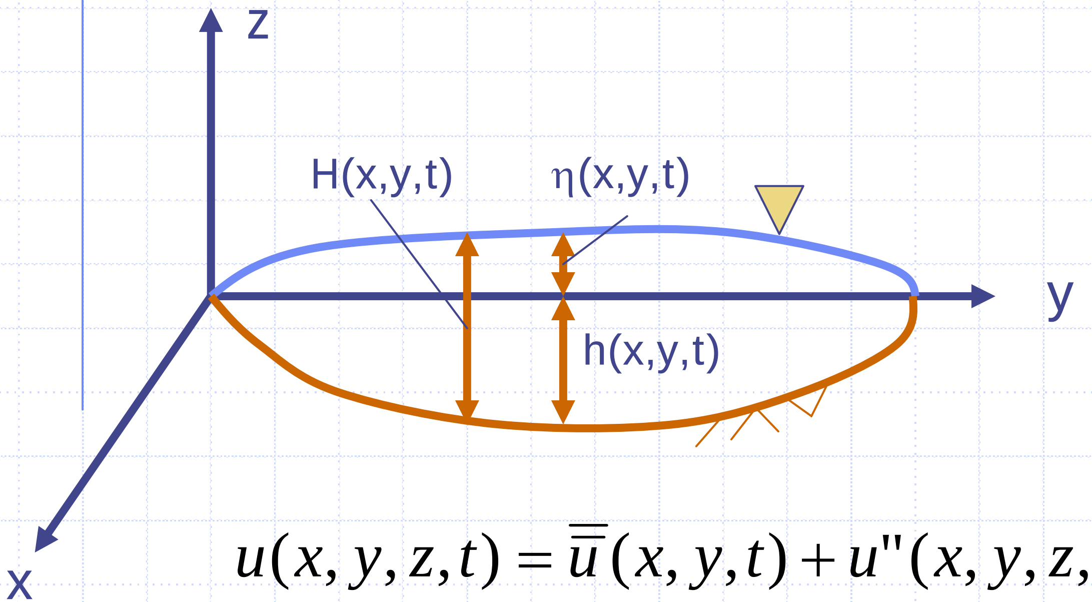
1-D horizontal average

$\phi(z, t)$ deep lake/reservoir ocean

Comments

- Models of reduced dimension achieved by spatial averaging or direct formulation (advantages of both)
- Demonstration of vertical averaging (integrate over depth then divide by depth, leading to 2D depth-averaged models)
- Discussion of cross-sectional averaging (river models)

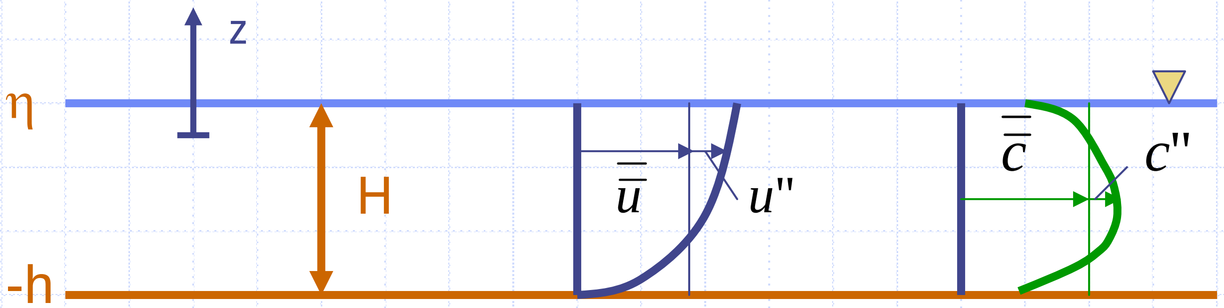
Vertical Integration => 2D (depth-averaged) eqns



$$u(x,y,z,t) = \overline{\overline{u}}(x,y,t) + u''(x,y,z,t)$$

$$u(x,y,z,t) = U_h(x,y,t) + u''(x,y,z,t)$$ (notes use)

Vertical Integration, cont'd



$$\overline{u}(x,y,t) = \frac{1}{H}\int_{-h}^{\eta} u(x,y,z,t)\,dz$$

In analogy with Reynolds averaging, decompose velocities and concentrations into $\overline{u} + u''$, $\overline{c} + c''$, etc. and spatially average

Depth-averaged eqns

Continuity

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\overline{u}H) + \frac{\partial}{\partial y}(\overline{v}H) = 0$$

from $\frac{\partial w}{\partial z}$ + kinematic surface bc of $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$

Mass and Momentum

straight forward except for NL terms

$$\int_{-h}^{\eta} \frac{\partial}{\partial x}(u^2)dz = \frac{\partial}{\partial x}\int \left(\overline{u} + u''\right)\left(\overline{u} + u''\right)dz = \frac{\partial}{\partial x}\int \overline{u}^2 dz + 2\int \overline{u}u''\,dz + \frac{\partial}{\partial x}\int u''^2\,dz$$

$$= \frac{\partial}{\partial x}\left(\overline{u}^2 H\right) + \frac{\partial}{\partial x}\left(\overline{\overline{u''^2}} H\right)$$

momentum dispersion

$$\int_{-h}^{\eta} \frac{\partial}{\partial x}(uc)dz = \frac{\partial}{\partial x}\left(\overline{u}\,\overline{c}H\right) + \frac{\partial}{\partial x}\left(\overline{\overline{u''c''}}H\right)$$

mass dispersion

Depth-averaged eqns, cont'd

x-momentum

$$\frac{\partial}{\partial t}\left(\bar{\bar{u}}H\right)+\frac{\partial}{\partial x}\left(\bar{\bar{u}}^{2}H\right)+\frac{\partial}{\partial y}\left(\bar{\bar{u}}\bar{\bar{v}}H\right)-f\bar{\bar{v}}H=-gH\frac{\partial \eta}{\partial x}$$

$$+\frac{\partial}{\partial x}\left[H\varepsilon_{L}\frac{\partial \bar{\bar{u}}}{\partial x}\right]+\frac{\partial}{\partial y}\left[H\varepsilon_{T}\frac{\partial \bar{\bar{v}}}{\partial x}\right]+\frac{\tau_{sx}}{\rho}-\frac{\tau_{bx}}{\rho}$$

$$\frac{\partial}{\partial x}\left[H\overline{\overline{\varepsilon_{x}\frac{\partial u}{\partial x}}}-\overline{\overline{u''^{2}}}H\right] \qquad \frac{\partial}{\partial y}\left[H\overline{\overline{\varepsilon_{y}\frac{\partial u}{\partial y}}}-\overline{\overline{u''v''}}H\right]$$

