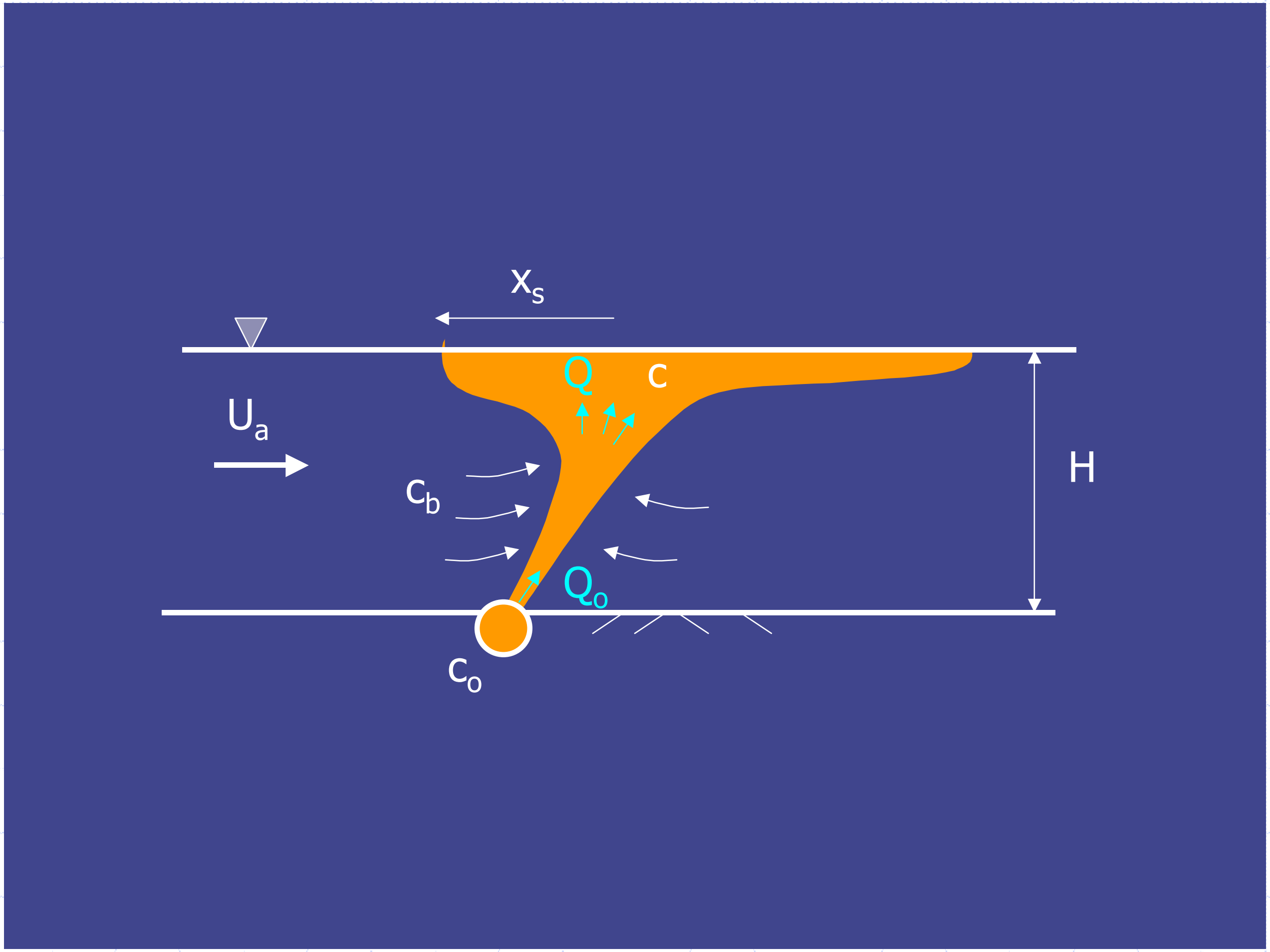
2 Turbulent Diffusion

- Turbulence
- Turbulent (eddy) diffusivities
- Simple solutions for instantaneous and continuous sources in 1-, 2-, 3-D.
- Boundary Conditions
- Fluid Shear
- Field Data on Horizontal & Vertical Diffusion
- Atmospheric, Surface water & GW plumes

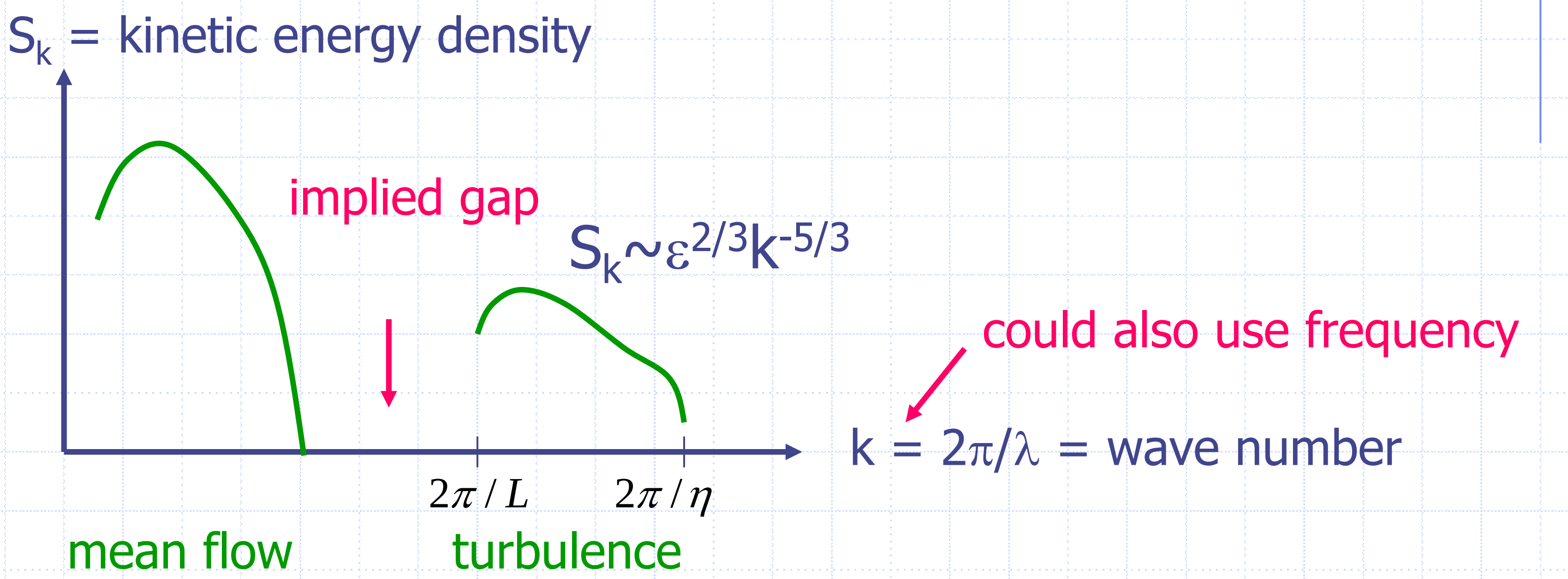
Turbulence

- Turbulent flow (unstable, chaotic) vs laminar flow (stable)
- Turbulent sources: internal (grid, wake), boundary shear, wind shear, convection
- Turbulent mixing caused by water movement, not molecular diffusion
- Two-way exchange--contrast with initial mixing (one-way process)

Initial mixing



Kinetic Energy Spectrum



S_k = kinetic energy/mass-wave number $[U^2/L^{-1} = L^3/T^2]$

L = size of largest eddy

ε = energy dissipation rate $[U^2/T = U^2/(L/U) = U^3/L = L^2/T^3]$

η = Kolmogorov (inner) scale = $(\nu^3/\varepsilon)^{1/4}$ [L]

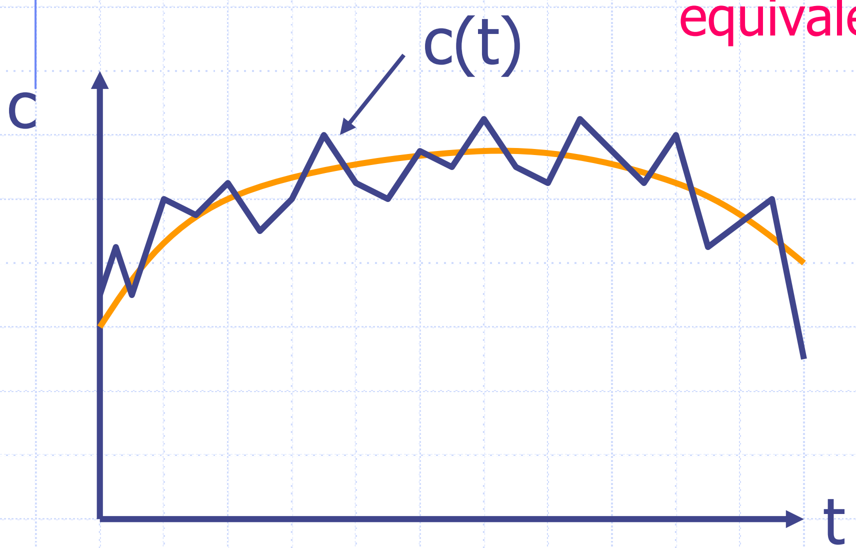
Turbulent Averaging

$$\frac{\partial c}{\partial t} + \nabla \cdot \left(\vec{q} c\right) = D \nabla^2 c$$

Conservative mass transport eq.

Both q and c fluctuate on scales smaller than environmental interest

Therefore average. Two choices: time average, ensemble average; equivalent if ergotic.



$\overline{c}$	Time average
$\langle c \rangle$	Ensemble average

$$c'(t) = c(t) - \overline{c}$$

$$q'(t) = q(t) - \overline{q}$$

$$c'(t) = c(t) - \langle c \rangle$$

$$q'(t) = q(t) - \langle q \rangle$$

Turbulent Averaging, cont'd

$$\frac{\partial c}{\partial t} + \nabla \cdot (\vec{q} c) = D \nabla^2 c$$

Expand q and c; time average

$$\overline{\nabla \cdot (\vec{q} c)} = \overline{\nabla \cdot \left[(\overline{\vec{q}} + q')(\overline{c} + c') \right]}$$

$$= \overline{\nabla \cdot \left[\overline{\vec{q}}\,\overline{c} + \overline{\vec{q}} c' + \vec{q}' \overline{c} + \vec{q}' c' \right]}$$

$$= \overline{\vec{q}} \cdot \nabla \overline{c} + \nabla \cdot \left(\overline{\vec{q}' c'} \right)$$

Continuity $\nabla \cdot \overline{\vec{q}} = 0$

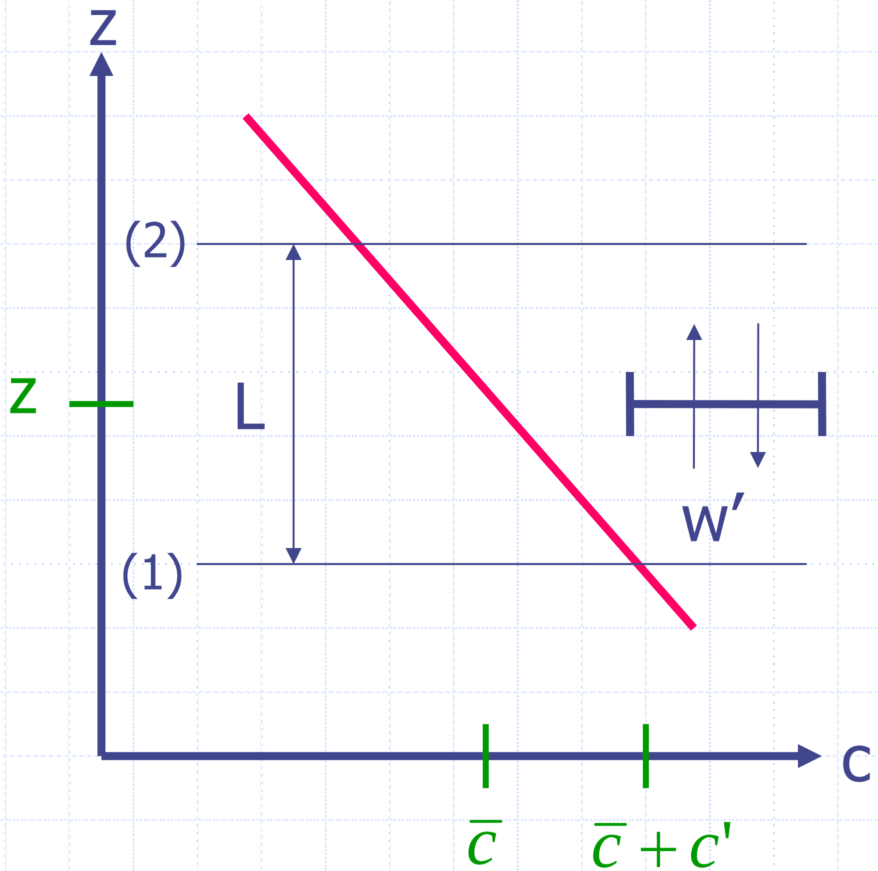
$$\frac{\partial \overline{c}}{\partial t} + \overline{\vec{q}} \cdot \nabla \overline{c} = -\nabla \cdot \left(\overline{\vec{q}' c'} \right) + D \nabla^2 c$$

$$\nabla \cdot \left(\overline{\vec{q}' c'} \right) = \frac{\partial}{\partial x} \overline{u' c'} + \frac{\partial}{\partial y} \overline{v' c'} + \frac{\partial}{\partial z} \overline{w' c'}$$

Closure problem: we only want $\overline{c}$ but we must deal with c'

$\overline{w' c'}$ Eddy correlation.

Turbulent Diffusion



Inst. flux (M/L^2-T)

$$J_z = w' c_1$$
$$= w'(\bar{c} + c')$$

Mean flux (M/L^2-T)

$$\bar{J}_z = |w'|(c_1 - c_2)$$

$$c_2 = c_1 + \frac{\partial \bar{c}}{\partial z} L$$

$$\bar{J}_z = -\underbrace{|w'| L}_{E_{zz}} \frac{\partial \bar{c}}{\partial z}$$

E_{zz}

Eddy (or turbulent) diffusivity

Eddy Diffusivity $E \sim |u'|L$

- Structurally similar to molecular diffusivity D, but much larger (due to fluid motion, not molecular motion) => often drop D
- E is a tensor (9 components, E_{xx}, E_{xy}, etc.) but often treated as a vector (E_x, E_y, E_z)
- Depends on nature of turbulence; in general neither isotropic nor uniform
- Eddy diffusivity ~ conductivity ~ viscosity
- Individual plumes not always Gaussian; but ensemble averages -> Gaussian

Turbulent transport eqn

$$\frac{\partial c}{\partial t} + \overline{\vec{q}} \cdot \nabla \overline{c} = \nabla \left(E \nabla \overline{c} \right) + \sum r$$

$$\frac{\partial c}{\partial t} + \overline{u} \frac{\partial \overline{c}}{\partial x} + \overline{v} \frac{\partial \overline{c}}{\partial y} + \overline{w} \frac{\partial \overline{c}}{\partial z}$$

Cartesian coordinates;
diagnolized diffusivity

$$= \frac{\partial}{\partial x} \left(E_x \frac{\partial \overline{c}}{\partial x} \right) + \frac{\partial}{\partial y} \left(E_y \frac{\partial \overline{c}}{\partial y} \right) + \frac{\partial}{\partial z} \left(E_z \frac{\partial \overline{c}}{\partial z} \right) + \sum r$$

How to measure eddy diffusivity

- Measure u′, c′, etc. and correlate
- Measure something else (e.g. dissipation) that correlates with E
- Measure concentration distribution and calibrate E (more later)
- Model it

Less direct

Models of turbulent diffusion

$$u' \sim \sqrt{k} \qquad k = TKE = \frac{\overline{u'^2}}{2} + \frac{\overline{v'^2}}{2} + \frac{\overline{w'^2}}{2}$$

turbulent kinetic energy (don't confuse with wave number)

1) *k*-L model (two eqn model)

$$\mathrm{E} \sim \sqrt{k}\,L$$

$$\frac{\partial k}{\partial t} = \cdots \pm \text{sources \& sinks of k}$$

$$\frac{\partial L}{\partial t} = \cdots \pm \text{sources \& sinks of L}$$

2) *k* model (one eqn; solve only for *k*; L is hardwired)

Models of turbulent diffusion, cont'd

$$u' \sim \sqrt{k} \quad k = TKE = \frac{\overline{u'^2}}{2} + \frac{\overline{v'^2}}{2} + \frac{\overline{w'^2}}{2}$$

turbulent kinetic energy

3) k-ε model (two eqn model)

ε = turbulent dissipation rate: $\frac{\partial k}{\partial t} = \cdots - \varepsilon$

ε ~ k/τ ; τ = time scale of turb. ~ L/$k^{1/2}$

ε ~ $k^{3/2}$/L or L ~ $k^{3/2}/\varepsilon$

$$E \sim \sqrt{k}L \sim k^2/\varepsilon$$

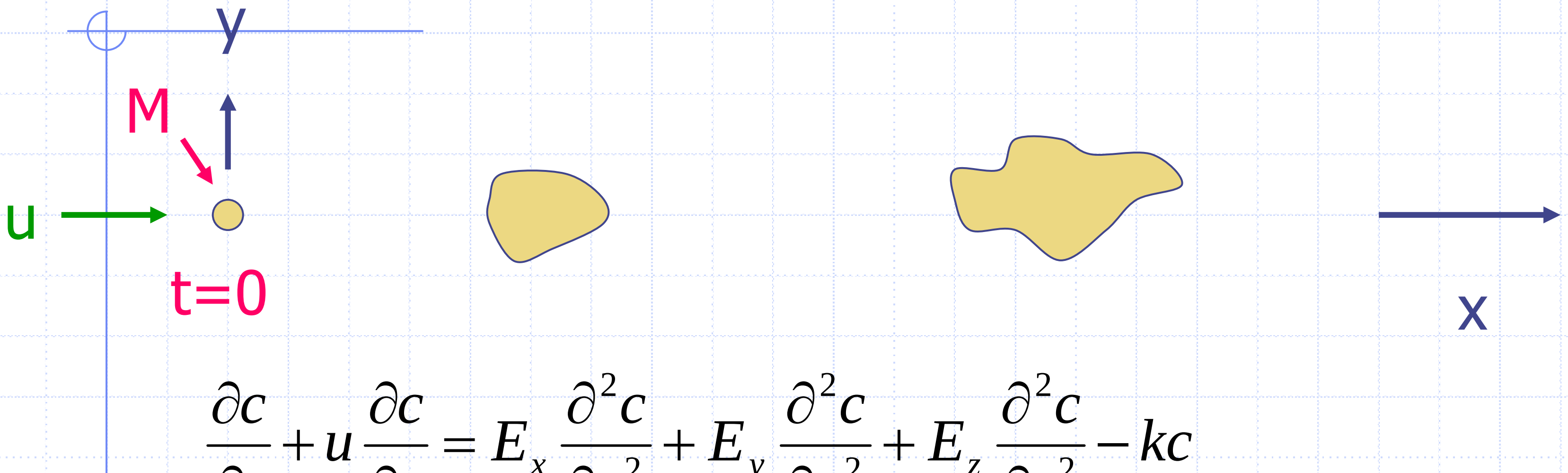
$$\frac{\partial \varepsilon}{\partial t} = \cdots \pm \text{sources \& sinks of } \varepsilon$$

A gazillion analytical solutions

- Instantaneous Point Source (Sec 2.2)
- Instantaneous Line Source (Sect 2.3)
- Instantaneous Plane Source (Sect 2.4)
- Continuous Point Source (Sect 2.5)
- Continuous Line Source (Sect 2.6)
- Continuous Plane Source (Sect 2.7)