Depth-ave long. diff

Long. dispersion

Depth integrated eqns, cont'd

Mass Transport

$$\frac{\partial}{\partial t}\left(\bar{\bar{c}}H\right)+\frac{\partial}{\partial x}\left(\bar{\bar{u}}\bar{\bar{c}}H\right)+\frac{\partial}{\partial y}\left(\bar{\bar{v}}\bar{\bar{c}}H\right)=$$

$$+\underbrace{\frac{\partial}{\partial x}\left[HE_L\frac{\partial\bar{\bar{c}}}{\partial x}\right]}\quad+\quad\underbrace{\frac{\partial}{\partial y}\left[HE_T\frac{\partial\bar{\bar{c}}}{\partial y}\right]}+E_z\left.\frac{\partial c}{\partial z}\right|_s-E_z\left.\frac{\partial c}{\partial z}\right|_b$$

$$\frac{\partial}{\partial x}\left[\overline{\overline{HE_x\frac{\partial c}{\partial x}}}-\overline{\overline{u''c''}}H\right]\qquad\frac{\partial}{\partial y}\left[\overline{\overline{HE_y\frac{\partial c}{\partial y}}}-\overline{\overline{v''c''}}H\right]$$

Depth-ave
long. diff

Long dispersion

Comments

- ε_L, ε_T, E_L , E_T are longitudinal and transverse momentum and mass shear viscosity/dispersion coefficients.
- $\varepsilon_L >> \varepsilon_T$ and $E_L >> E_T$, but relative importance depends on longitudinal gradients
- Dispersion process represented as Fickian (explained shortly)

Boundary Conditions: momentum

$$\frac{\tau_{sx}}{\rho} = C_D \sqrt{U_w^{\,2} + V_w^{\,2}}\, U_w$$

Surface shear stress due to wind (components U_w and V_w); external input (unless coupled air-water model)

Assumes $U_w >> u_s$; C_D = drag coefficient ~ 10^{-3} (more in Ch 8)

$$\frac{\tau_{bx}}{\rho} = C_f \sqrt{\overline{\overline{u}}^{\,2} + \overline{\overline{v}}^{\,2}}\, \overline{\overline{u}}$$

$$\cong u_*^{\,2}$$

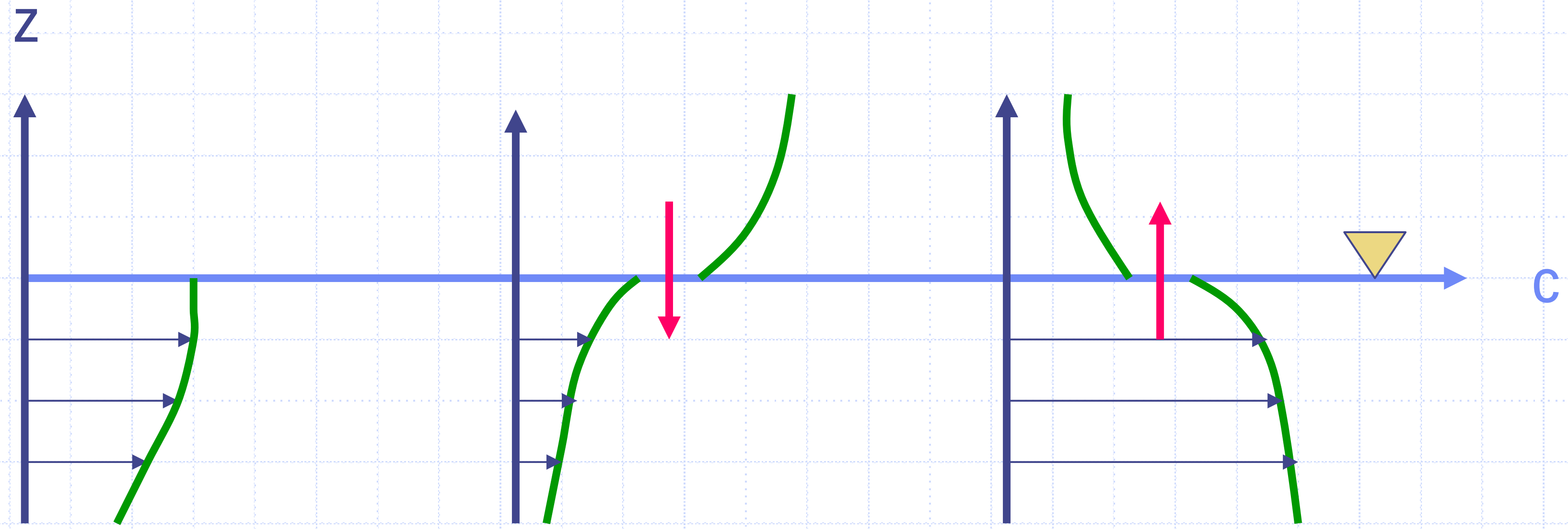
Bottom shear stress caused by flow (computed by model)

Different models for C_f (Darcy-Weisbach f; Manning n; Chezy C), e.g., $\tau_{bx} = \frac{f}{8} \rho \overline{\overline{u}}^{\,2}$

Boundary Conditions: mass transport

$$E_z \left.\frac{\partial c}{\partial z}\right|_s$$

Surface mass transfer
(air-water exchange)



$$\left.\frac{\partial c}{\partial z}\right|_s = 0$$

No flux
(dye, salt)

$$\left.\frac{\partial c}{\partial z}\right|_s > 0$$

Source
(DO)

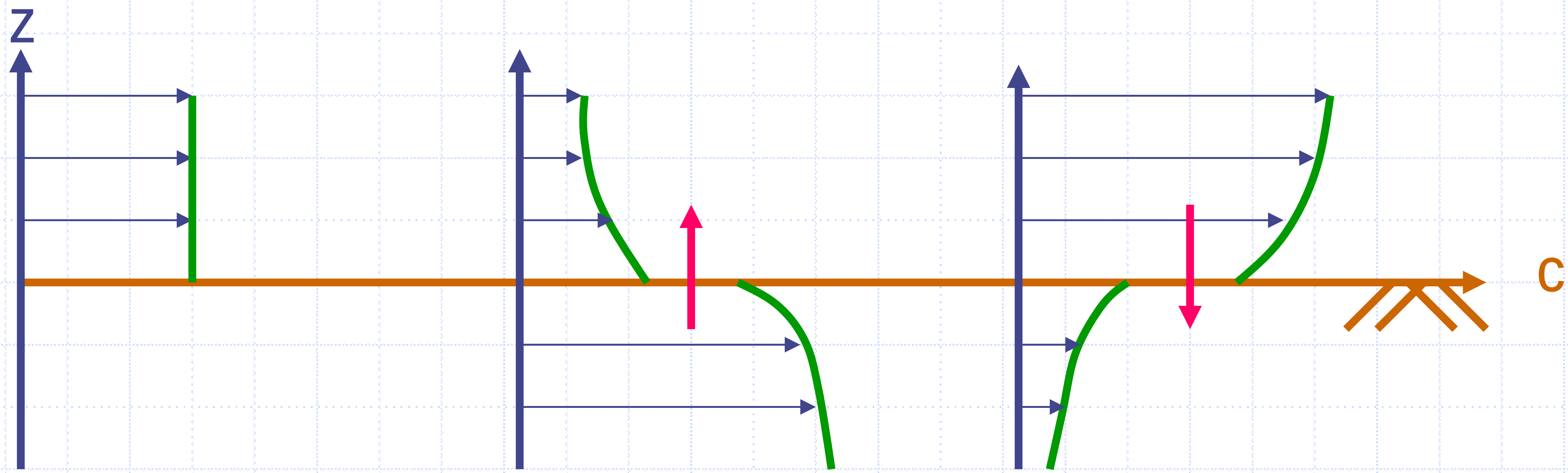
$$\left.\frac{\partial c}{\partial z}\right|_s < 0$$