Simple ones, e.g., u = const, given in following

Instantaneous (point) source in 3D



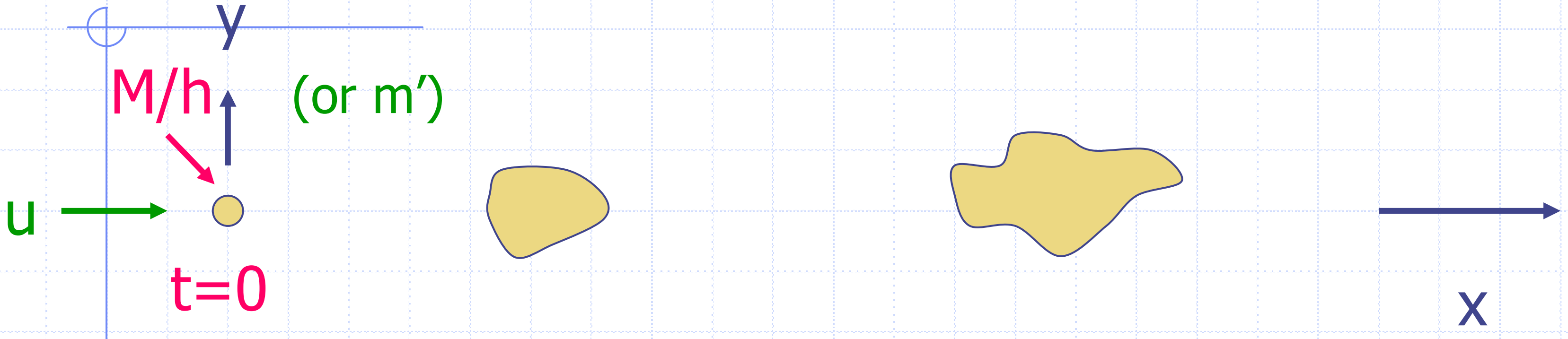
$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = E_x \frac{\partial^2 c}{\partial x^2} + E_y \frac{\partial^2 c}{\partial y^2} + E_z \frac{\partial^2 c}{\partial z^2} - kc$$

$$c = \frac{M}{8(\pi t)^{3/2}(E_x E_y E_z)^{1/2}} \exp - \left\{ \frac{(x-ut)^2}{4E_x t} + \frac{y^2}{4E_y t} + \frac{z^2}{4E_z t} + kt \right\}$$

$$c = \frac{M}{(2\pi)^{3/2}\sigma_x \sigma_y \sigma_z} \exp - \left\{ \frac{(x-ut)^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2} + kt \right\}$$

Instantaneous (line) source in 2D

(e.g. extending over epilimnion)

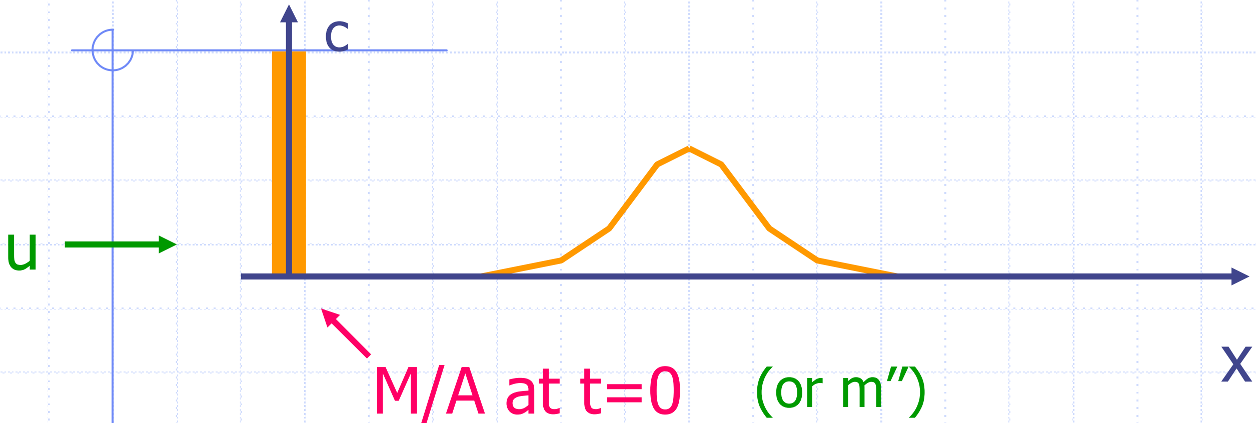


$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = E_x \frac{\partial^2 c}{\partial x^2} + E_y \frac{\partial^2 c}{\partial y^2} - kc$$

$$c = \frac{M/h}{4(\pi t)(E_x E_y)^{1/2}} \exp - \left\{ \frac{(x-ut)^2}{4E_x t} + \frac{y^2}{4E_y t} + kt \right\}$$

$$c = \frac{M/h}{(2\pi)\sigma_x \sigma_y} \exp - \left\{ \frac{(x-ut)^2}{2\sigma_x^{\ 2}} + \frac{y^2}{2\sigma_y^{\ 2}} + kt \right\}$$

Instantaneous (plane) source in 1D

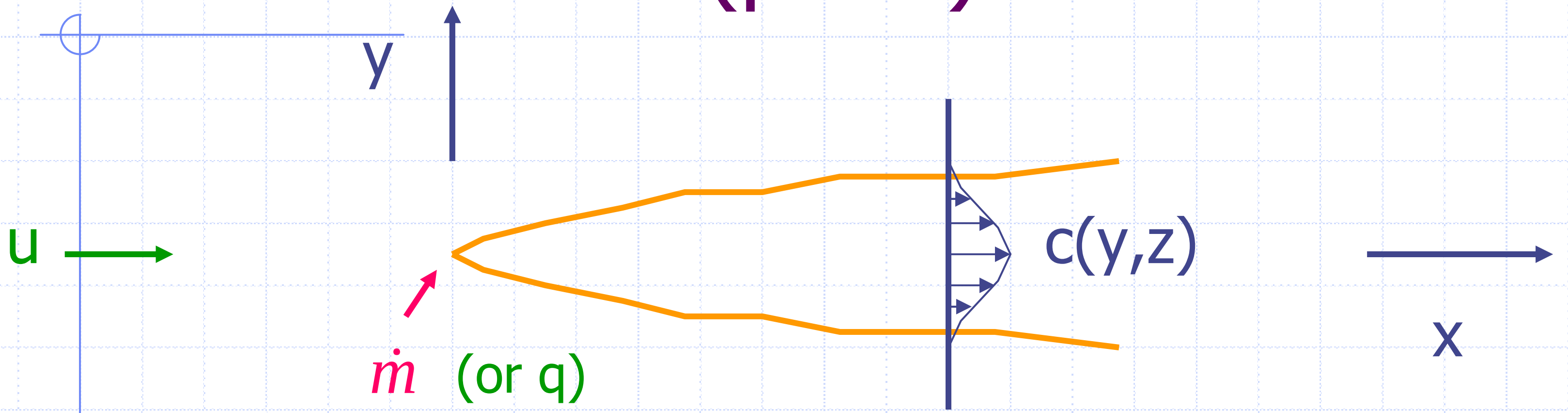


$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}=E_x\frac{\partial^2 c}{\partial x^2}-kc$$

$$c=\frac{M}{2(\pi t)^{1/2}E_x^{\ 1/2}}\exp-\left\{\frac{(x-ut)^2}{4E_x t}+kt\right\}$$

$$c=\frac{M}{(2\pi)^{1/2}\sigma_x}\exp-\left\{\frac{(x-ut)^2}{2\sigma_x^{\ 2}}+kt\right\}$$

Continuous (point) source in 3D



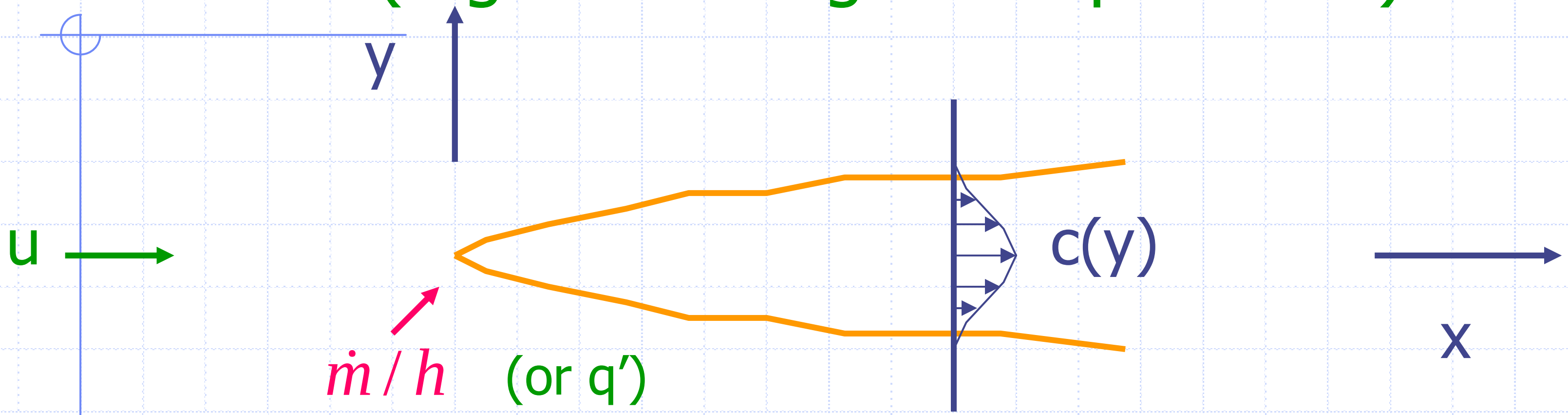
$$u\frac{\partial c}{\partial x} = E_x\frac{\partial^2 c}{\partial x^2} + E_y\frac{\partial^2 c}{\partial y^2} + E_z\frac{\partial^2 c}{\partial z^2} - kc$$

$$c = \frac{\dot{m}}{4\pi x(E_x E_z)^{1/2}}\exp-\left\{\frac{y^2 u}{4E_y x} + \frac{z^2 u}{4E_z x} + k\frac{x}{u}\right\}$$

$$c = \frac{\dot{m}/u}{(2\pi)\sigma_y\sigma_z}\exp-\left\{\frac{y^2 u}{2\sigma_y^{\ 2}} + \frac{z^2 u}{2\sigma_z^{\ 2}} + k\frac{x}{u}\right\}$$

Continuous (line) source in 2D

(e.g. extending over epilimnion)

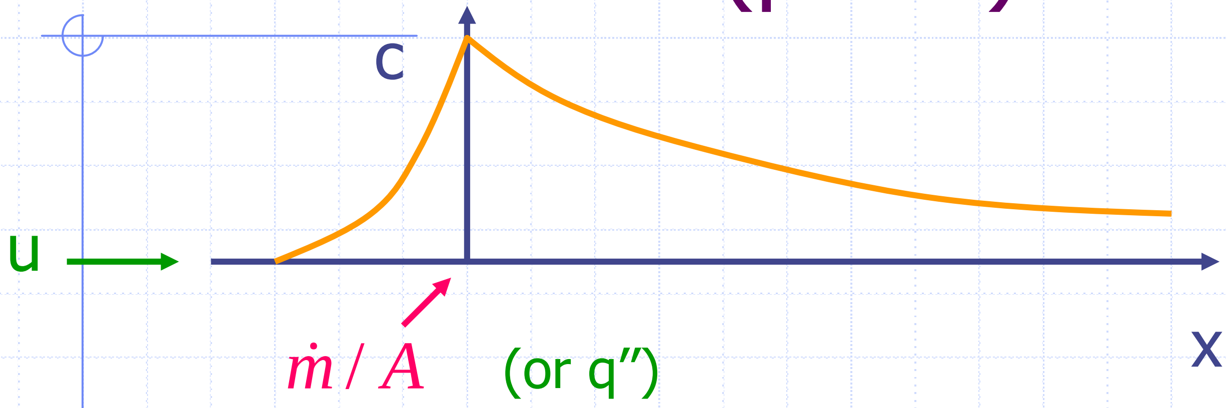


$$u\frac{\partial c}{\partial x} = E_y \frac{\partial^2 c}{\partial y^2} - kc$$

$$c = \frac{\dot{m}/h}{2(\pi u E_y x)^{1/2}} \exp - \left\{ \frac{y^2 u}{4E_y x} + k\frac{x}{u} \right\}$$

$$c = \frac{\dot{m}/uh}{(2\pi)^{1/2}\sigma_y} \exp - \left\{ \frac{y^2 u}{2\sigma_y x} + k\frac{x}{u} \right\}$$

Continuous (plane) source in 1D



$$u\frac{\partial c}{\partial x} = E_x \frac{\partial^2 c}{\partial x^2} - kc$$

$$c \cong \frac{\dot{m}}{Au}\exp-\left\{\frac{xk}{u}\right\} \qquad x > 0$$

$$c \cong \frac{\dot{m}}{Au}\exp\left\{\frac{xu}{E_x}\right\} \qquad x < 0$$

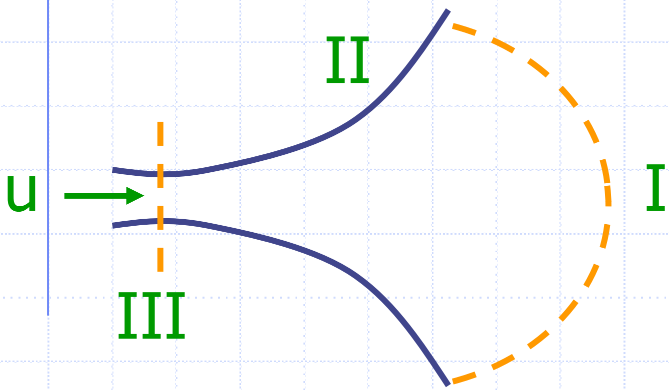
A few comments re solutions

- Spatial integration of point source => line source => plane source
- Temporal integration of inst source => Continuous source
- Relationship between σ's and E's found from spatial moments (as before)

Comments, cont'd

- $c \sim t^{-1/2}$, t^{-1}, $t^{-3/2}$ for instantaneous 1, 2, 3-D sources
- $c \sim x^{-0}$, $x^{-1/2}$, x^{--1} for continuous 1, 2, 3-D sources (difference: negligible E_x)
- Assumes E's are constant. If not, E's are 'apparent' (more later)
- Most common method to determine E is to fit to measured concentration distribution (tracer, drogues)

Boundary Conditions



Type I:

$$c = const, \quad e.g., \quad c = 0$$

$$on \quad open \quad bndry$$

Type II:

$$\frac{\partial c}{\partial n} = const, \quad e.g. = \frac{\partial c}{\partial n} = 0$$

$$on \quad solid \quad bndry$$

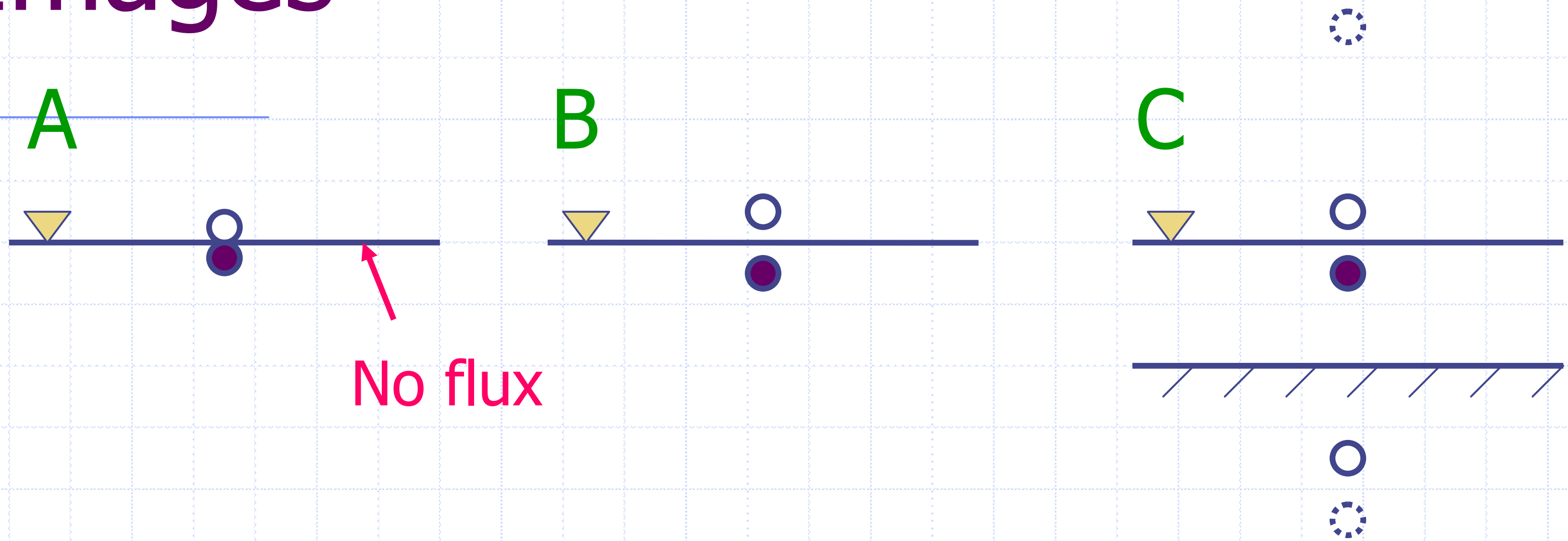
Type III (mixed):

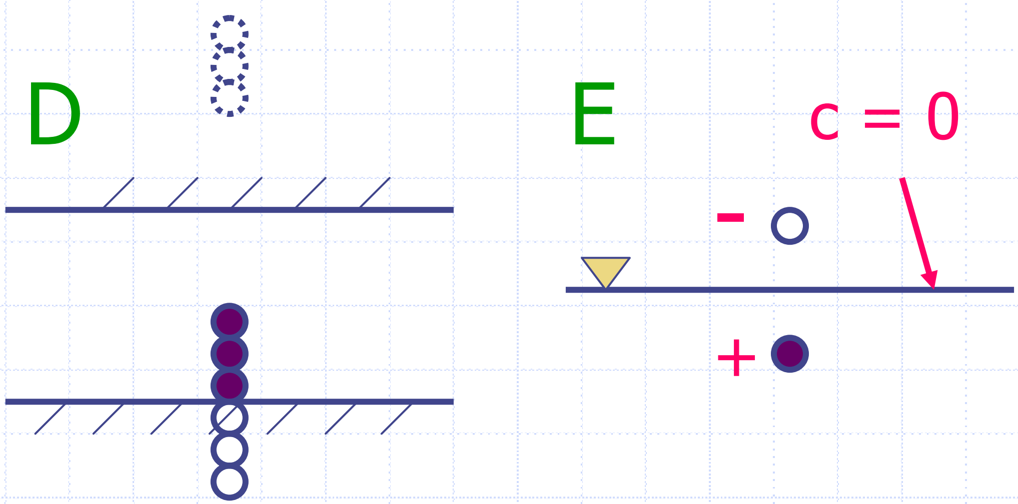
$$ac + b\frac{\partial c}{\partial n} = const,$$

$$e.g. \quad uc\Big|_{0-} = \left(uc - E_x \frac{\partial c}{\partial x}\right)_{0+}$$

$$at \quad inlet$$

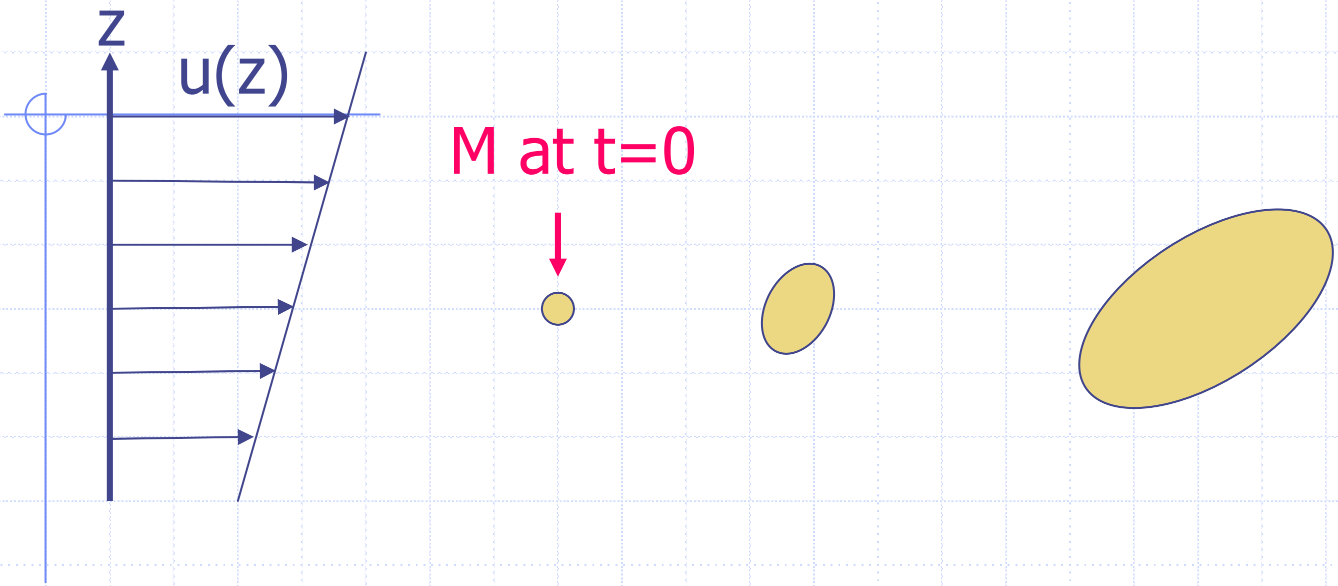
Images





$$C = C_{real} + \sum C_{image}$$

Inst. Pt. Source in Linear Shear



$$u = u_o(t) + \lambda_y y + \lambda_z z \qquad \lambda_y = \partial u / \partial y \quad \lambda_z = \partial u / \partial z$$

$$c(x,y,z,t) = \frac{M}{(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}\left[1+(\phi t)^2\right]^{1/2}} \exp - \left[\frac{\left\{ x - \int_0^t u_o(t')dt' - \frac{1}{2}(\lambda_y y + \lambda_z z)t \right\}^2}{4E_x t\left[1+(\phi t)^2\right]} + \frac{y^2}{4E_y t} + \frac{z^2}{4E_z t} + kt \right]$$

$$\phi^2 = \frac{1}{12}\left(\lambda_y{}^2 \frac{E_y}{E_x} + \lambda_z{}^2 \frac{E_z}{E_x} \right)$$