Sink
(VOC)

Boundary Conditions: mass transport

$$-E_z \left.\frac{\partial c}{\partial z}\right|_b$$

Benthic mass transfer
(sediment-water exchange)



$$-\left.\frac{\partial c}{\partial z}\right|_b = 0$$

No flux
(dye, salt)

$$-\left.\frac{\partial c}{\partial z}\right|_b > 0$$

Source (pore
water diffusion)

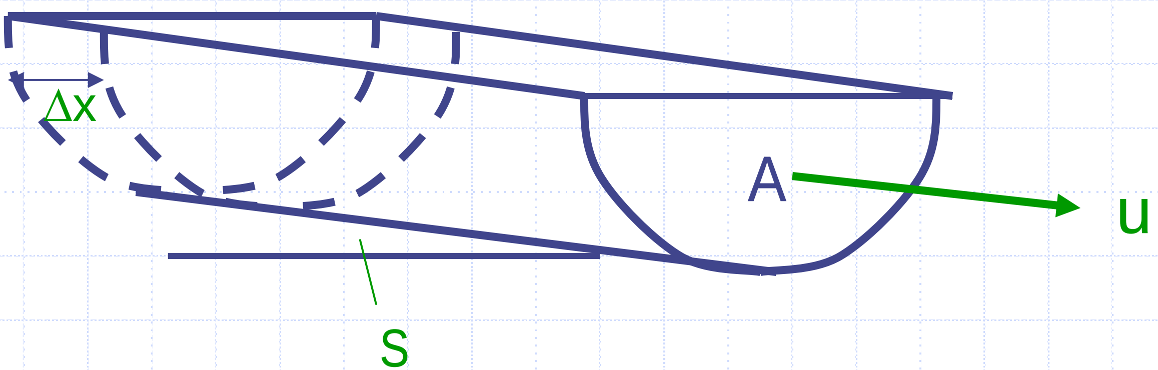
$$-\left.\frac{\partial c}{\partial z}\right|_b < 0$$

Sink (trace metals bound
by anoxic sediments)

Magnitude of terms: E_z

- $E_z \sim u'L \sim u_* H$

$$u_* = \text{shear velocity} = \sqrt{\tau_b / \rho} = \sqrt{\frac{f}{8}}\,\overline{\overline{u}}, \quad f \cong 0.02 \Rightarrow u_* = 0.05\overline{\overline{u}}$$



$$F_g = A\Delta x \rho S g \quad = \quad F_f = \tau_b p \Delta x = \rho u_*^{\,2} p \Delta x$$

Normal flow; gravity balances friction

$$u_*^{\,2} = \frac{ASg}{p} = R_H S g; \quad u_* \cong \sqrt{gHS};$$

$$R_H = A / p \quad = \quad \textit{hydraulic radius} \cong H$$

E_z cont'd

$$\overline{\overline{E_z}} = 0.07 u_* H$$

Seen previously; from analogy of mass and momentum conservation (Reynolds' analogy) and log profile for velocity

$$\tau_{vm} \cong \frac{0.5H^2}{\overline{\overline{E_z}}} \cong \frac{0.5H^2}{0.07u_*H} \cong \frac{7H}{u_*}$$

$$x_{vm} = \overline{\overline{u}}\,\tau_{vm} \cong \frac{7H\overline{\overline{u}}}{u_*}; \quad if\ \overline{\overline{u}} = 20u_* \quad x_{vm} \cong 150H$$

Transverse mixing: E_T

$$\frac{E_T}{u_* H} \cong 0.08 - 0.24$$

(say 0.15)

Laboratory rectangular channels

$$\frac{E_T}{u_* H} \cong 0.2 - 4.6$$

(say 0.6)

Real channels (irregularities, braiding, secondary circulation)

$$\tau_{tm} \cong \frac{0.5B^2}{E_T} = \frac{0.5B^2}{0.6 u_* H}$$

$$x_{tm} = \frac{0.5B^2}{0.6 u_* H}\overline{\overline{u}}; \quad if\ \overline{\overline{u}} = 20 u_*$$

$$= \frac{17B^2}{H}$$

B = channel width

Example

B = 100 m, H = 5m, u = 1 m/s

X_{vm} = 150H = 750 m

$x_{tm} = 17B^2/H = (17)(100)^2/5 = 34{,}000$ m (34 km)

It may take quite a while before concentrations can be considered laterally (transversally) uniform

Simplifications

Steady state; depth-averaged; no lateral advection or long dispersion; no boundary fluxes

$$\frac{\partial}{\partial t}\left(\bar{\bar{c}}H\right)+\frac{\partial}{\partial x}\left(\overline{\overline{u\,c}}H\right)+\frac{\partial}{\partial y}\left(\overline{\overline{v\,c}}H\right)=\frac{\partial}{\partial x}\left[HE_L\frac{\partial \bar{\bar{c}}}{\partial x}\right]+\frac{\partial}{\partial y}\left[HE_T\frac{\partial \bar{\bar{c}}}{\partial y}\right]+E_z\left.\frac{\partial c}{\partial z}\right|_s-E_z\left.\frac{\partial c}{\partial z}\right|_b$$

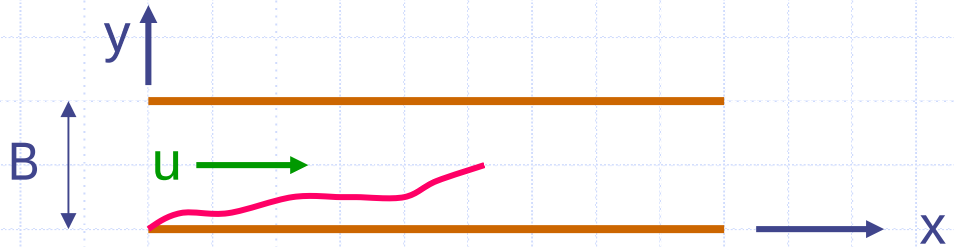
$$H\bar{\bar{u}}\frac{\partial \bar{\bar{c}}}{\partial x}=\frac{\partial}{\partial y}\left(HE_T\frac{\partial c}{\partial y}\right)$$

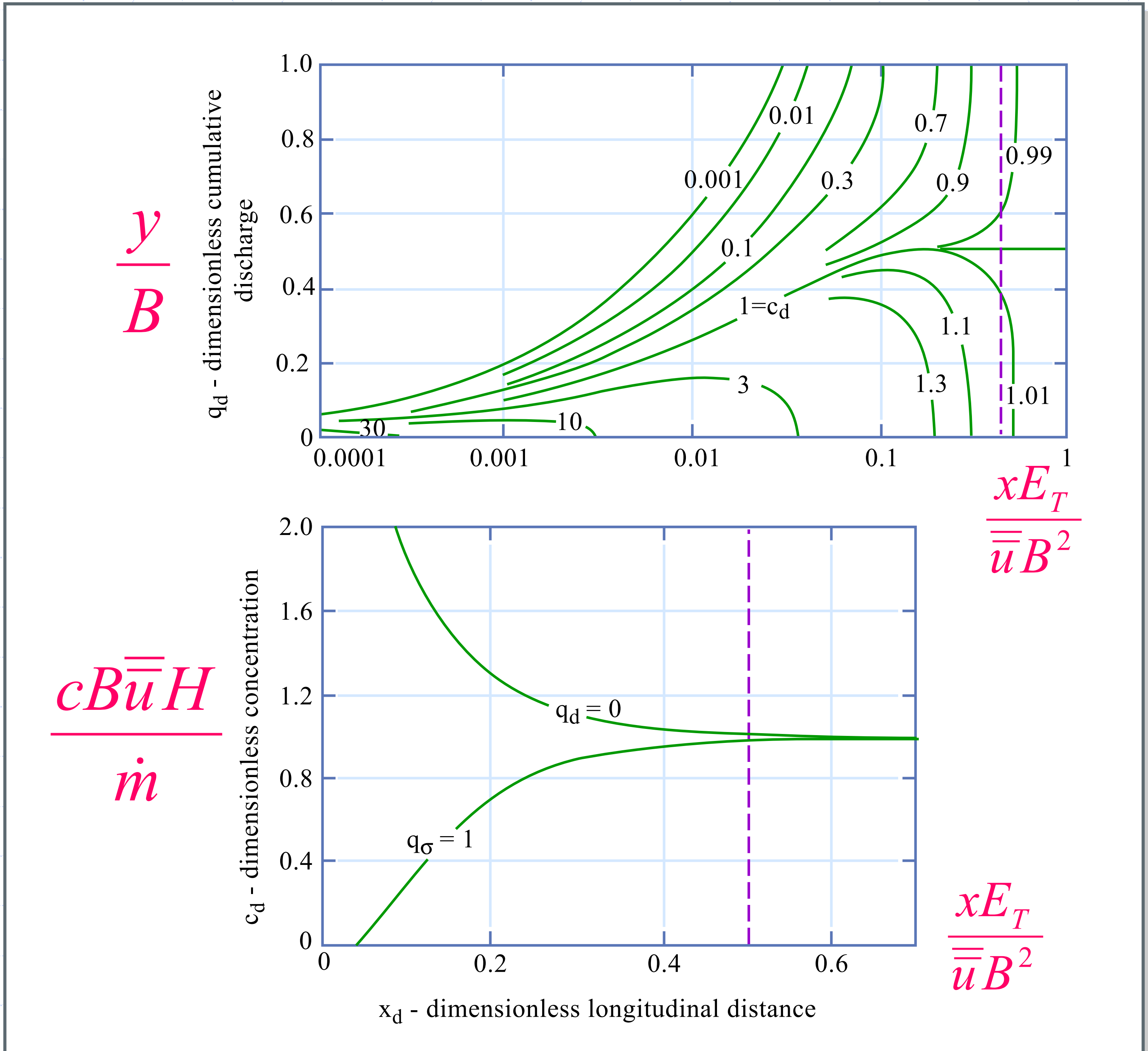
Uniform channel

$$\frac{\partial \bar{\bar{c}}}{\partial x}=\frac{E_T}{\bar{\bar{u}}}\frac{\partial^2 c}{\partial y^2}$$

const

Simple diffusion equation: solutions for continuous source at x=y=0





Holly and Jirka (1986)

Note:

$$x_{tm} \cong \frac{0.5B^2\overline{\overline{u}}}{E_T}$$

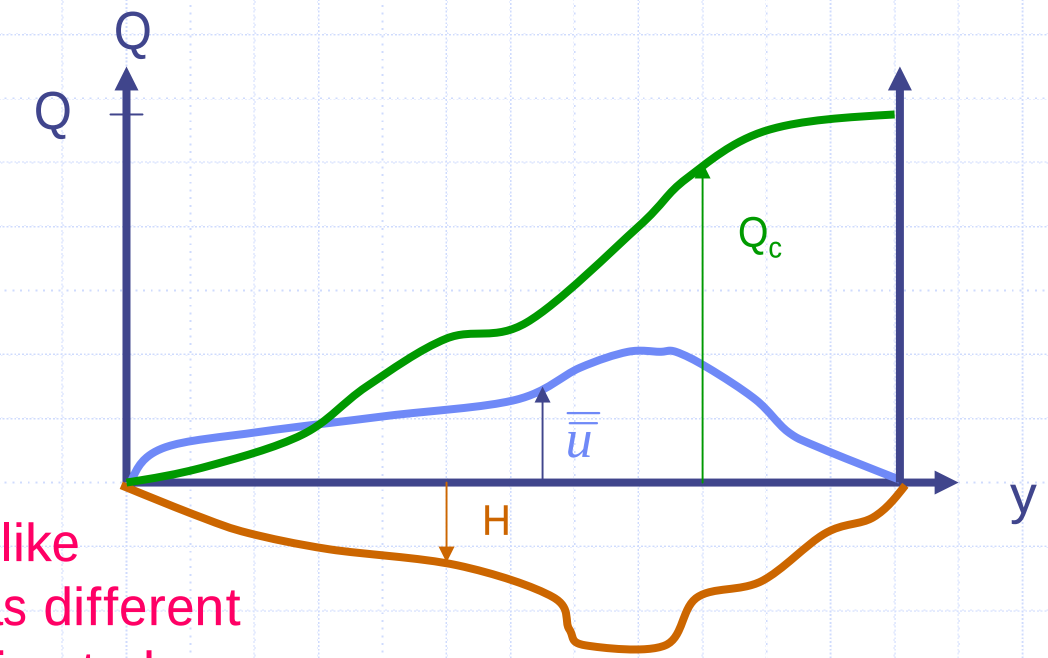
Figure by MIT OCW.