Inst. Pt. Source in Linear Shear, cont'd

$$E_x' = E_x\left(1+\phi^2 t^2\right)$$

$$\text{small}\ t, \quad E_x' \to E_x$$

C ~ $t^{-3/2}$

$$\text{large}\ t, \quad E_x' \to E_x \phi^2 t^2$$

$$= \frac{t^2}{12}\left[\left(\frac{\partial u}{\partial y}\right)^2 E_y + \left(\frac{\partial u}{\partial z}\right)^2 E_z\right]$$

C ~ $t^{-5/2}$

(1) (2) (1) (2)

Longitudinal (Shear) Dispersion

(1) differential longitudinal advection

(2) transverse mixing

E_x' is really a dispersion coefficient

Okubo (1970)

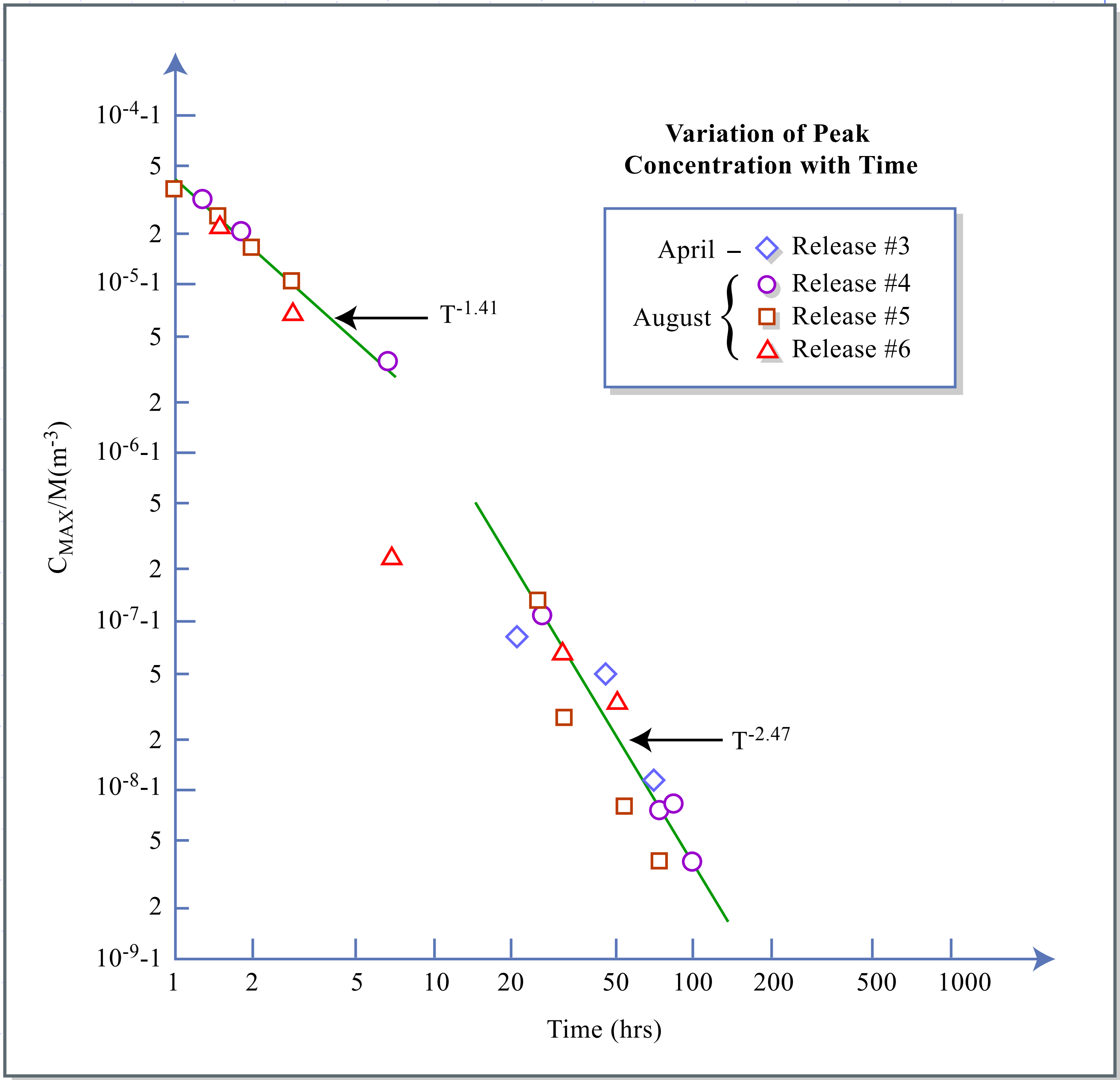


Figure by MIT OCW.

Fluorescent Tracer

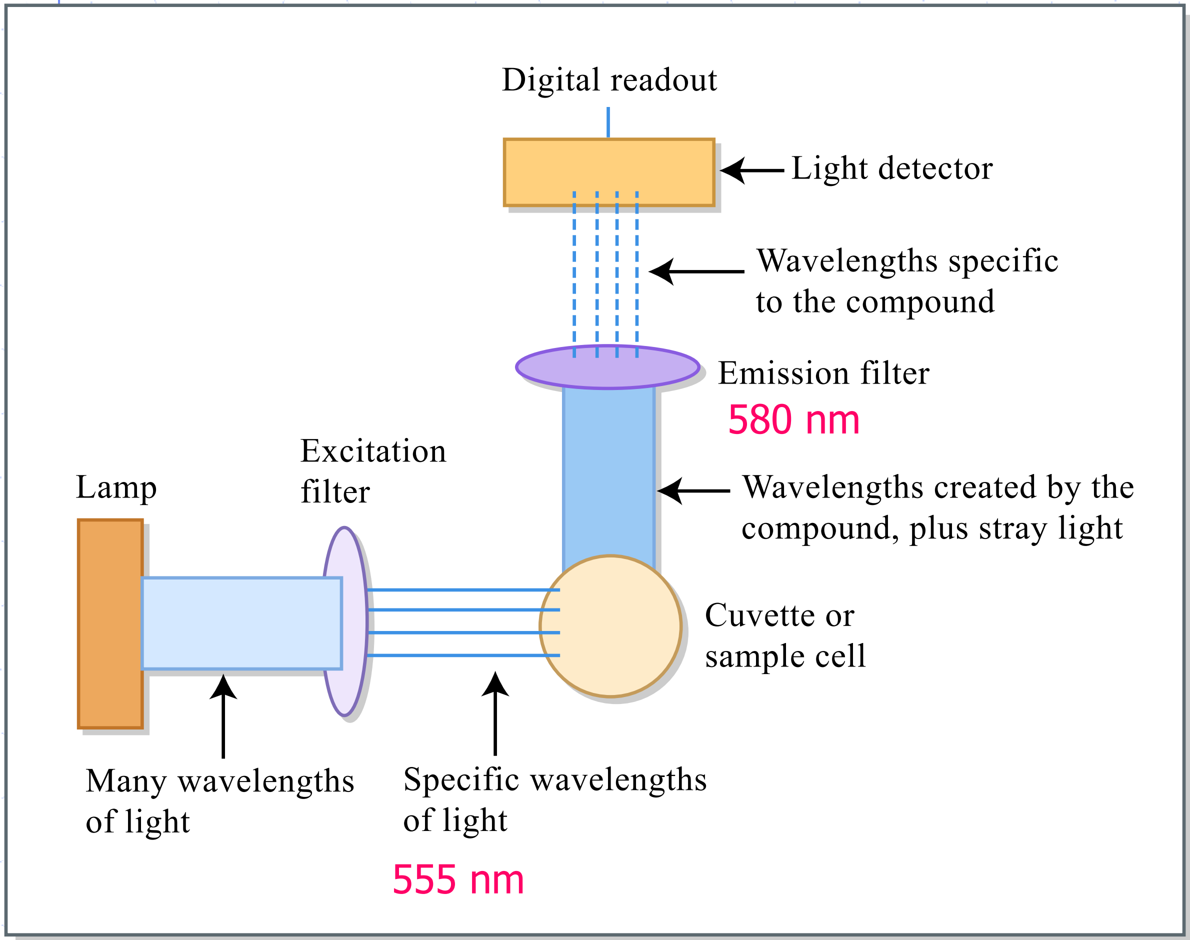


Figure by MIT OCW.

- Rhodamine WT (red dye; fluoresces orange)
- Injected as neutrally buoyant liquid
- Flow thru or *in situ* fluorometer (I ~ c)
- Detection ~ 10^{-10}

Example

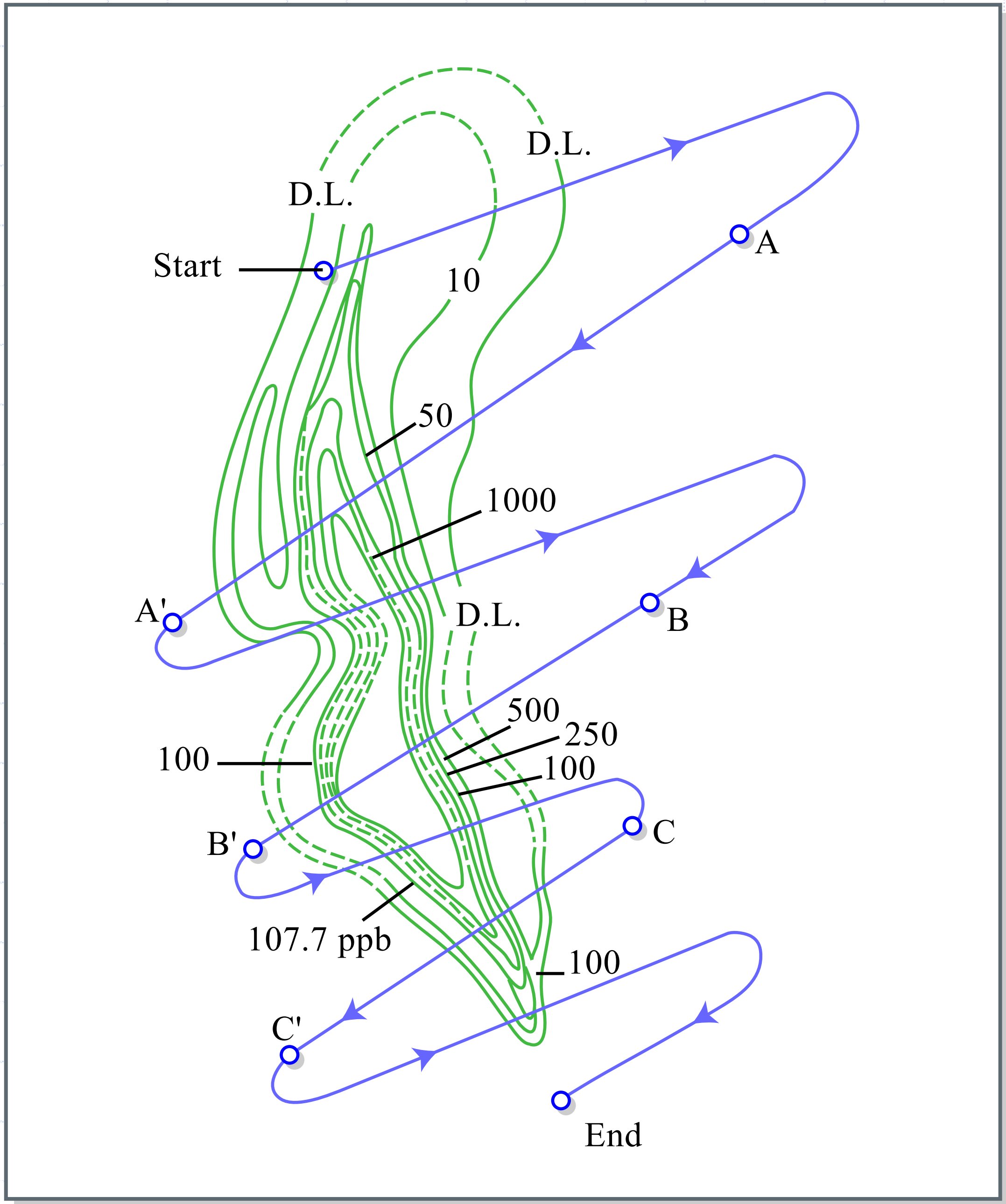


Figure by MIT OCW.

SF_6

- Injected as gas dissolved in water
- Sampled with Niskin bottle or equiv (profiles collected w/ Rosette sampler)
- Analyzed w/ shipboard GC w/ electron capture
- Detection ~10^{-17}

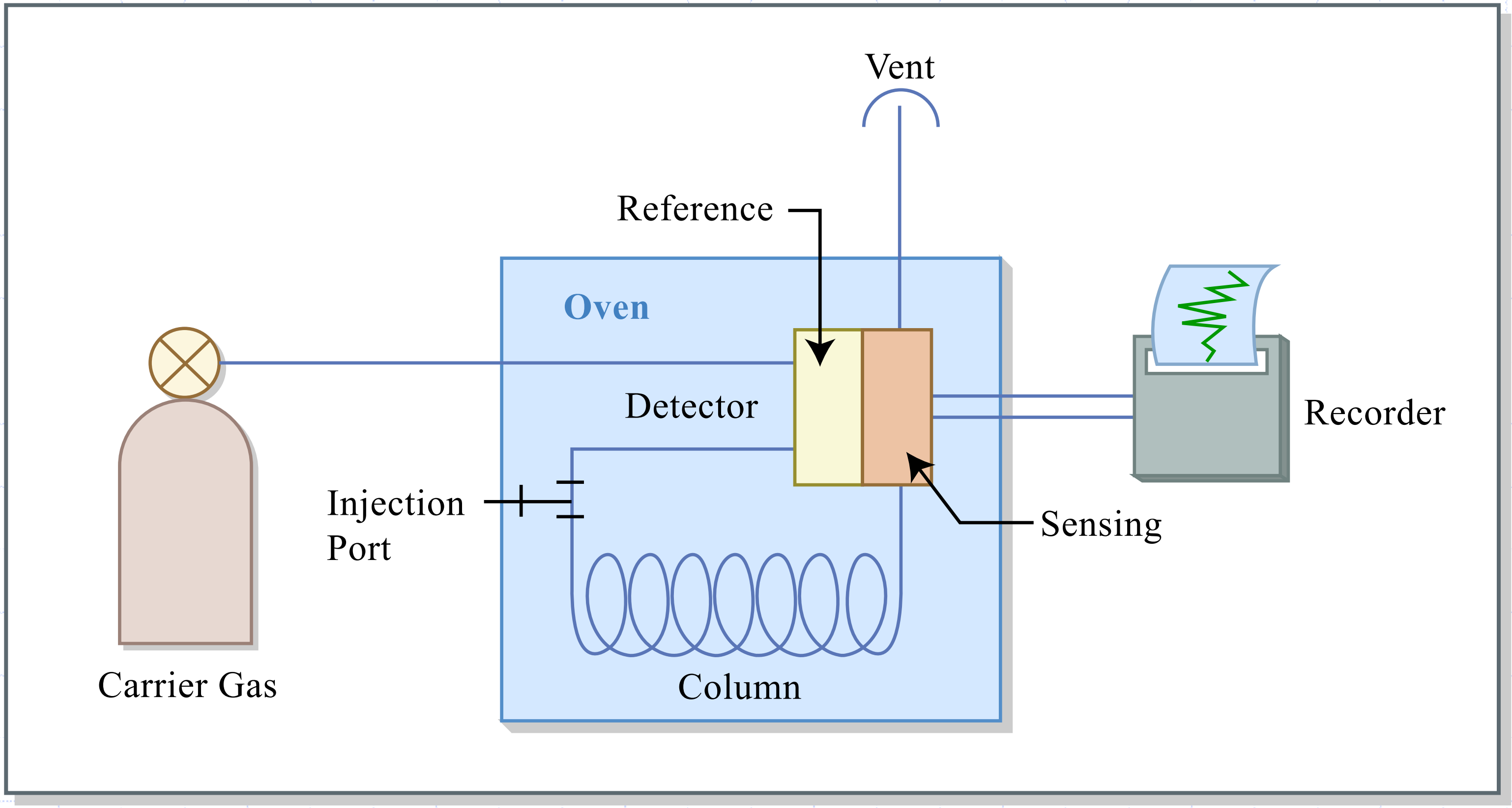
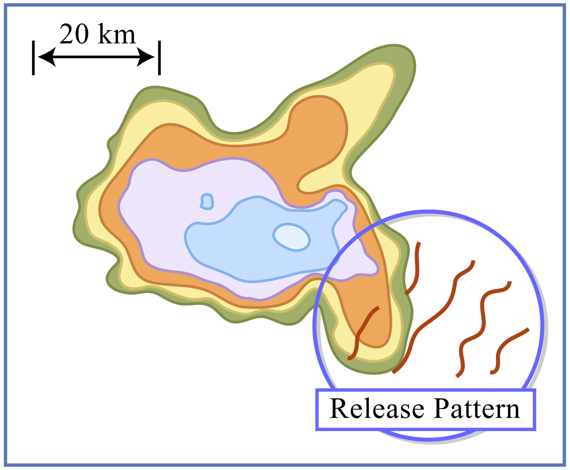
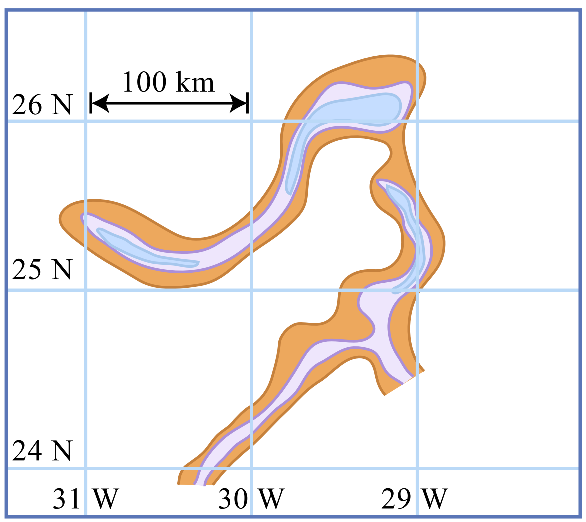


Figure by MIT OCW.

North Atlantic Tracer Release Experiment (NATRE)



Two Weeks After Release



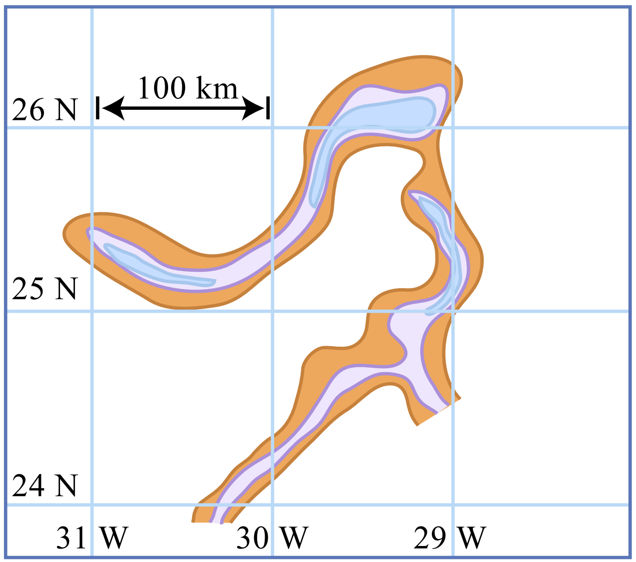
Six Months After Release

- Mass of SF_6: 139 kg
- Location: 1200 km W of Canary Is.
- Depth = 310 m
- Time: 5-13 May, 1992
- References:
  - Ledwell et al., *Nature*, 1993
  - Ledwell et al., *JGR*, 1998

Images: Kim Van Scoy

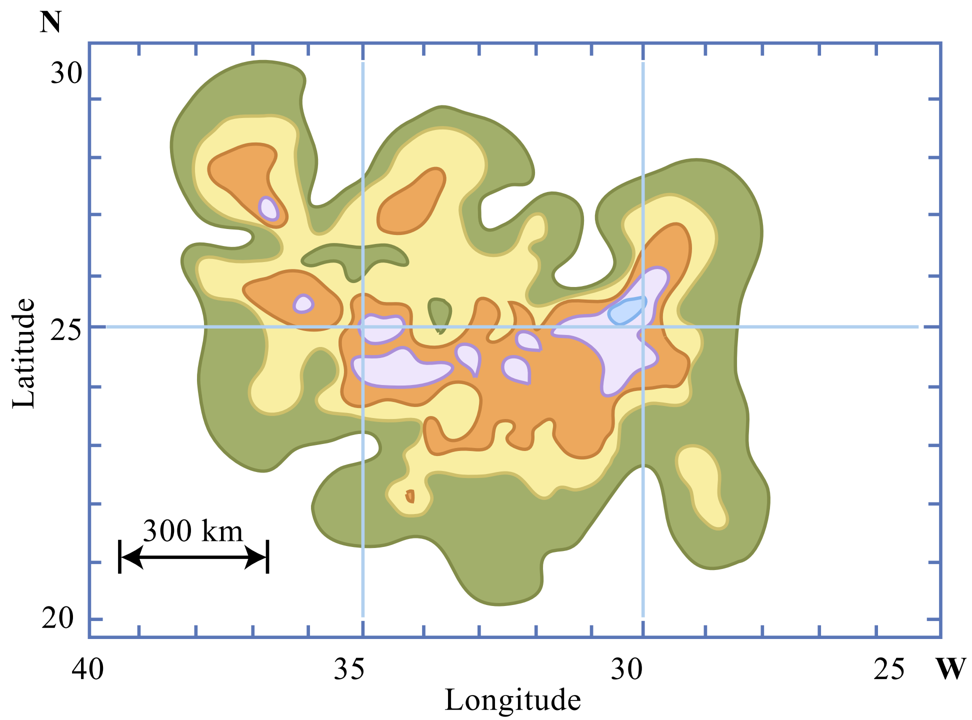
Figures by MIT OCW.