A useful extension: cumulative discharge approach (Yotsukura & Sayre, 1976)

Use cumulative discharge (Q_c) instead of y as lateral variable

$$Q_c = \int_0^y H(y')\overline{\overline{u}}(y')dy'$$

$$\frac{\partial \overline{\overline{c}}}{\partial x} = \frac{\partial}{\partial Q_c}\left[\underbrace{H^2\overline{\overline{u}}E_T}_{\mathbf{D}} \frac{\partial \overline{\overline{c}}}{\partial Q_c}\right]$$



D behaves mathematically like diffusion coefficient, but has different dimensions; can be approximated as constant (cross-sectional average):

$$\mathbf{D} = \frac{1}{Q}\int_0^{Q_c} H^2\overline{\overline{u}}E_T\, dQ_c \quad => \quad \frac{\partial \overline{\overline{c}}}{\partial x} = \mathbf{D}\frac{\partial^2 \overline{\overline{c}}}{\partial Q_c^{\ 2}}$$

Can use previous analysis

Longitudinal Shear Dispersion

Why is longitudinal dispersion Fickian?

Original analysis by Taylor (1953, 1954) for flow in pipes; following for 2D flow after Elder (1959)

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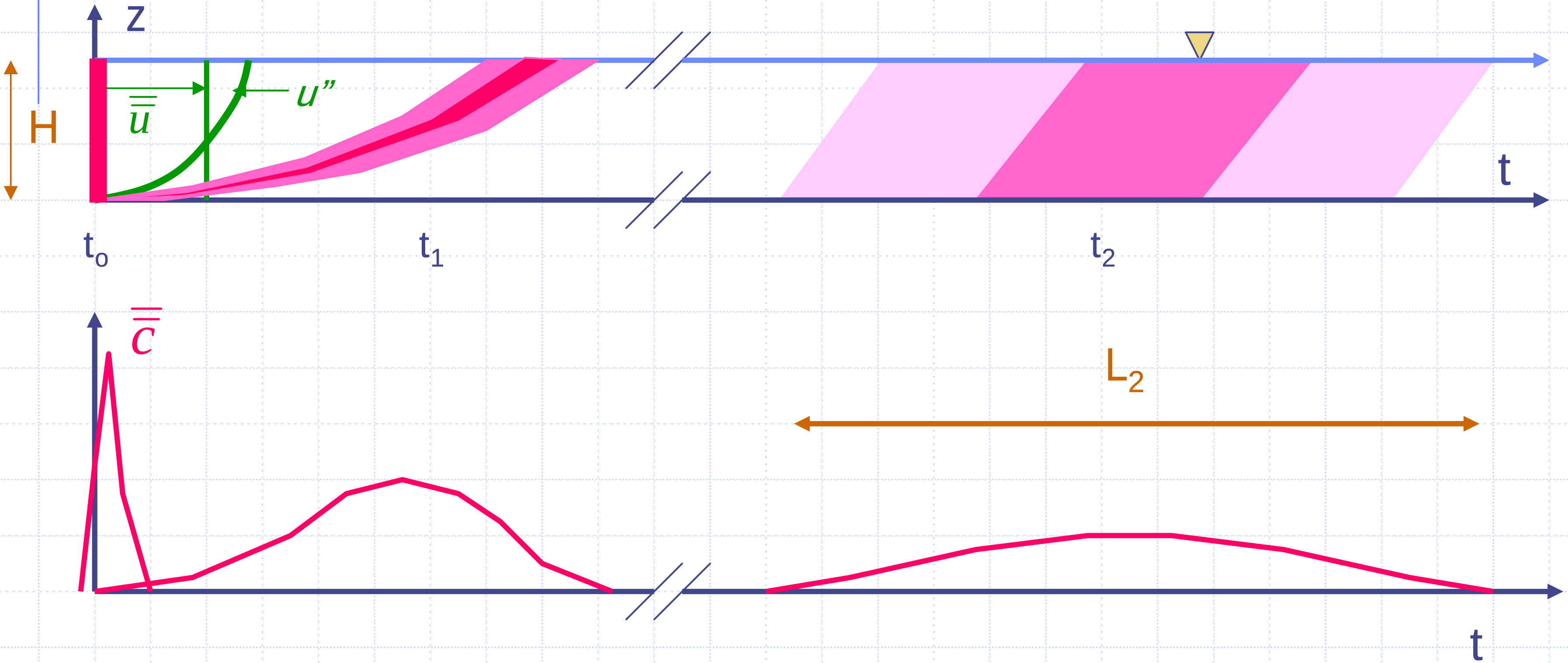
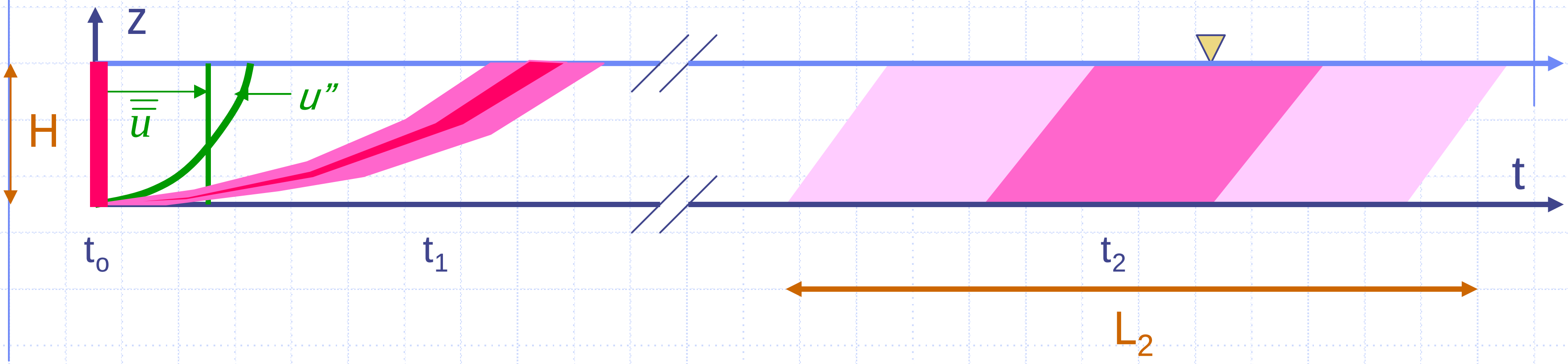


Shear dispersion, cont'd



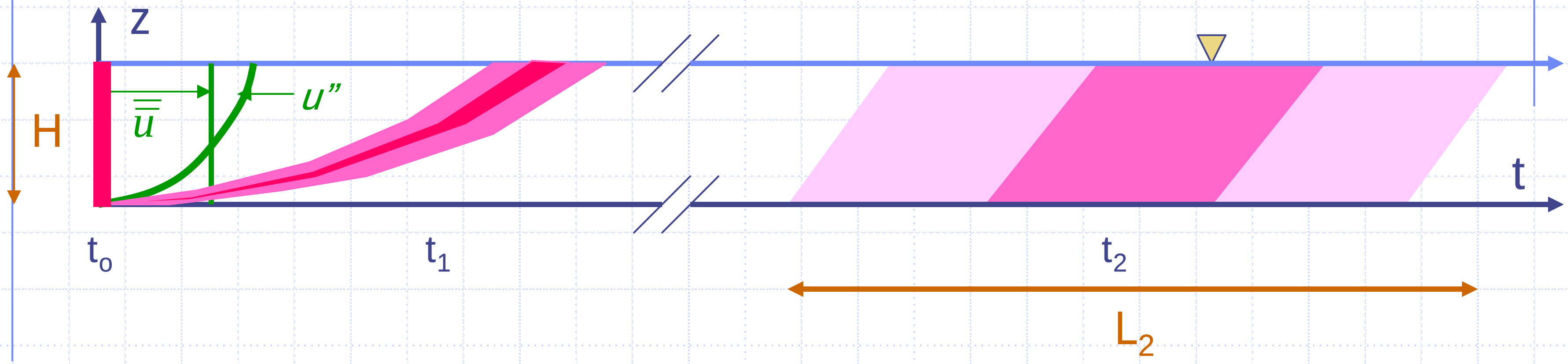
$$\frac{\partial c}{\partial t} + u(z)\frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\left(E_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(E_y \frac{\partial c}{\partial y}\right)$$

$$u = \overline{\overline{u}} + u''; \quad c = \overline{\overline{c}} + c''; \quad x = \zeta + \overline{\overline{u}}\, t; \quad t = \tau$$

$$\frac{\partial \overline{\overline{c}}}{\partial \tau} + \frac{\partial c''}{\partial \tau} + u''\frac{\partial \overline{\overline{c}}}{\partial \zeta} + u''\frac{\partial c''}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(E_x \frac{\partial \overline{\overline{c}}}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(E_x \frac{\partial c''}{\partial \zeta}\right) + \cancel{\frac{\partial}{\partial z}\left(E_z \frac{\partial \overline{\overline{c}}}{\partial z}\right)} + \frac{\partial}{\partial z}\left(E_z \frac{\partial c''}{\partial z}\right)$$

1 2 3 4 5 6 7

Shear dispersion, cont'd



$a)\ L >> H \quad => \quad$ longitudinal dispersion << vertical — 5, 6 << 7

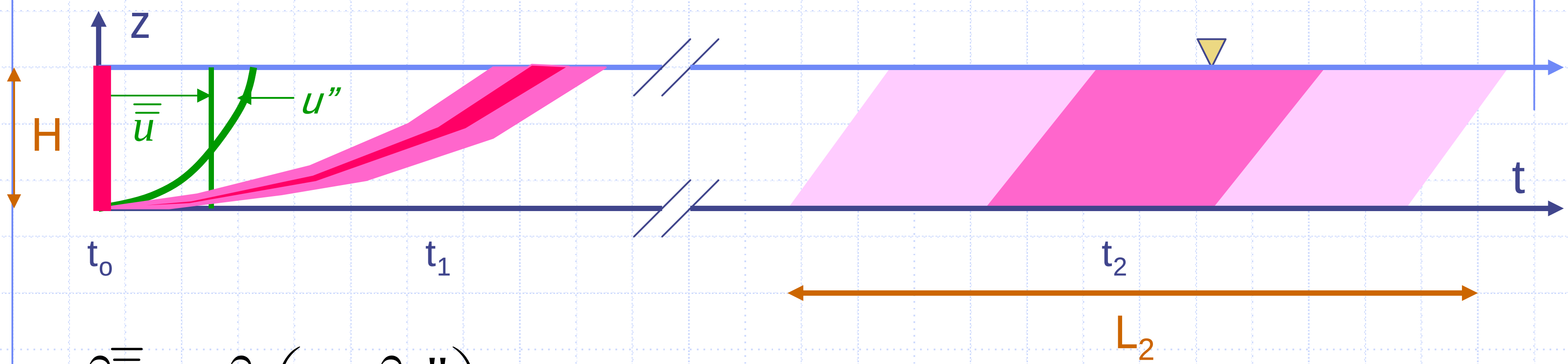
$b)\ \overline{\overline{c}} >> c'' \quad => \quad \partial c''/\partial \zeta << \partial \overline{\overline{c}}/\partial \zeta$ — 4 << 3

$c)\ x >> L \quad => \quad \partial/\partial \tau << u''\partial/\partial \zeta$ — 1, 2 << 3

$$\frac{\partial \overline{\overline{c}}}{\partial \tau} + \frac{\partial c''}{\partial \tau} + u''\frac{\partial \overline{\overline{c}}}{\partial \zeta} + u''\frac{\partial c''}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(E_x \frac{\partial \overline{\overline{c}}}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(E_x \frac{\partial c''}{\partial \zeta}\right) + \frac{\partial}{\partial z}\left(E_z \frac{\partial c''}{\partial z}\right)$$

1 2 3 4 5 6 7

Shear dispersion, cont'd



$$u''\frac{\partial \overline{\overline{c}}}{\partial \zeta} = \frac{\partial}{\partial z}\left(E_z \frac{\partial c''}{\partial z}\right)$$

3 7

Differential advection balanced by transverse diffusion

We want $-\overline{\overline{u''c''}}$ and to show that it is ~ $E_L \frac{\partial \overline{\overline{c}}}{\partial x}$

Integrate over z twice to get c"; multiply by u"; integrate again and divide by H (depth average); add minus sign

Shear dispersion, cont'd

$$-\overline{\overline{u''c''}} = -\frac{1}{H}\int_0^H u''c''\,dz = -\frac{1}{H}\frac{\partial\overline{\overline{c}}}{\partial\zeta}\int_0^H u''\int_0^z \frac{1}{E_z}\int_0^z u''\,dz\,dz\,dz$$

E_L!!!

$$E_L = I_h \frac{H^2}{\overline{\overline{E_z}}}\overline{\overline{u''^2}}$$

I_h = dimensionless triple integration ~ 0.07

$u'' \sim u_*; \quad \overline{\overline{E_z}} \sim u_* H$

$E_L = 5.9 u_* H$ Elder (1959) using log profile for u"(z)

$E_L = 10 u_* r_o$ Taylor (1954) turbulent pipe flow (r_o= radius)

Use E_L to compute σ_L or use measured σ_L to deduce E_L

Comments

- E_L involves differential advection (u”) with transverse mixing in direction of advection gradient (E_z)
- E_L ~ 1/E_z; perhaps counter-intuitive, but look at time scales:

$$E_L \sim \frac{H^2}{\overline{\overline{E_z}}} \overline{\overline{u''^2}}$$

T_c U_c^2

Recall Taylor’s Theorem

$$D \sim \overline{u^2} \int_0^\infty R(\tau) d\tau$$

A thought experiment

Consider the trip on the Mass Turnpike from Boston to the NY border (~150 miles).