NATRE, cont'd



Six Months After Release

Figure by MIT OCW.



One Year After Release

Figure by MIT OCW.

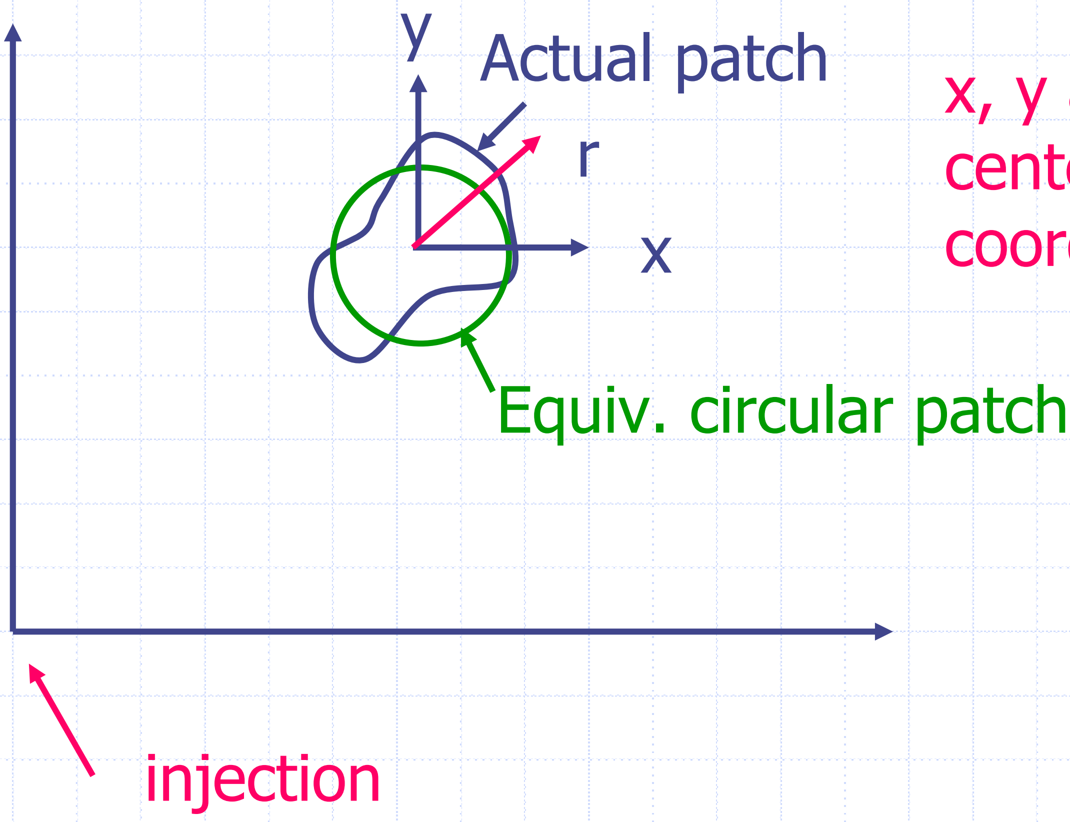
Images: Kim Van Scoy

Drogues (drifters)

- Floats w/ large drag at constant depth
- Have flag or periodically rise to surface
- Position viewed from above or recorded using GPS

Horizontal Diffusion

Historically analyzed using vertical line source in cylindrical coordinates (rather than x, y)



x, y are relative (to center of mass) coordinates

Cylindrical Coordinates, cont'd

$$x^2 + y^2 = r^2 \quad => \quad \sigma_x{}^2 + \sigma_y{}^2 = \sigma_r{}^2$$

$$E_x = E_y = E_r$$

$$u = v = 0$$

Diffusivity assumed horizontally isotropic, independent of coordinate system

$$m' = M / h$$

vertical line source

$$\sigma_r{}^2 = \frac{M_2}{M_o} = \frac{\int_0^\infty 2\pi r c r^2 dr}{\int_0^\infty 2\pi r c dr} = 4E_r$$

4 (vs 2) because $\sigma_r{}^2 = 2\sigma_x{}^2$

If E_r is const. (or treated as such)

$$c = \frac{(M/h)e^{-kt}}{\pi \sigma_r{}^2} e^{-\frac{r^2}{\sigma_r{}^2}} = \frac{(M/h)e^{-kt}}{4\pi E_r t} e^{-\frac{r^2}{4E_r t}}$$

Gaussian; if E_r = const

$c_{max} \sim t^{-1}$ but obs show

$c_{max} \sim t^{-2}$ or t^{-3}

Horizontal Diffusion Diagram (Okubo 1971)

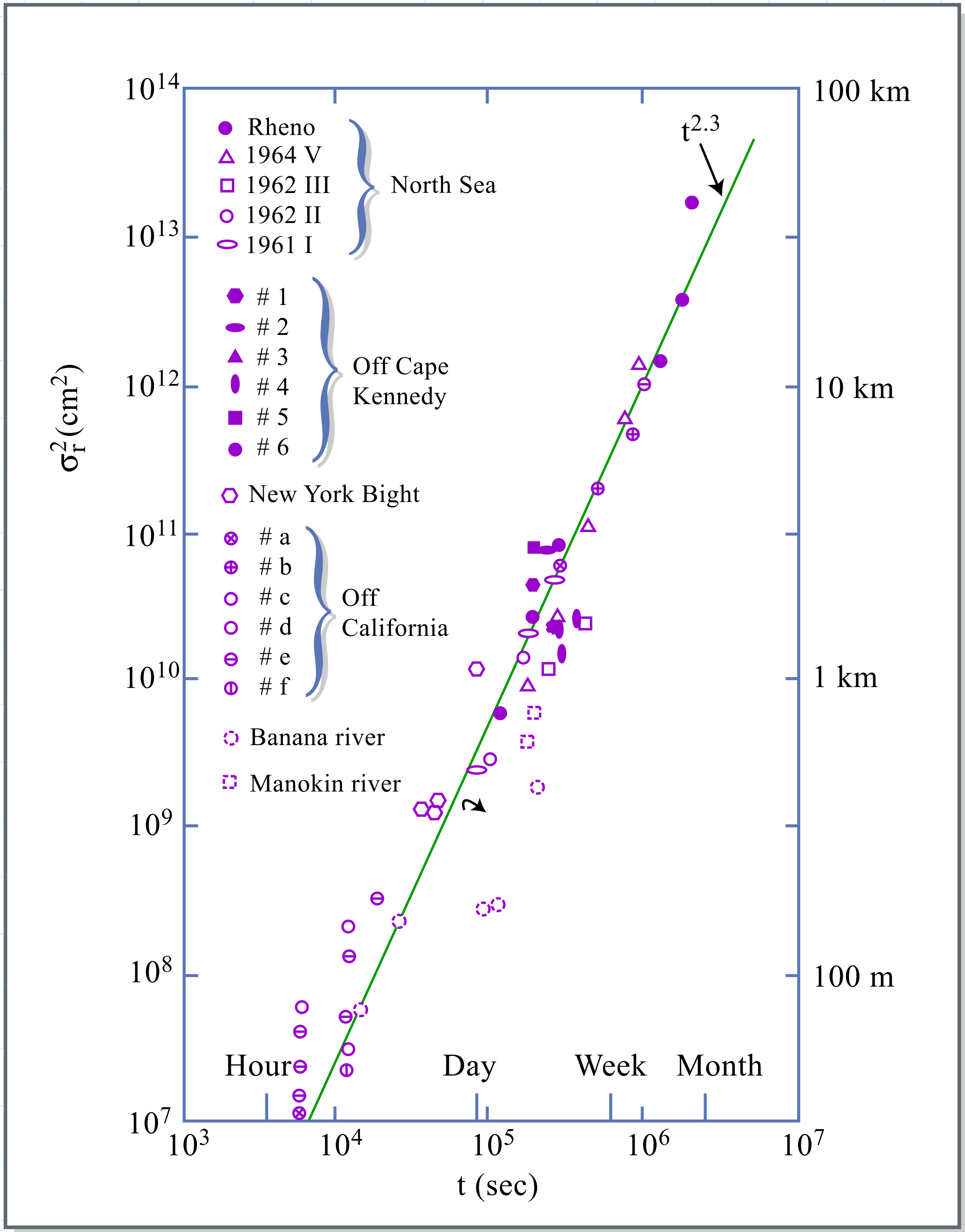


Figure by MIT OCW.

Horizontal Diffusion Summary

$$\sigma_r^2 = 0.011 t^{2.34}$$

cgs units; some data from pt source, some from line source (not quite proper but…)

$$E_r = \frac{d\sigma_r^2}{4dt} = 0.006 t^{1.34}$$

$$E_r = 0.085 \sigma_r^{1.15}$$

$$E_r = 0.017 \ell^{1.15}$$

$$\ell = 4\sigma_r$$

arbitrary length scale of patch

Example

100 kg of paint spilled in Mass Bay over a depth of 10m; how widely will it have spread in one week?

$\sigma_r^2 = 0.011\, t^{2.34} = 3.7 \times 10^{11}\ cm^2 = 3.7 \times 10^7\ m^2$

$\sigma_r = 6000\ m$

Peak concentration?

$$c = \frac{M/h}{\pi \sigma_r^2} e^{-kt} e^{-r^2/\sigma_r^2}$$

k = r = 0; h = 10m; M = 100 kg; t =86400x7=600,000 s

$c = 8.6 \times 10^{-8}\ kg/m^3 = 8.6 \times 10^{-5}\ mg/L$

Gaussian fit; actual peak may be higher

$\sigma_r^2 = 3.7 \times 10^{11}\ cm^2$

$\sigma_r = 6000\ m$

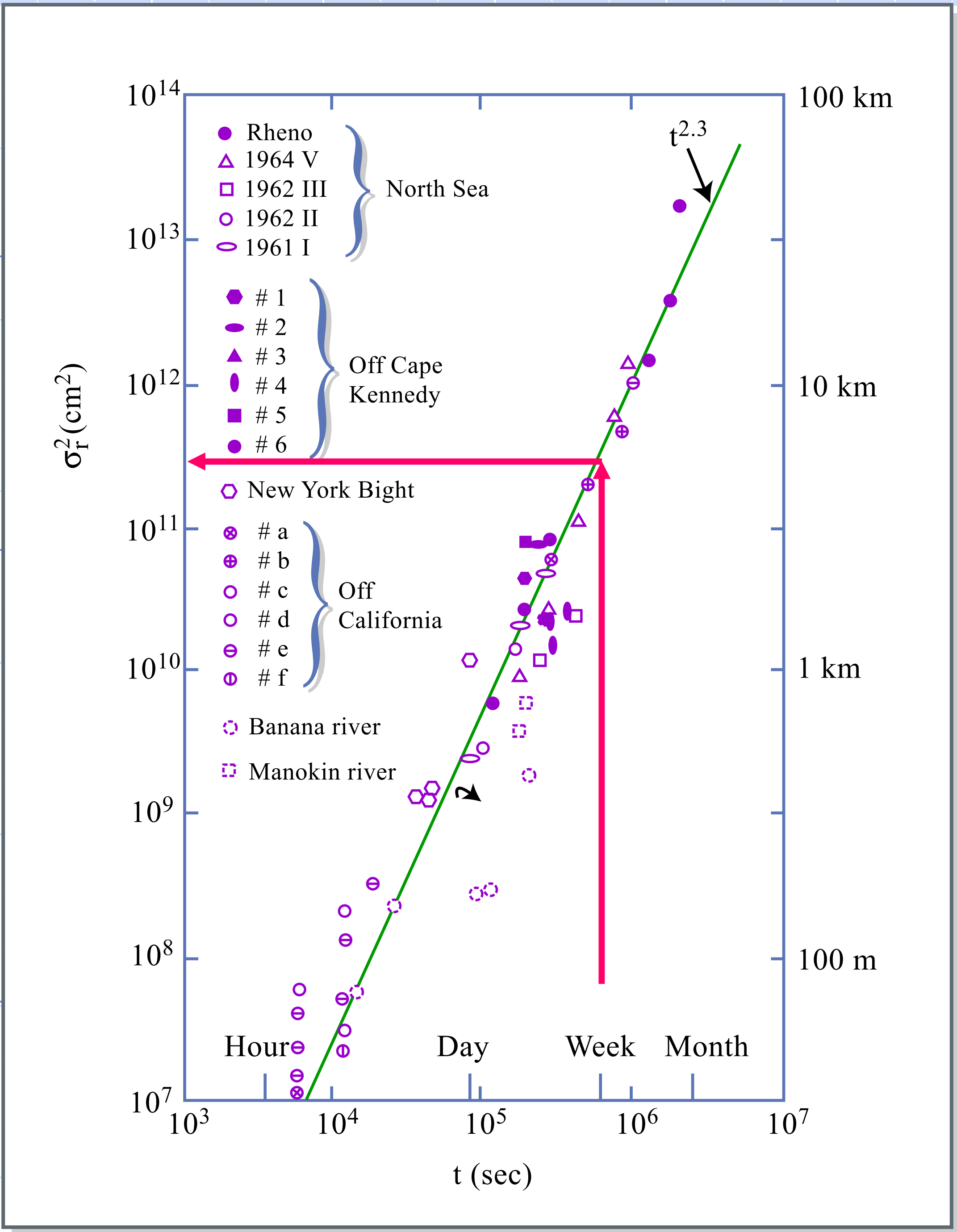


Figure by MIT OCW.

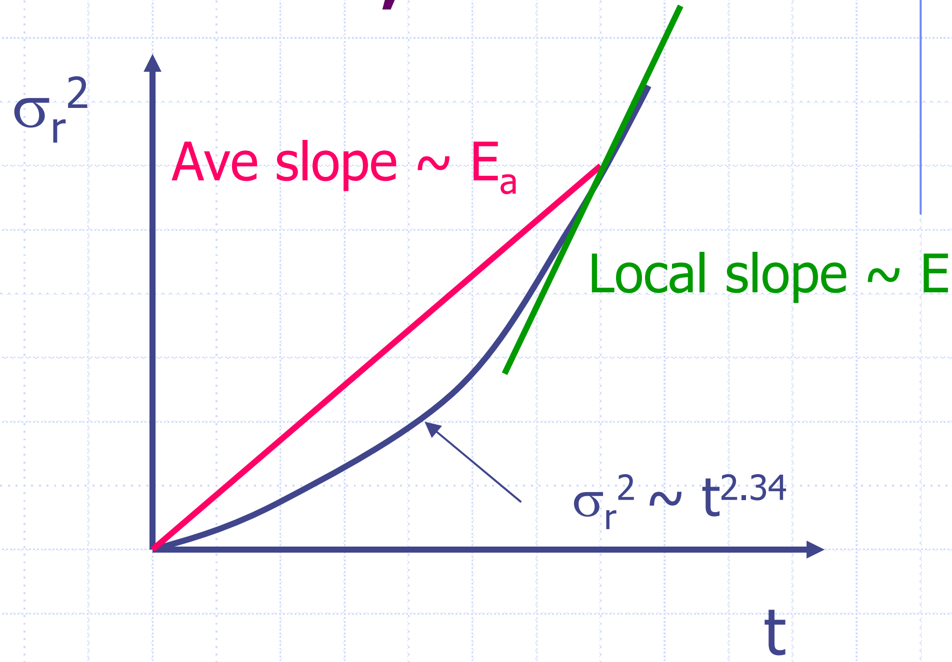
A few more comments

- Three ways to relate tracer spreading: σ(t), E(t), E(σ)
- E(σ) => scale dependent diffusion.
- Not truly stationary => ensemble average not same as individual realization (absolute vs relative diffusion; more later)
- $\ell = 4\sigma_r$ is arbitrary; others choose $\ell = 3.5\sigma, \quad \ell = \sqrt{12}\sigma$

A few more comments, cont'd

$$E_r = \frac{d\sigma_r^2}{4dt}$$ Diffusivity

$$E_{ra} = \frac{\sigma_r^2}{4t}$$ Apparent diffusivity



Richardson's 4/3 law

$$E \sim \varepsilon^{1/3} \ell^{4/3}$$

4/3 rather than 1.15; theoretical (but not empirical) basis

Interpretation of Scale-dependent horizontal diffusivity & 4/3 law

- Eddy Soup: As patch increases in size it encounters eddies of increasing size (eddies smaller than patch spread patch while larger eddies merely advect it)
- 4/3 Law interpreted as shear dispersion:

$$(\phi t)^2 >> 1$$

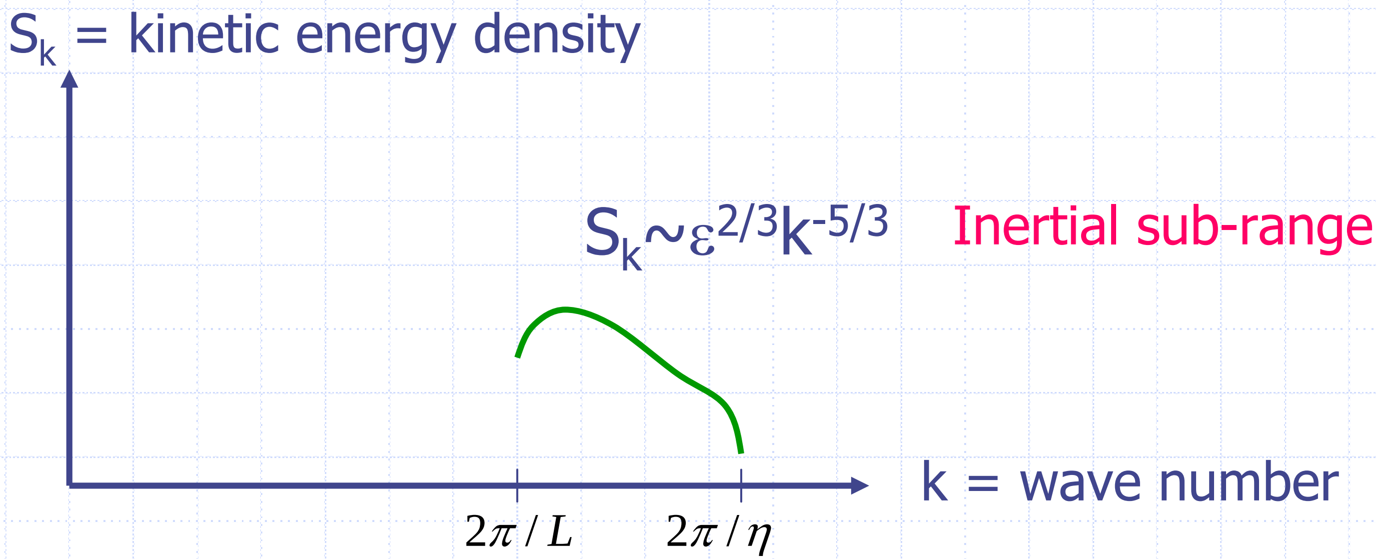
$$E \sim t^2$$

$$E \sim \frac{d\sigma^2}{dt} => \sigma^2 \sim t^3 => t \sim \sigma^{2/3}$$

$$E \sim \sigma^{4/3}$$

Interpretation of 4/3 law, cont'd

- 4/3 law in inertial sub-range



E = diffusivity ~ u′L

ε= dissipation rate ~ d*k*/dt ~ $u'^2/t \sim u'^2/(L/u') \sim u'^3/L$ = const

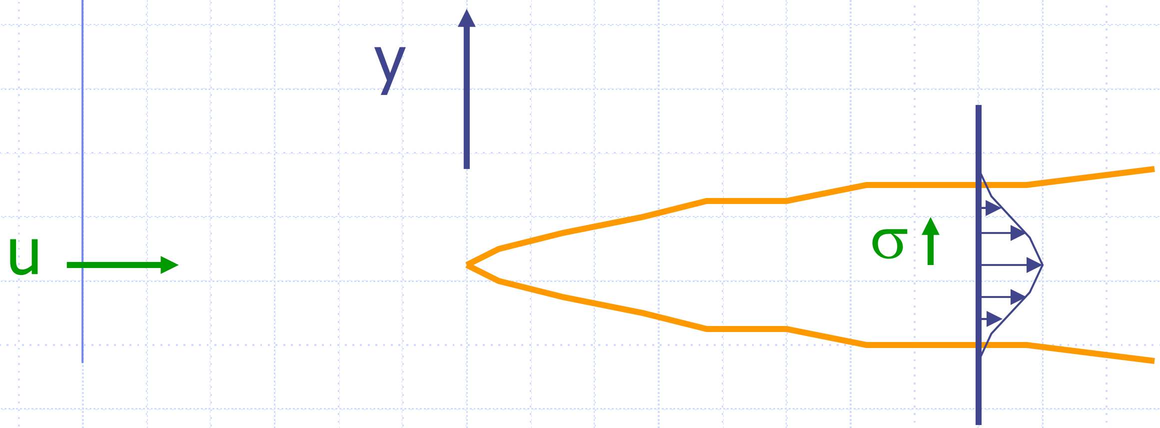
$u' \sim L^{1/3}$

$E \sim u'L \sim L^{4/3}$

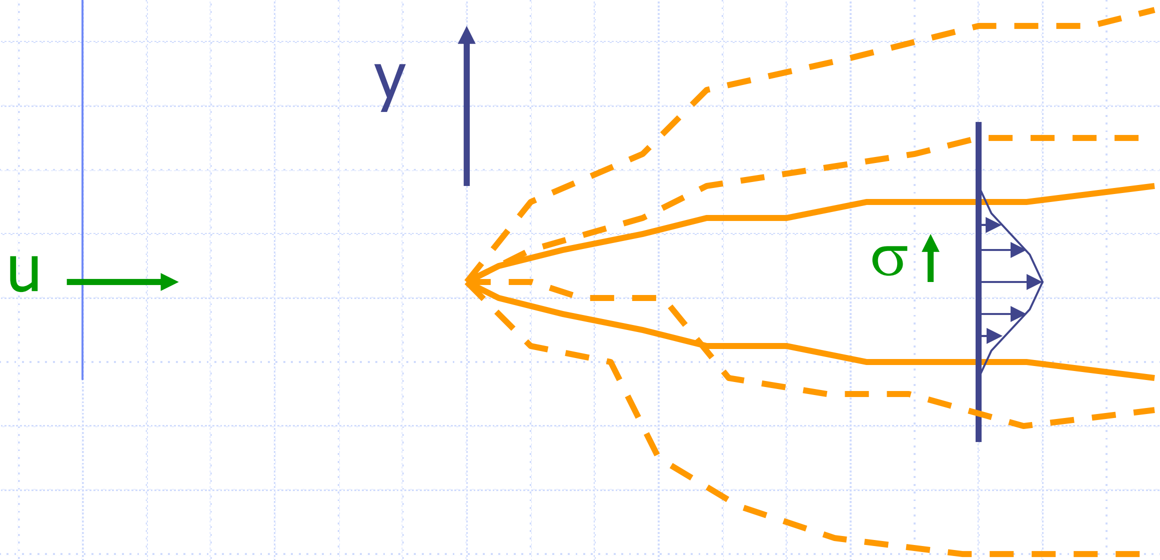
Summary

	Fickian	Okubo	4/3 Law	Gen'l
$\sigma^2(t)$	$\sigma^2 \sim t$	$\sigma^2 \sim t^{2.34}$	$\sigma^2 \sim t^3$	$\sigma^2 \sim t^q$
$E(t)$	$E \sim const$	$E \sim t^{1.34}$	$E \sim t^2$	$E \sim t^{q-1}$
$E(\sigma)$	$E \sim const$	$E \sim \sigma^{1.15}$	$E \sim \sigma^{4/3}$	$E \sim \sigma^{(2q-2)/q}$

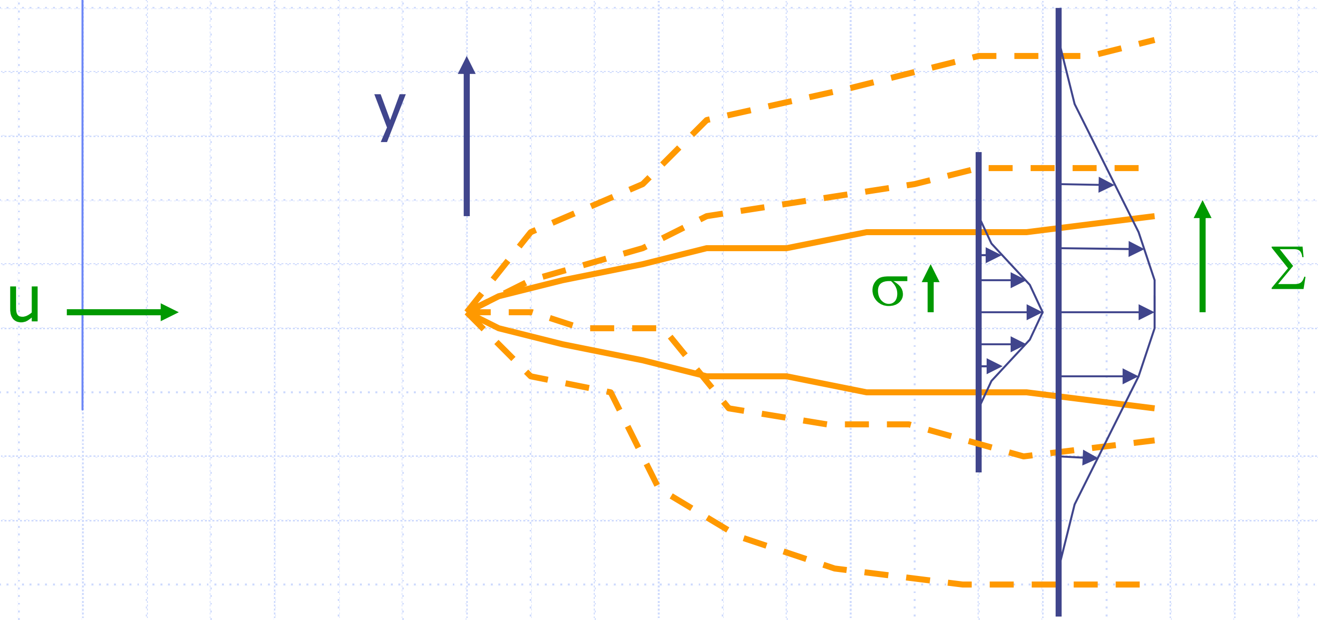
Absolute vs Relative Diffusion



Absolute vs Relative Diffusion

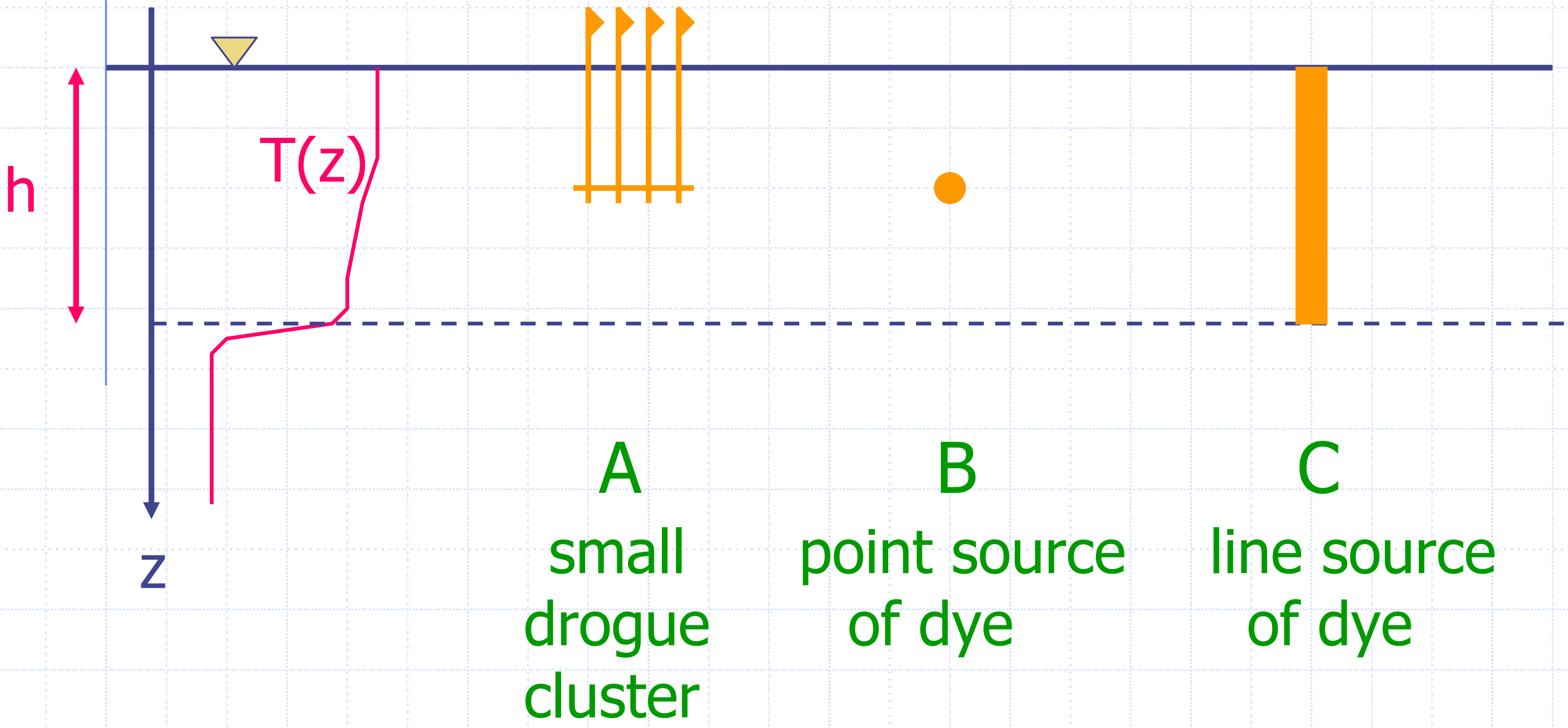


Absolute vs Relative Diffusion

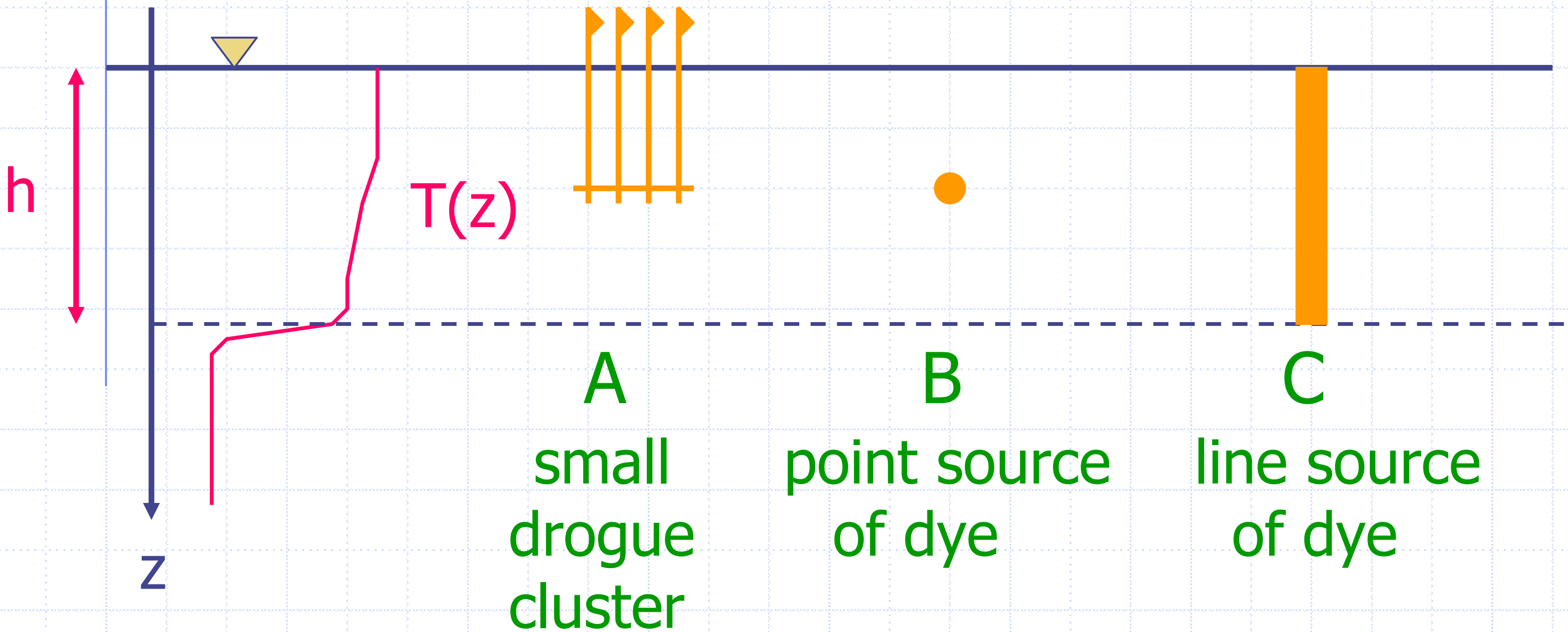


Absolute diffusion (Σ^2) > Relative diffusion (σ^2); ratio decreases with time

Do the values of E_r differ (and if so, which is right)?



Do the values of E_r differ (and if so, which is right)?



E_r (drogue) < E_r (point) < E_r (line)

Okubo et al. (1983)

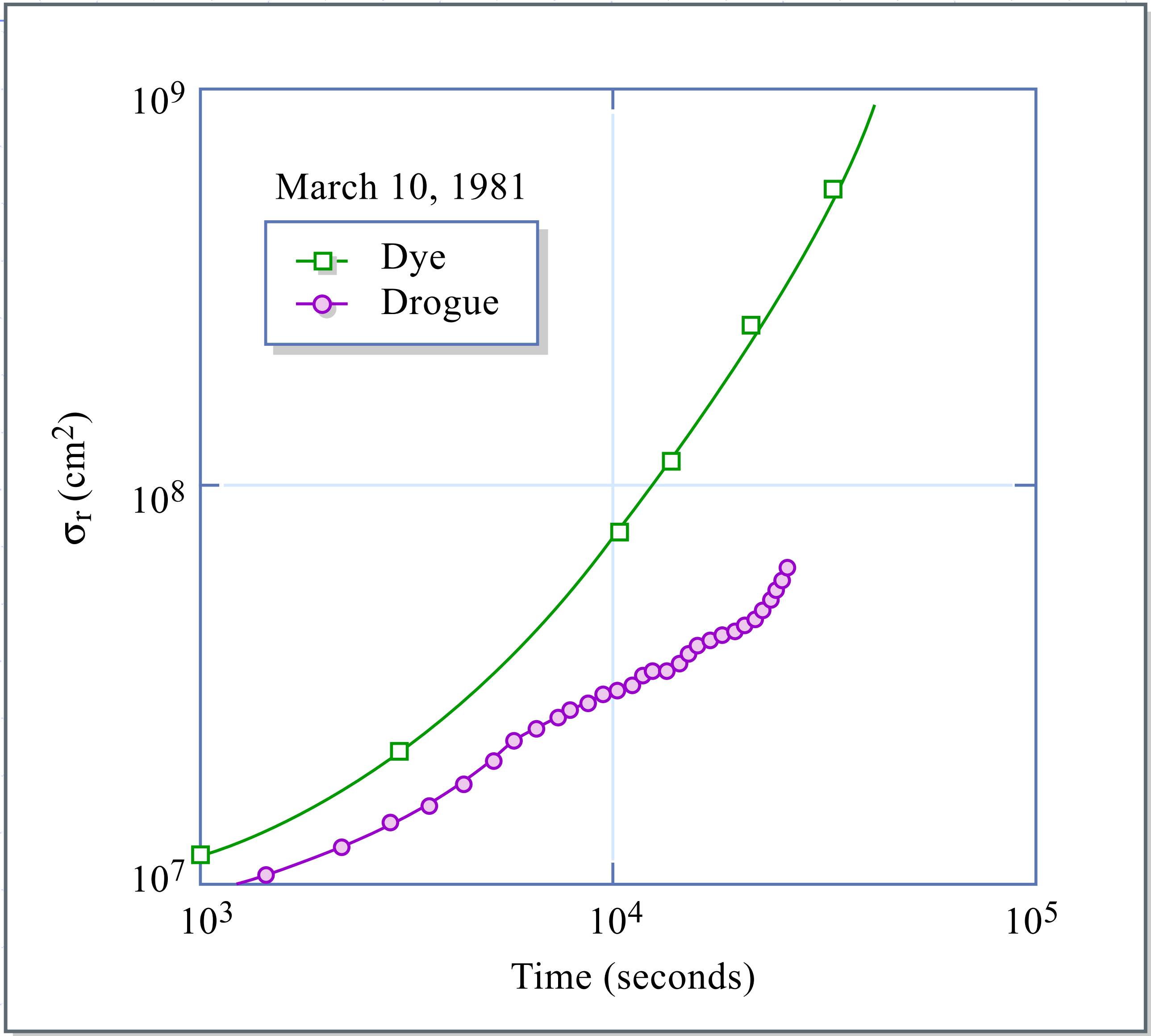
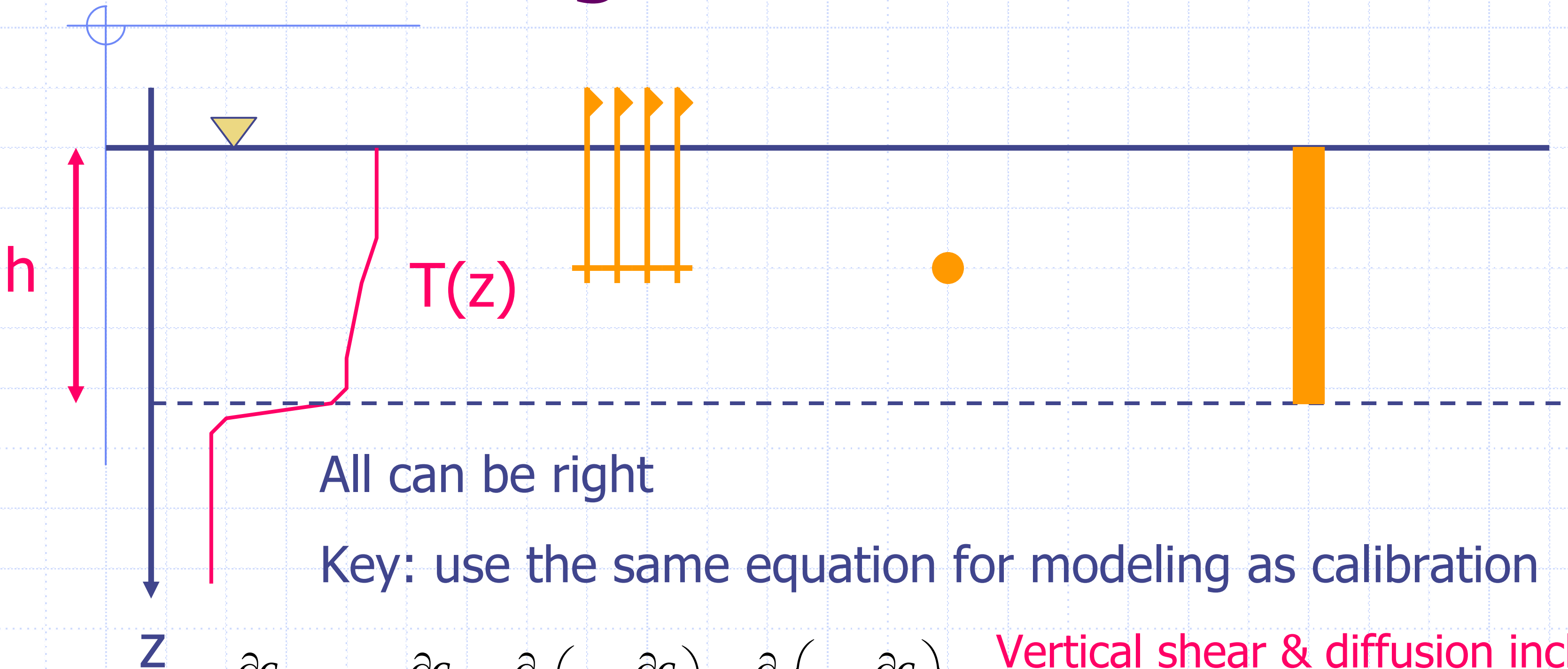


Figure by MIT OCW.