Assume two lanes in each direction, and that cars in left lane always travel 65 mph, while those in the right lane travel 55 mph.

At the start 50 cars in each lane have their tops painted red and a helicopter observes the "dispersion" in their position as they travel to NY

1) How does this dispersion depend on the frequency of lane changes?

2) Would dispersion increase or decrease if there were a third (middle) lane where cars traveled at 60 mph?

Thought experiment, cont'd

$$E_L \sim \frac{\overline{H^2}}{\overline{\overline{E_z}}} \overline{\overline{(u'')^2}}$$

What are the analogs of H, E_z and u”

Thought experiment, cont'd

$$E_L \sim \frac{H^2}{E_z}\overline{(u'')^2}$$

H ~ number of lanes

E_z ~ frequency of lane changing

u" ~ difference between average and lane-specific speed

1) Decreasing E_z increases dispersion (as long as there is some E_z)

2) Increasing lanes increases H, decreases mean square u",

$$\overline{(u'')^2} = \frac{(65-60)^2 + (55-60)^2}{2} = 50/2$$ (2 lanes)

$$\overline{(u'')^2} = \frac{(65-60)^2 + (60-60)^2 + (55-60)^2}{3} = 50/3$$ (3 lanes)

If E_z is constant, net effect is increase in E_L by $(3/2)^2(2/3)$ = 50%

1D (river) dispersion

$$u = \overline{\overline{u}} + u''; \quad c = \overline{\overline{c}} + c''$$

$$c = \frac{1}{A}\int_A c\,dA$$

Insert into GE and spatial average

Continuity

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = q_L$$

q_L = lateral inflow/length [L^2/T]

Mass Conservation (conservative form)

$$\frac{\partial}{\partial t}(Ac) + \frac{\partial}{\partial x}(Auc) = \frac{\partial}{\partial x}\left[A\overline{\overline{E_x}}\frac{\partial c}{\partial x} - A\overline{\overline{u''c''}} \right] + Ar_i + q_L c_L$$

$$AE_L \frac{\partial c}{\partial x}$$

Longitudinal dispersion again

1D (river) dispersion, cont'd

NC form from conservative equation minus c times continuity

$$\frac{\partial}{\partial t}(Ac) + \frac{\partial}{\partial x}(Auc) = \frac{\partial}{\partial x}\left[A\overline{\overline{E}}_x \frac{\partial c}{\partial x} - A\overline{\overline{u''c''}}\right] + Ar_i + q_L c_L$$

$$- \quad c \cdot \left\{\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = q_L\right\}$$

Mass Conservation (NC form)

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = \frac{1}{A}\frac{\partial}{\partial x}\left(AE_L \frac{\partial c}{\partial x}\right) + r_i + \frac{q_L}{A}(c_L - c)$$

Note: if $c_L > c$, c increases;
if $c_L < c$, c decreases (dilution)

E_L for rivers

Elder formula accounts for vertical shear (OK for depth averaged models that resolve lateral shear); here we need to parameterize lateral and vertical shear. Analysis by Fischer (1967)

$$-\overline{\overline{u''c''}} = \underbrace{I_b \frac{B^2}{E_T}\overline{\overline{u''^2}}}_{E_L} \frac{\partial \overline{\overline{c}}}{\partial x}$$

B = river width

E_T = transverse dispersion coefficient

I_b – ND triple integration (across A) ~ 0.07

Same form as Elder, but now time scale is B^2/E_T, rather than H^2/E_z.
$E_T > E_z$, but $B^2 >> H^2$ => this E_L is generally much larger

E_L for rivers, cont'd

Using approximations for u", E_T, etc.

$$E_L \cong 0.01 \frac{\overline{\overline{u}}^2 B^2}{u_* H}$$

Fischer (1967); useful for reasonably straight, uniform rivers and channels

or

$$\frac{E_L}{u_* H} \cong 0.01 \left(\frac{\overline{\overline{u}}}{u_*} \right)^2 \left(\frac{B}{H} \right)^2$$

if $\overline{\overline{u}} \cong 20 u_*$

$$\frac{E_L}{u_* H} \cong 4 \left(\frac{B}{H} \right)^2$$

Magnitude of terms, revisited

$$\frac{\overline{\overline{E_z}}}{u_* H} \cong 0.07$$ Vertical Diffusion

$$\frac{E_T}{u_* H} \cong 0.6$$ Transverse Diffusion in Channels

$$\frac{E_L}{u_* H} \cong 6$$ Longitudinal Dispersion (depth-averaged flow)

$$\frac{E_L}{u_* r_o} \cong 10$$ Longitudinal Dispersion (turbulent pipe flow)

$$\frac{E_L}{u_* H} \cong 0.01 \left(\frac{\overline{\overline{u}}}{u_*} \right)^2 \left(\frac{B}{H} \right)^2$$ Longitudinal Dispersion (rivers)

Previous Example, revisited

B = 100 m, H = 5m, u = 1 m/s, u_* = 0.05u =0.05

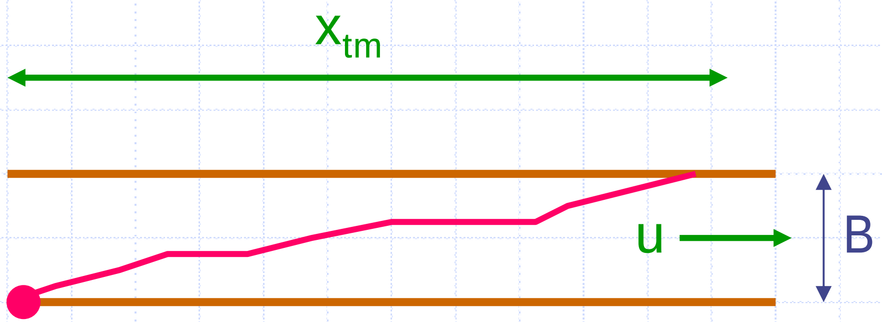
$$E_L = \frac{0.01u^2B^2}{u_*H} = \frac{(0.01)(1)^2(100)^2}{(0.05)(5)} = 400\ \mathrm{m}^2/\mathrm{s}$$

$$\frac{d}{dt}\sigma_x^{\,2} = 2E_L$$

=> Gaussian Distribution; but only after cross-sectional mixing

$$x_{tm} = \frac{0.5B^2}{E_T}u$$

x_{tm} = 34 km if point source on river bank;