OK, the values of E_r differ, but which is right?



All can be right

Key: use the same equation for modeling as calibration

$$\frac{\partial c}{\partial t} + u(z)\frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\left(E_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial z}\left(E_z \frac{\partial c}{\partial_z}\right)$$

Vertical shear & diffusion included explicitly in g.e. => don’t want them influencing E_x => use drogues

$$\frac{\partial c}{\partial t} + u_{ave}\frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\left(E_x \frac{\partial c}{\partial x}\right)$$

Shear & diffusion excluded from g.e. include effects in E_x calibration => use line source of dye

Diffusivities in numerical models (with finite grid sizes)

$$E_h = \alpha \Delta x \Delta y \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + 0.5 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \right]^{1/2}$$

Smagorinski and Lilly (1963)

α = Smagorinski coefficient (0.1-2.0)

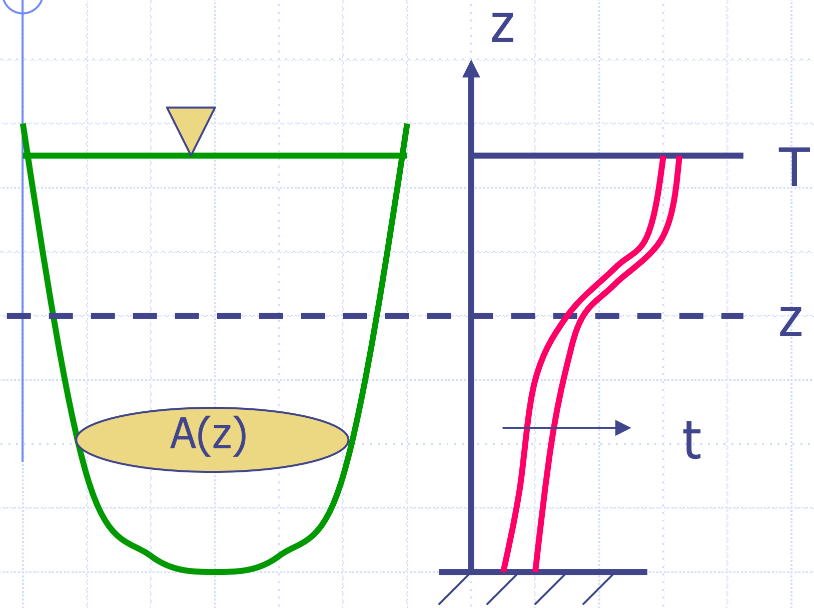
α = 0.16 theoretically; higher empirical values account for vertical shear

Vertical Diffusion

- Fit to large scale property distributions
  - Flux gradient method (lakes & reservoirs)
  - Upwelling diffusion (ocean)
- Measured rate of spread of tracer second moment
- Rates of measured dissipation
- Others

Decreasing time scale

Flux-gradient method



$$E(z,t) = \frac{\frac{\partial}{\partial t}\int_0^z A(z)T(z,t)dz}{A(z)\left.\frac{\partial T}{\partial z}\right|_z}$$

- Below depth of other sources/sinks, thermal energy increases only by turbulent diffusion
- Applicable to relatively long time steps (e.g. weeks or more)

North Anna Power Station (WE2-1)

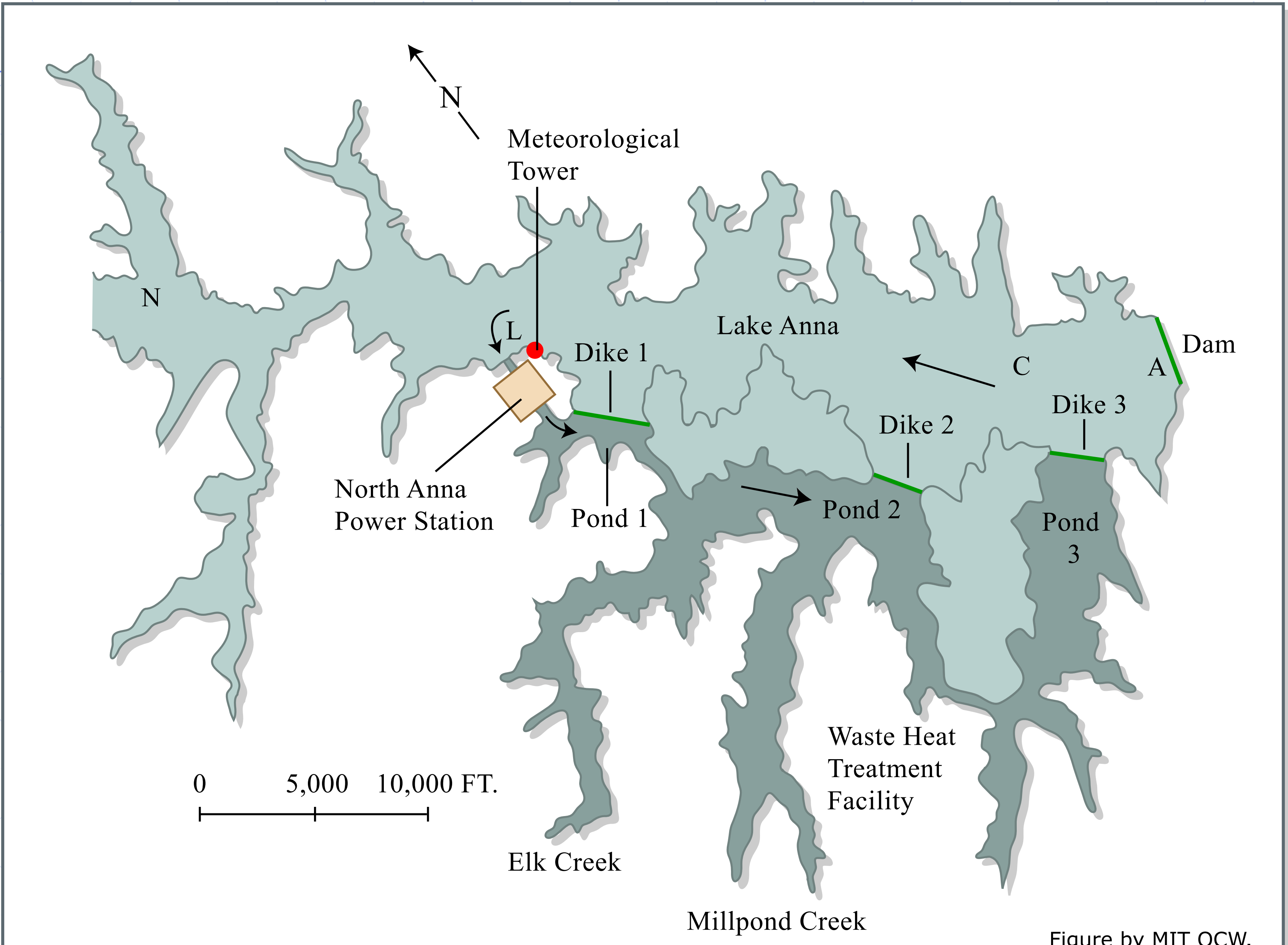
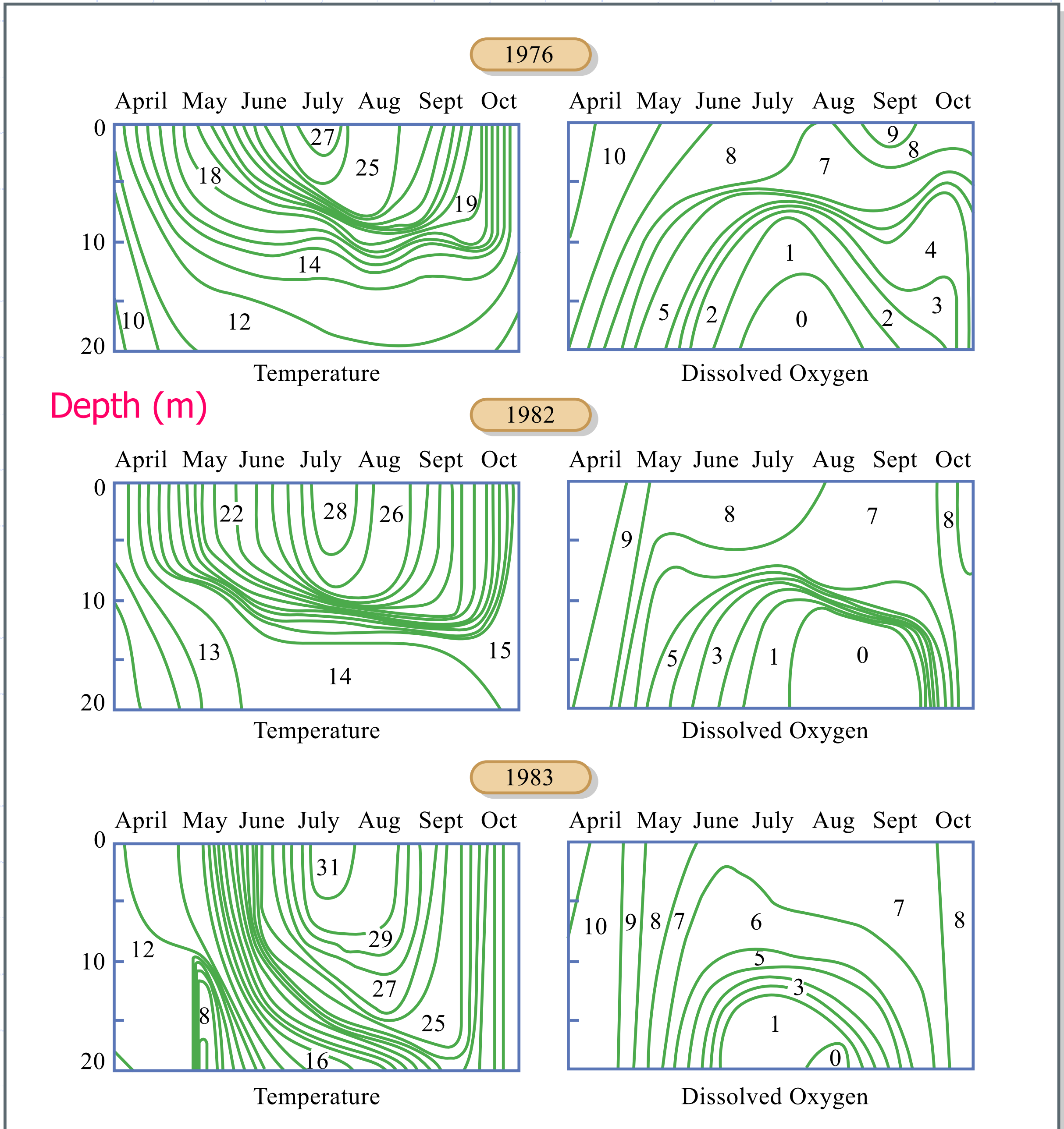


Figure by MIT OCW.

Temperature and DO profiles



June-August average E_z = 0.11 m²/d (0.013 cm²/s)

Pre-operational

0.14 m²/d (0.016 cm²/s)

< one unit

0.46 m²/d (0.053 cm²/s)

~ two units

Dominion Power Co.

Figure by MIT OCW.

Vertical Diffusion from NATRE

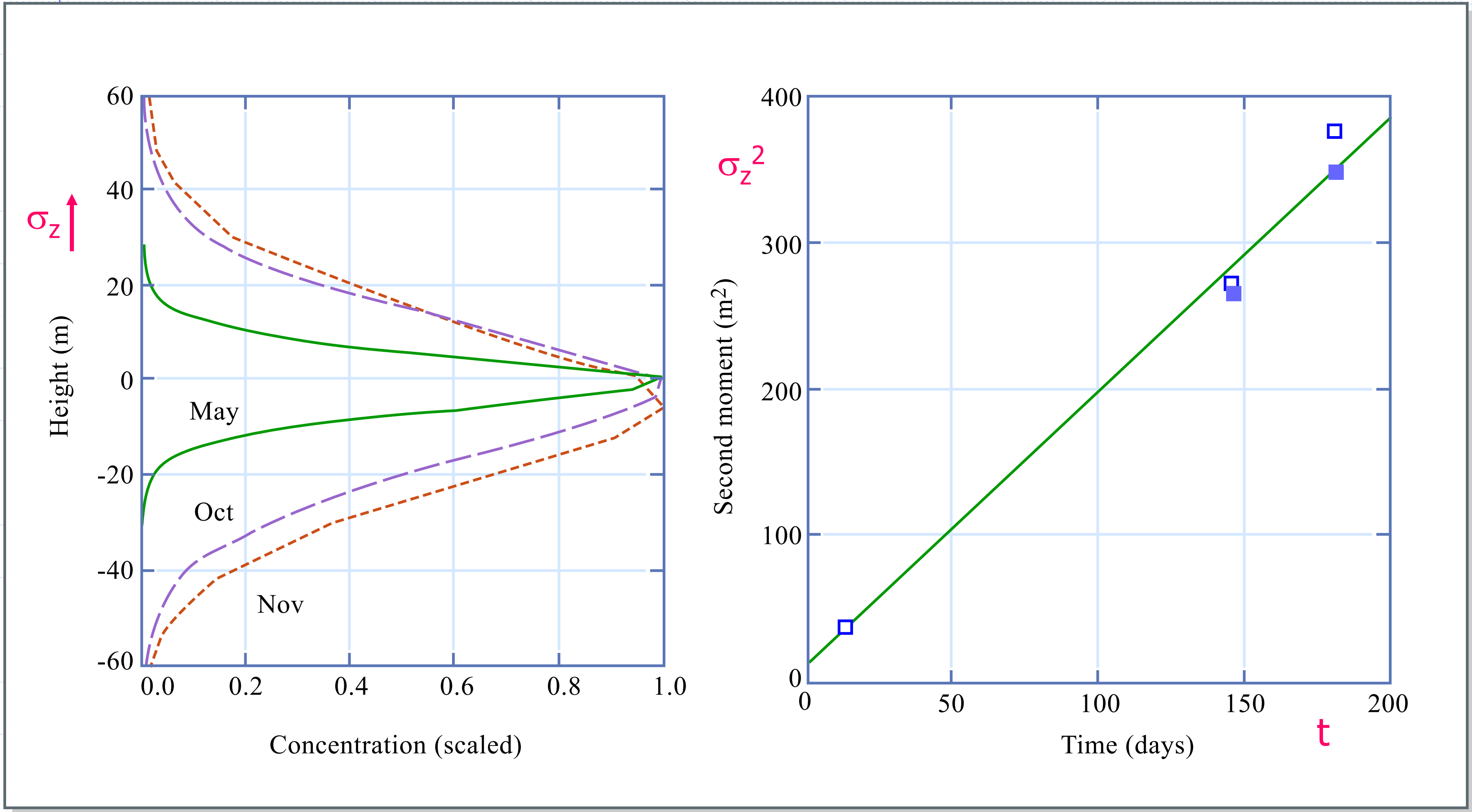


Figure by MIT OCW.

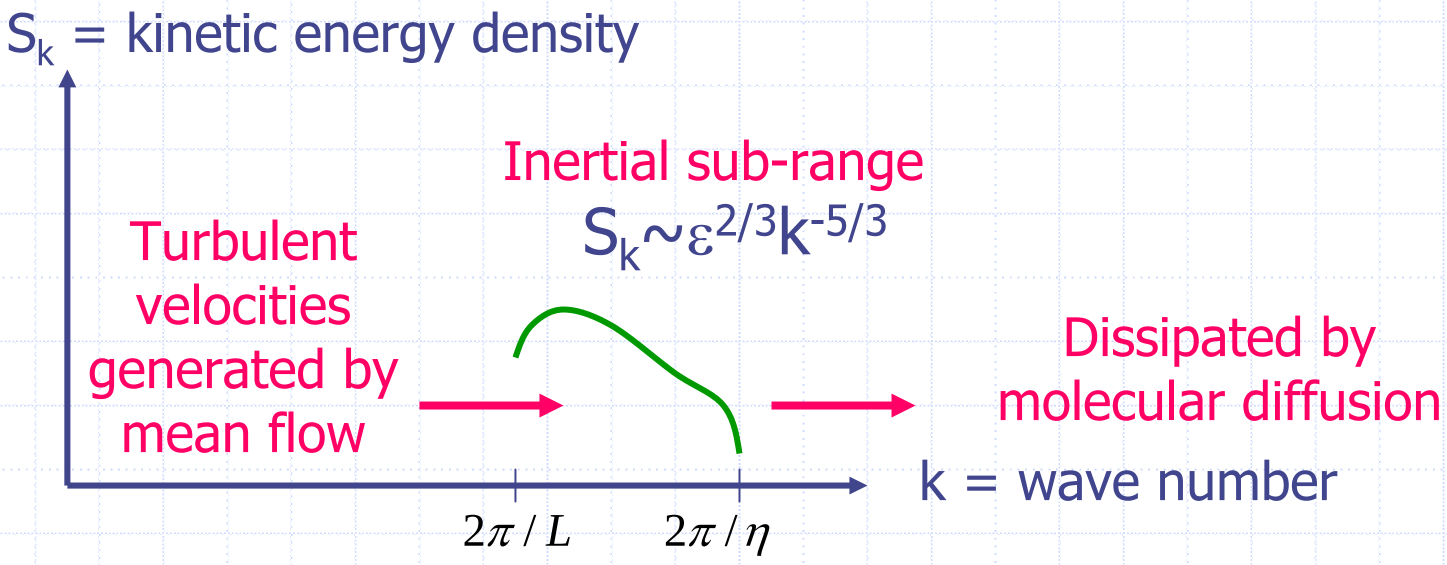
Ledwell, et al., 1993

$$E_z = \frac{z}{2t} \cong 0.11\ \mathrm{cm}^2/\mathrm{s} \quad (6\ \mathrm{mo})$$

$$\cong 0.17\ \mathrm{cm}^2/\mathrm{s} \quad (2\ \mathrm{yr})$$

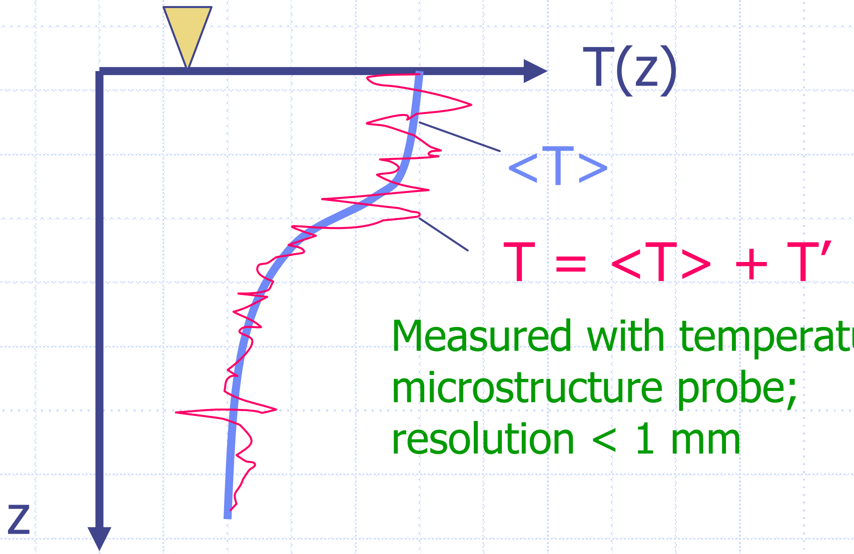
Measured dissipation

From previous discussion



Turbulent temperature variations similar to turbulent velocity variations

Temperature Micro-profile



Measured with temperature microstructure probe; resolution < 1 mm

Generation (of temp variance)

$\sim E_z \, (d\langle T\rangle/dz)^2$

Dissipation (of temp variance)

$\sim \kappa \, (dT'/dz)^2$

E_z = turbulent eddy diffusivity

κ = molecular thermal diffusivity

Formulae based on measured dissipation

Osborn-Cox (1972), Sherman-Davis (1995)

$$E_z = \frac{\langle \chi \rangle}{2\left(\partial \langle T \rangle / \partial z\right)^2}$$

$$\langle \chi \rangle = 2\kappa I \left\langle (\partial T'_z / \partial z)^2 \right\rangle$$

$$E_z = \frac{I\kappa \left\langle (\partial T'_z / \partial z)^2 \right\rangle}{\left(\partial \langle T \rangle / \partial z\right)^2}$$

χ = temp variance dissipation rate [K^2s^{-1}]

T(z) = <T> + T′

κ = molecular thermal diffusivity [m^2s^{-1}]

I ~ 3 (accounts for gradient in T′ in 3 directions)

Osborn (1980)

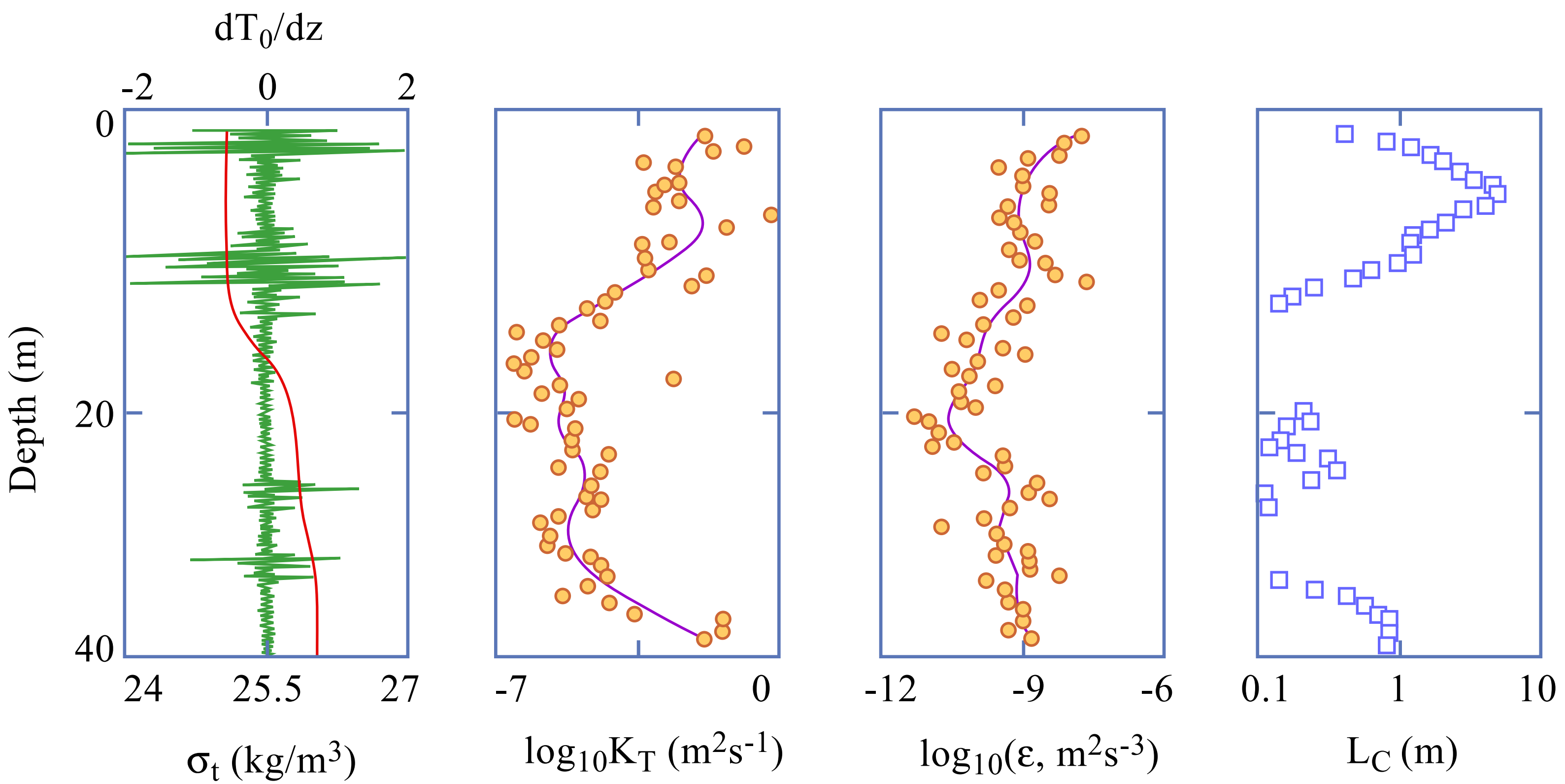
$$E_z = \frac{\gamma_{mix}\varepsilon}{N^2}$$

$N^2 = (g/\rho)(d\rho/dz)$ [s^{-2}]

ε = TKE dissipation rate [m^2s^{-3}]

γ_{mix} = const <= 0.2

Examples



Profiles of (a) density and temperature gradient, (b) K_T, (c) ε and (d) centered-displacement lengthscale L_C. The estimates of K_T and ε include low-pass filtered versions.

Figure by MIT OCW.

Stevens, et al. 2000

Langmuir Circulation

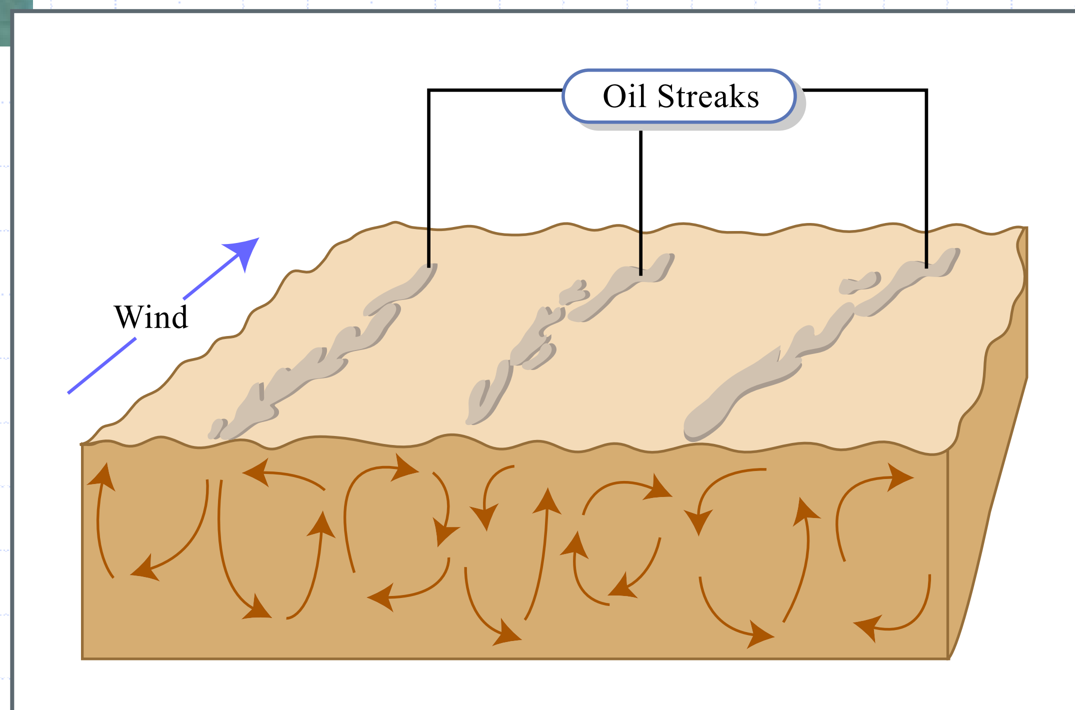


Figure by MIT OCW.

Formulae for E_z

Open waters, near surface

$$E_z = \frac{0.028 H_w^{\,2}}{T_w} e^{-4\pi z / L_w}$$

Ichiye (1967); z = depth; H_w, T_w, L_w = significant wave height, period and length

In presence of stratification and shear

$$E_z = E_{zo}\left[1 + \frac{10}{3} Ri\right]^{-3/2}$$

$$Ri = \frac{(g/\rho)\left|d\rho/dz\right|}{\left(du/dz\right)^2}$$

Munk & Anderson (1948);

Ri = gradient Richardson no

E_{zo} = value at neutral stratification

Formulae for E_z

Stratification only (near surface)

$$E_z = \frac{10^{-6}}{\left|\partial\rho / \partial z\right|}$$

Koh and Fan (1970)

[E_z in cm^2/s; $d\rho/dz$ in g/cm^4]

Stratification only (deep waters)

$$E_z = \frac{4x10^{-9}}{\left|\partial\rho / \partial z\right|}$$

Broecker and Peng (1982)

[E_z in cm^2/s; $d\rho/dz$ in g/cm^4]

Typical ocean

$E_z \cong 0.1$ (local) $E_z \cong 1$ (basin average) E_z in cm^2/s

Formulae, cont'd

Rivers

$$E_z = \kappa u_* z(1 - z/h)$$

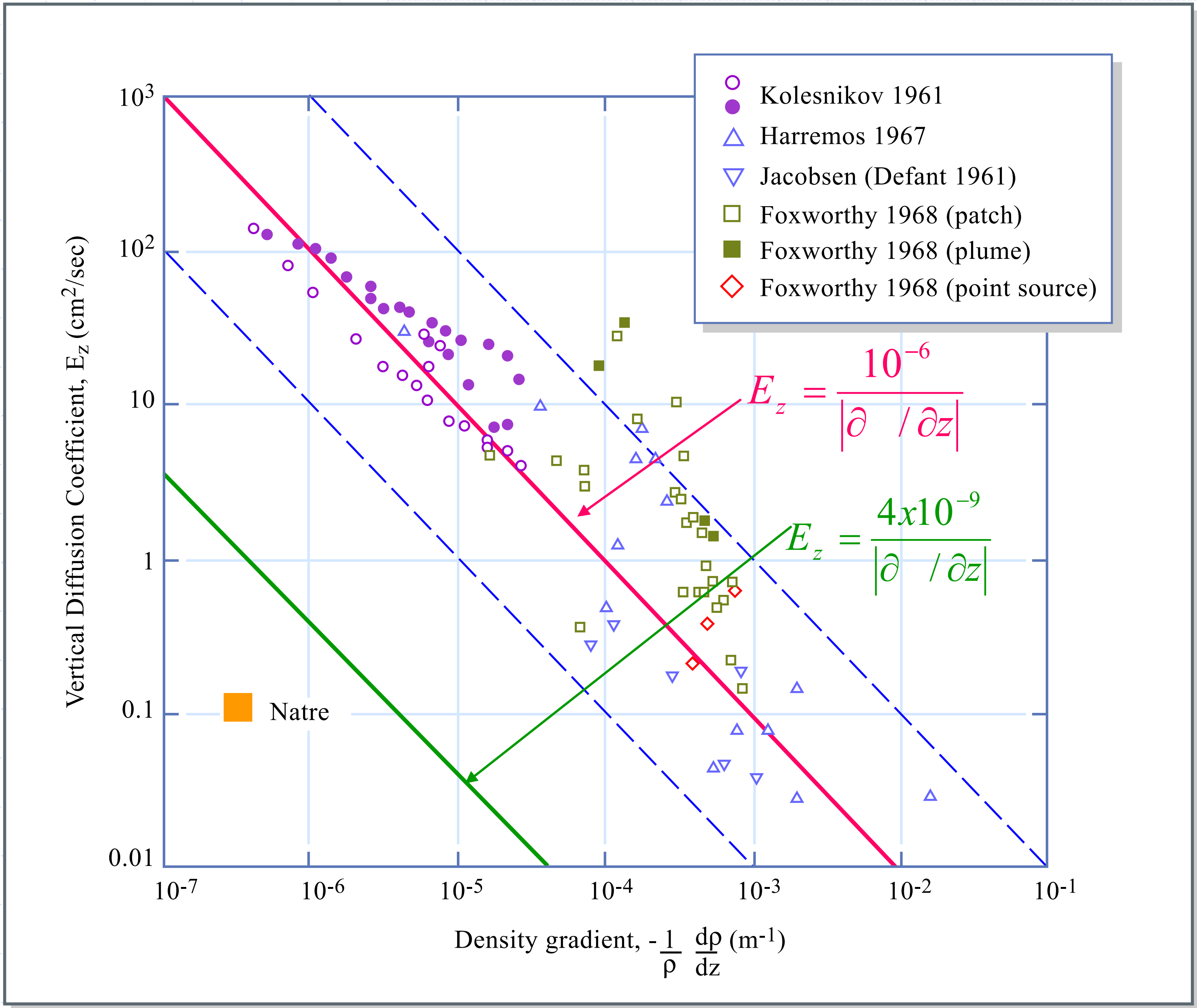
$$\overline{E_z} = 0.07 u_* h$$

u_* = friction velocity, h = water depth, z = height above bottom

Estuaries

$$E_z = \eta |\bar{u}| \frac{z^2 (h-z)^2}{h^3} (1 + \beta Ri)^{-2} + \zeta \frac{z(h-z)}{h} \frac{H_w}{T_w} e^{-2\pi z / L_w} (1 + \beta Ri)^{-2}$$

Pritchard (1971);
$\eta = 8.59 \times 10^{-3}$,
$\zeta = 9.57 \times 10^{-3}$,
$\beta = 0.276$
$\bar{u}$ = mean tidal speed



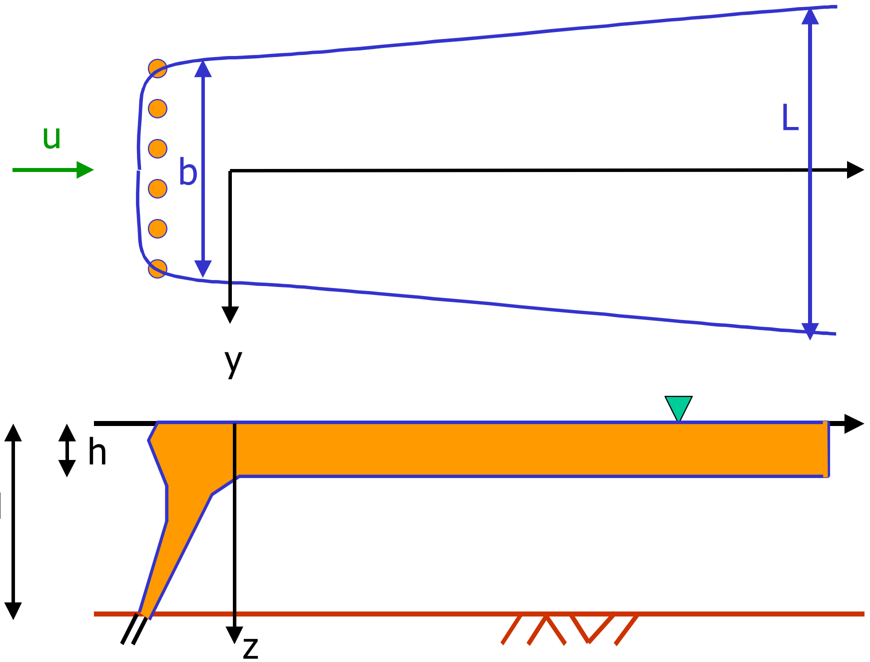
Koh and Fan (1970)

Figure by MIT OCW.

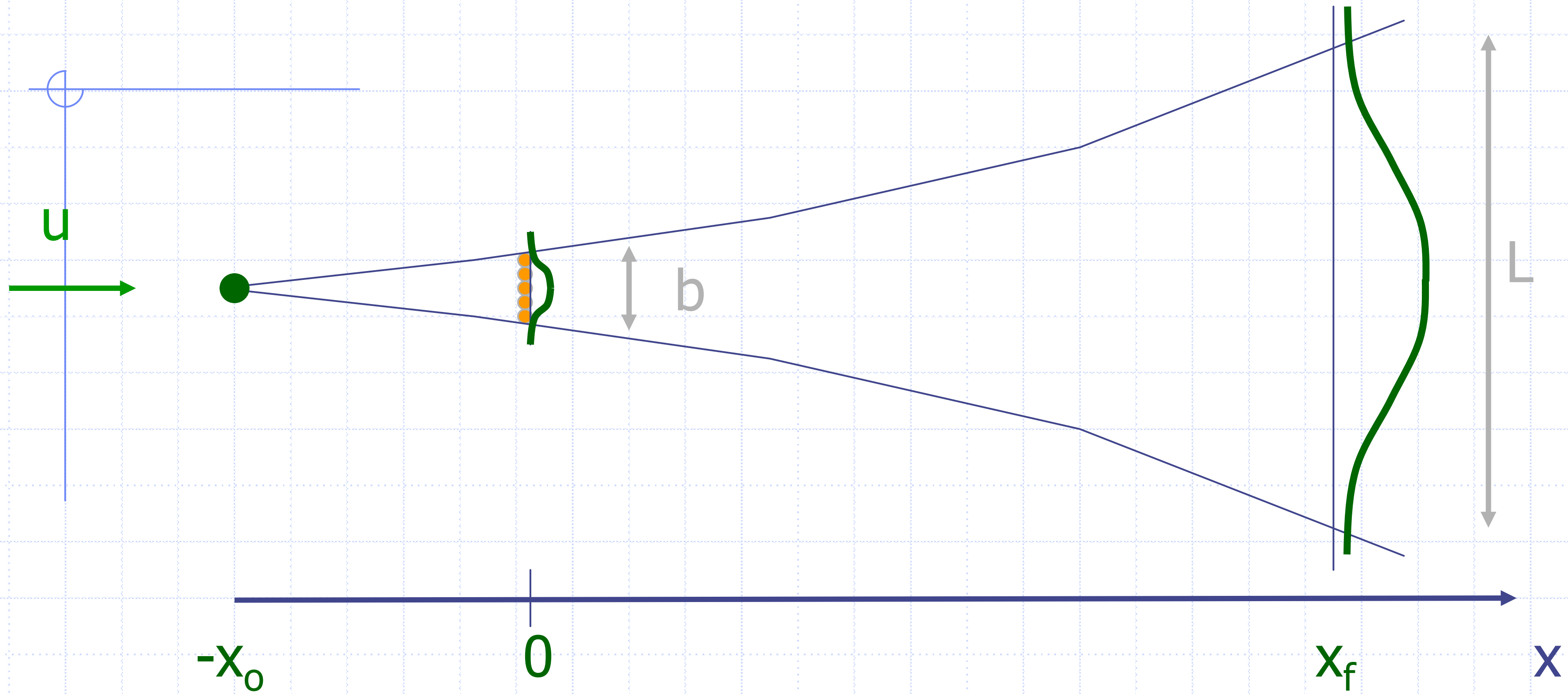
100(dρ/dz) (g/cm^4)

Application: coastal sewage discharge from multi-port diffuser

- Assume
- $b = 300$ m
- $H = 30$ m
- $h = 10$ m
- $u = 0.1$ m/s
- NF dilution $S_N = 100$
- How far ds until $S_F = 10$?
  ($S_T = S_N S_F = 1000$)
- Formal solution by Brooks in Section 2.8; approximate solution follows



Sewage discharge, cont'd

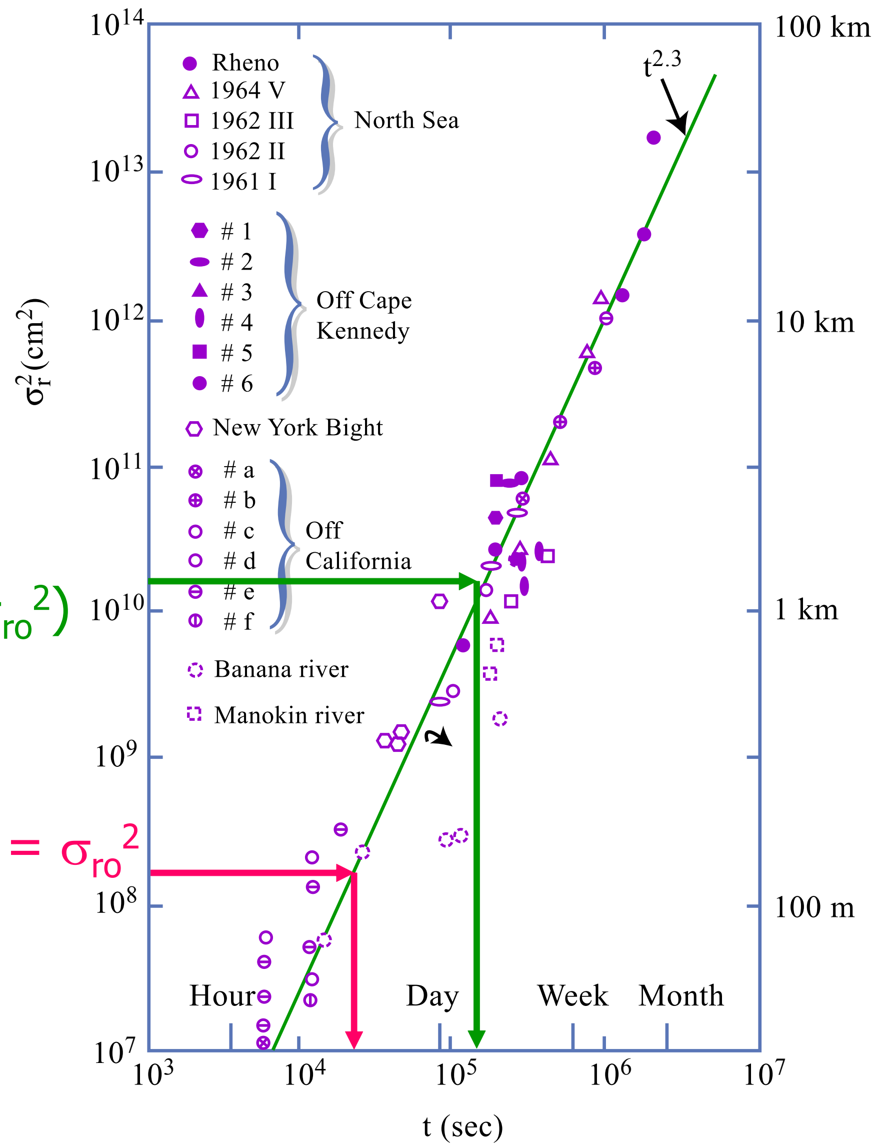


$$at\ x = 0,\ \sigma_{ro} \cong \frac{b}{\sqrt{6}} \cong 0.4b \cong \underline{12{,}000\ cm};\quad \sigma_{rf} = 10\sigma_{ro} = \underline{120{,}000\ cm}$$

$$\sigma_r^{\ 2} = 0.011t^{2.34} \Rightarrow t = \left[\frac{\sigma_r^{\ 2}}{0.011}\right]^{1/2.34};\quad t_o = \left[\frac{12{,}000^2}{0.011}\right]^{1/2.34} = 22{,}000s;\quad t_f = \left[\frac{120{,}000^2}{0.011}\right]^{1/2.34} = 162{,}000s$$

$$x = u(t_f - t_o) = (0.1)(162{,}000 - 22{,}000) = 14{,}000m = 14km$$

Contrast with 30 m!