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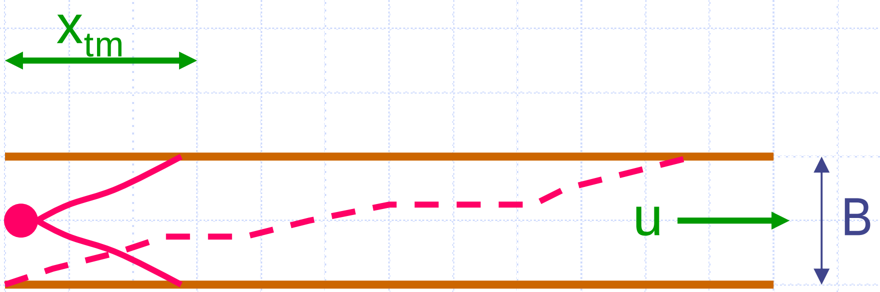
$$x_{tm} = \frac{0.5B^2}{E_T}u$$

4 times less if point source in mid-stream



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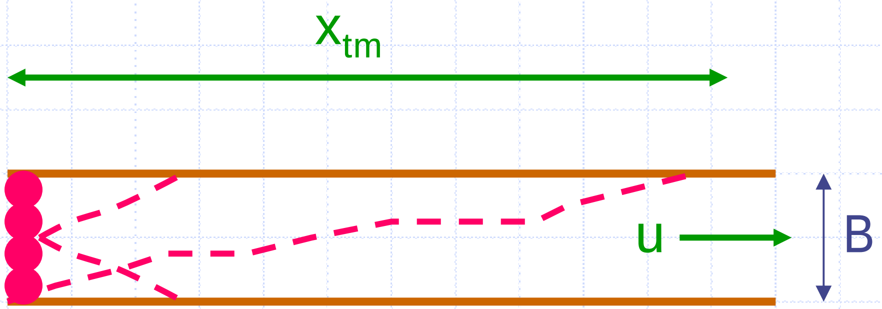
=> Gaussian Distribution; but only after cross-sectional mixing

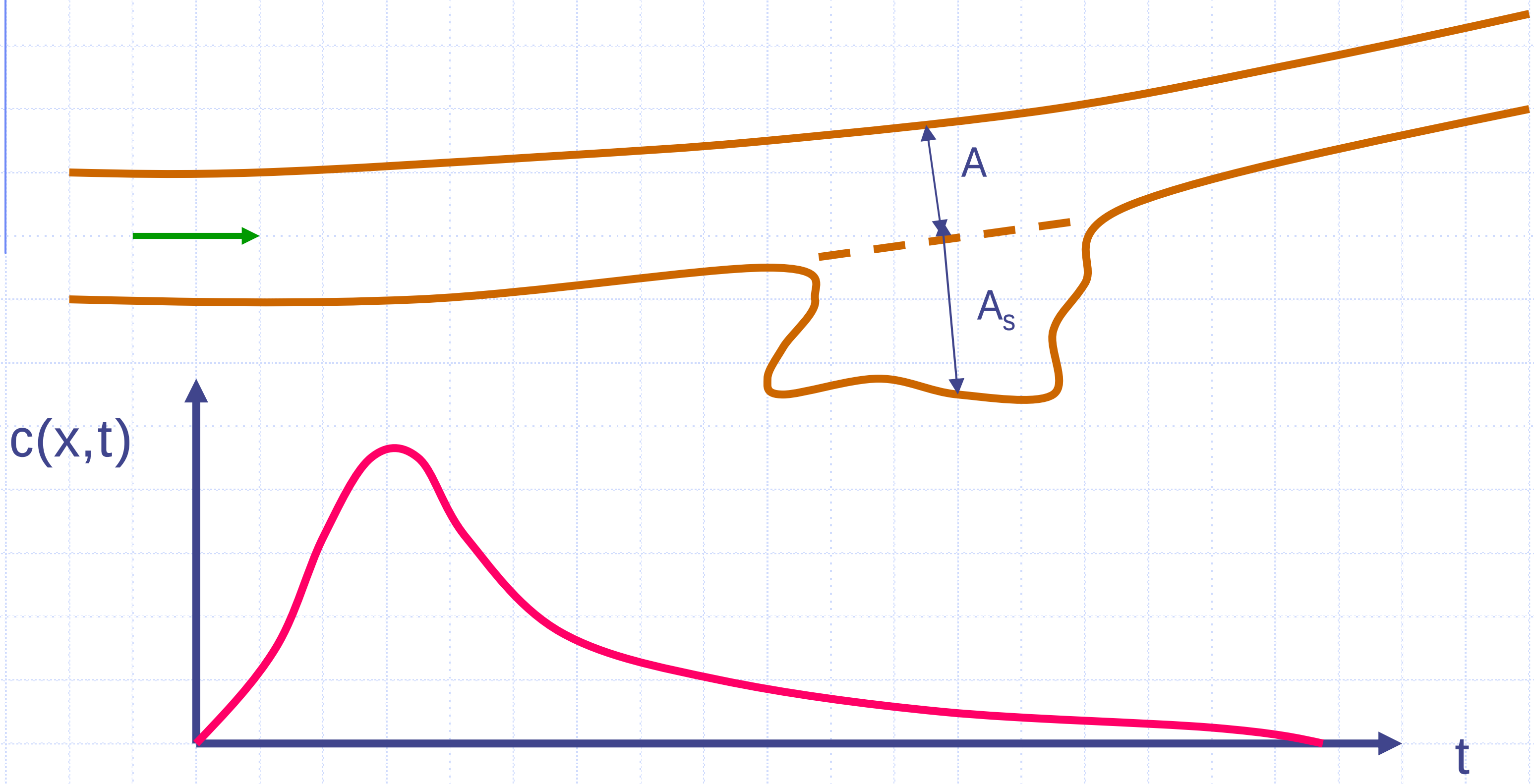
$$x_{tm} = \frac{0.5B^2}{E_T}u$$

Less still if distributed across channel (but not zero)



Storage zones

Real channels often have backwater (storage) zones that increase dispersion and give long tails to c(t) distribution



Storage zones, cont'd

1) $$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}=\frac{1}{A}\frac{\partial}{\partial x}\left(AE_L\frac{\partial c}{\partial x}\right)+\frac{q_L}{A}(c_L-c)+\alpha(c_s-c)$$ Main channel

2) $$\frac{dc_s}{dt}=-\alpha\frac{A}{A_s}(c_s-c)$$ Storage zone

$A(x)$ = cross-sectional area of main channel

A_s = cross-sectional area of storage zone

α = storage zone coefficient (rate, t^{-1}; like q_L/A)

If you multiply 1) by A and 2) by A_s, the exchange terms are $\alpha A(c_s-c)$ and $-\alpha A(c_s-c)$

Really the same process as longitudinal dispersion, but instead of cars in either fast or slow lane, some are in the rest stop.