$S_F = 10$

$(\sigma_{ro}^2 = 100\sigma_{ro}^2)$

$(120m)^2 = \sigma_{ro}^2$

Figure by MIT OCW.

$t_o = 22000s \quad t = 162000s$

$x = u(t-t_o) = (0.1)(162000-22000) = 14\ km$

Neglect of vertical diffusion

Reasonable?

$$\rho_s \cong 1.03 \quad \rho_o \cong 1.00\ g/cm^3$$

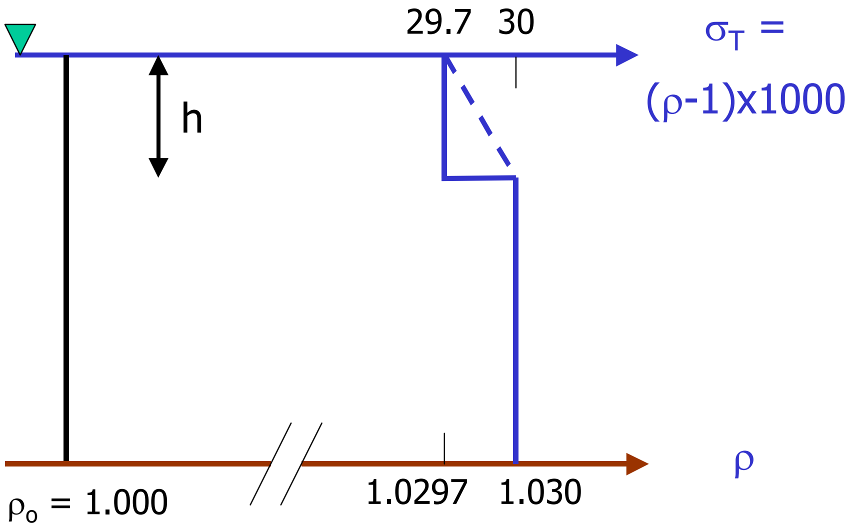
$$\frac{\Delta\rho_o}{\rho} \cong 0.03$$

$$\frac{\Delta\rho_o}{S_N\rho} \cong 0.0003$$

$$\overline{\frac{\partial\rho}{\partial z}} \cong 0.0003/1000 \cong 3x10^{-7}\ g/cm^4$$

conservatively small

$$E_z = \frac{10^{-6}}{\frac{\partial\rho}{\rho\partial z}} \cong \frac{10^{-6}}{3x10^{-7}} \cong 3\ cm^2/s$$



Vertical diffusion, cont'd

$$\sigma_{zo} = \frac{1}{\sqrt{12}}(2h) \cong 5.8\ m$$

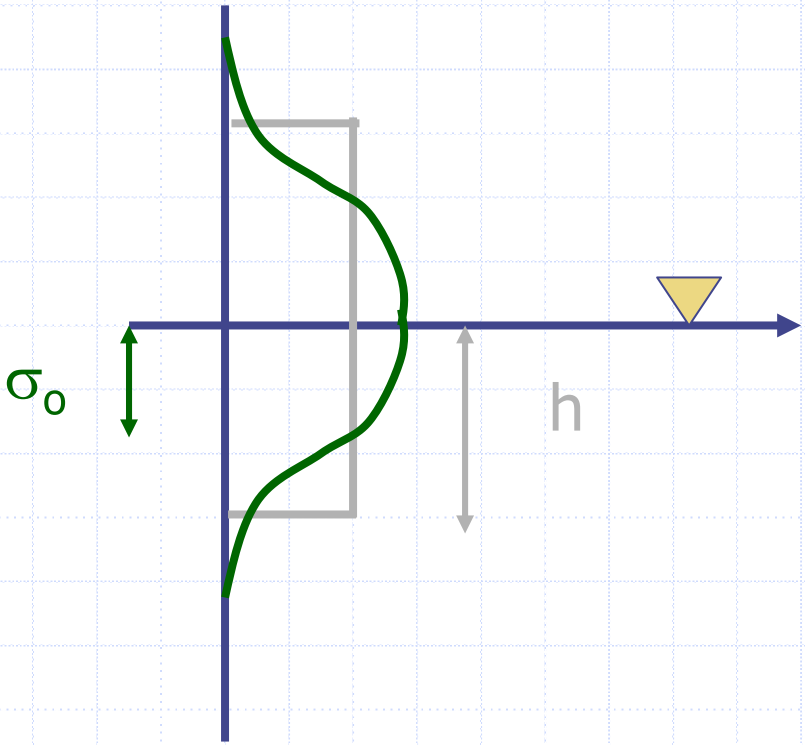
$$\sigma_z^{\ 2} = \sigma_{zo}^{\ 2} + 2E_z t$$

$$= 580^2 + (2)(3)(140000)$$

$$= 117000\ cm^2$$

$$\sigma_z = 10.8\ m$$

$$S_{Fv} = \frac{1080}{580} = 1.9$$



concentration reduction = 9% of that due to horizontal mixing; even smaller if stronger density gradient chosen

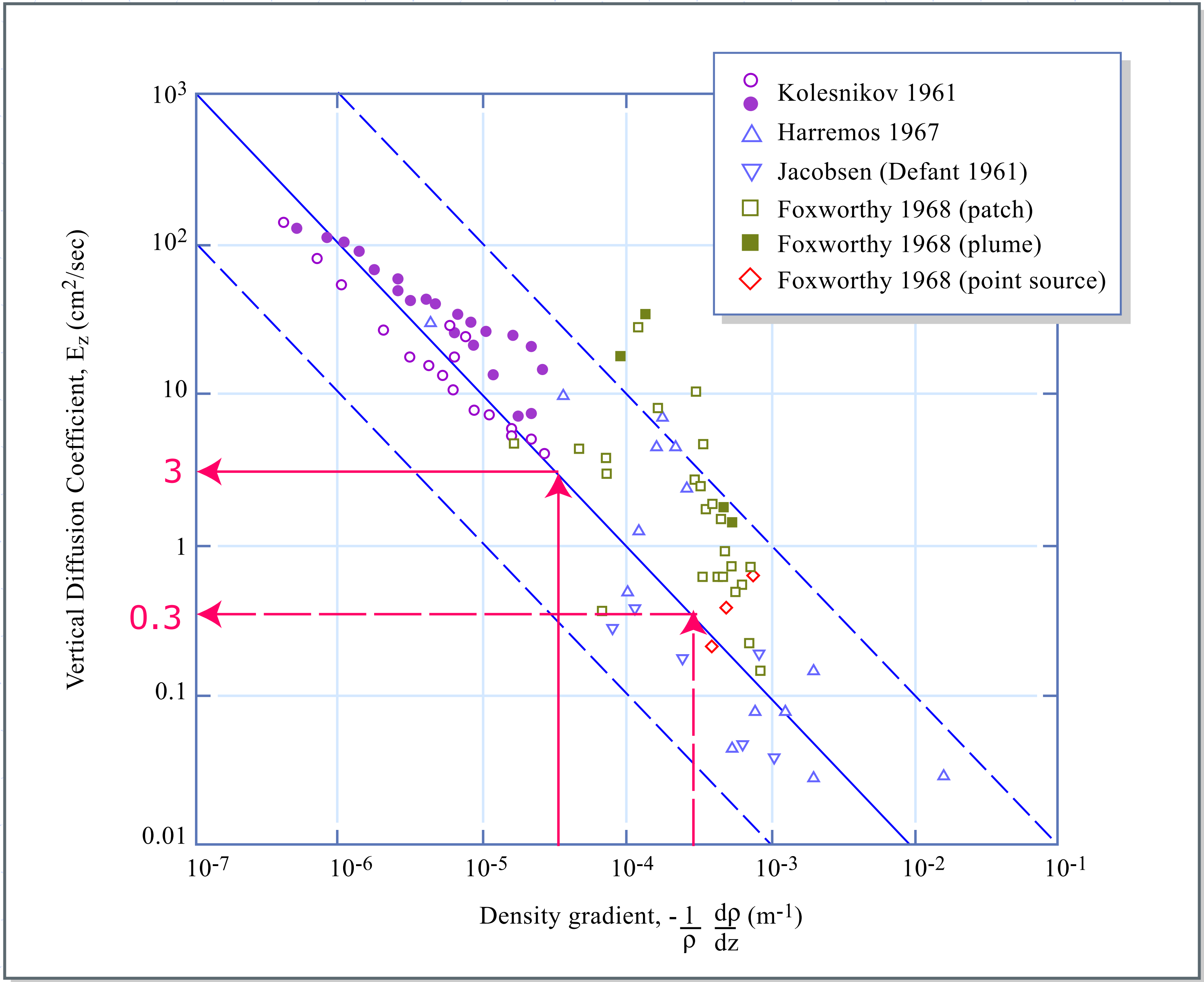


Figure by MIT OCW.

Koh and Fan (1970)

Atmospheric, surface water and ground water plumes

Similarities

- Same transport equation (porosity included in some GW terms)
- Scale-dependent dispersion. Similar mechanisms: non-uniform flow (differential longitudinal advection plus transverse mixing)
- $E_x > E_y >> E_z$

Differences, too

Atmospheric Plumes

- Modest NF mixing (wind quickly dominates)
- Often large "point" sources
- Time scales: minutes to days
- Non-uniform wind caused by shear and density stratification

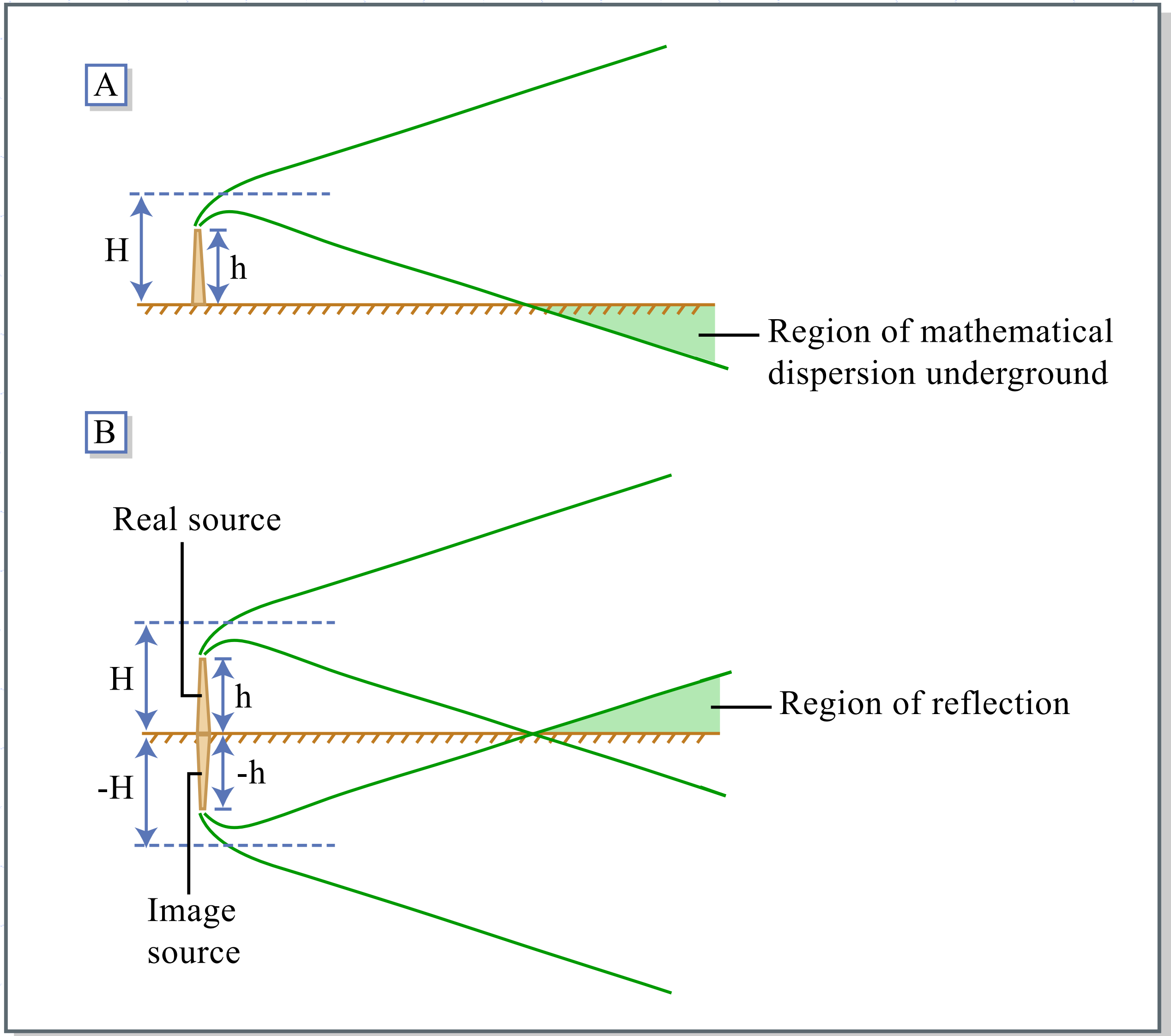
Image courtesy of usgs.gov.

Stratification

For examples of plume types, please see:

http://www.environmenthamilton.org/projects/stackwatch/plume_types.htm

Typical analysis



- Image source for ground level exposure
- NF mixing handled by virtual elevation
- Cooper and Alley (1994)

Figure by MIT OCW.

Diffusion diagrams

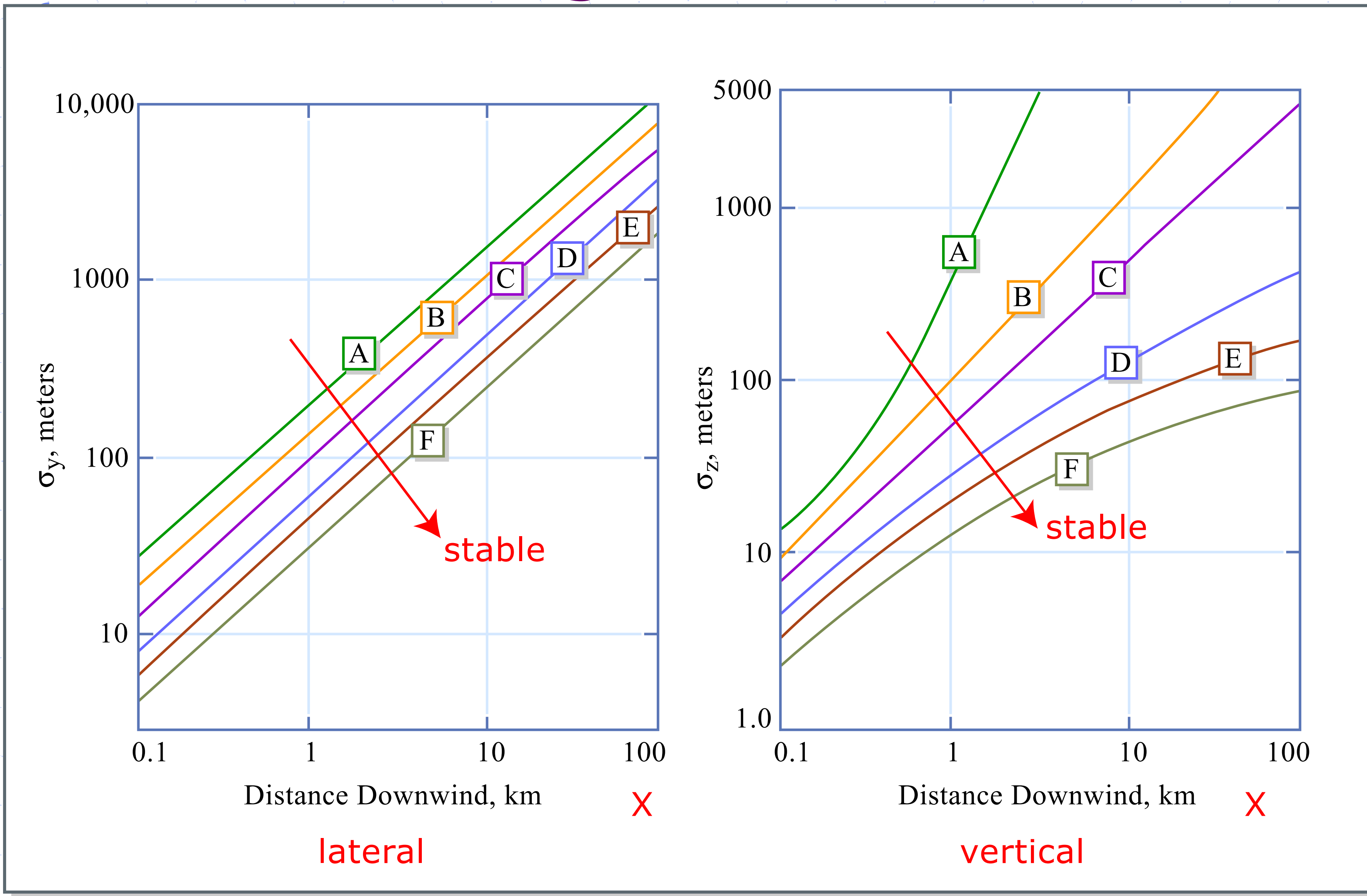


Figure by MIT OCW.

Turner (1970); Cooper and Alley (1994)

Surface Wind Speed[a] (m/s)	Day Incoming Solar Radiation			Night Cloudiness[e]	
	Strong[b]	Moderate[c]	Slight[d]	Cloudy ($\geq 4/8$)	Clear ($\leq 3/8$)
< 2	A	A-B[f]	B	E	F
2-3	A-B	B	C	E	F
3-5	B	B-C	C	D	E
5-6	C	C-D	D	D	D
> 6	C	D	D	D	D

unstable / stable

Notes:-

a) Surface wind speed is measured at 10 m above the ground.

b) Corresponds to clear summer day with sun higher than 60^o above the horizon.

c) Corresponds to a summer day with a few broken clouds, or a clear day with the sun $35\text{-}60^o$ above the horizon.

d) Corresponds to a fall afternoon, or a cloudy summer day, or clear summer day with the sun $15\text{-}35^o$.

e) Cloudiness is defined as the fraction of sky covered by clouds.

f) For A-B, B-C, or C-D conditions, average the values obtained for each.

* A = Very unstable, B = Moderately unstable, C = Slightly unstable, D = Neutral, E = Slightly stable, and F = Stable.

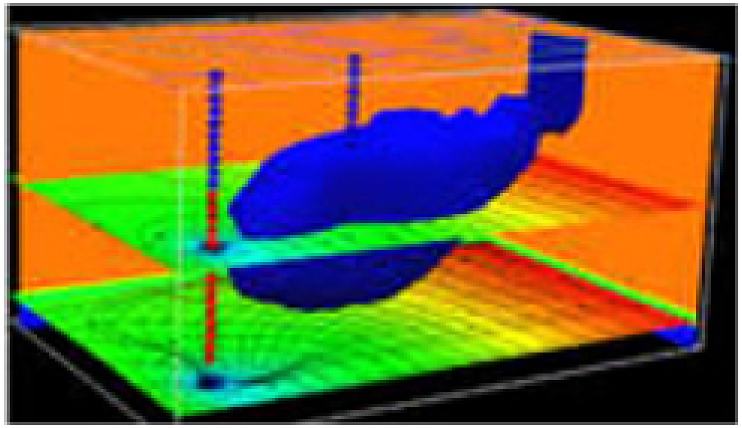
Regardless of wind speed, Class D should be assumed for overcast conditions, day or night.

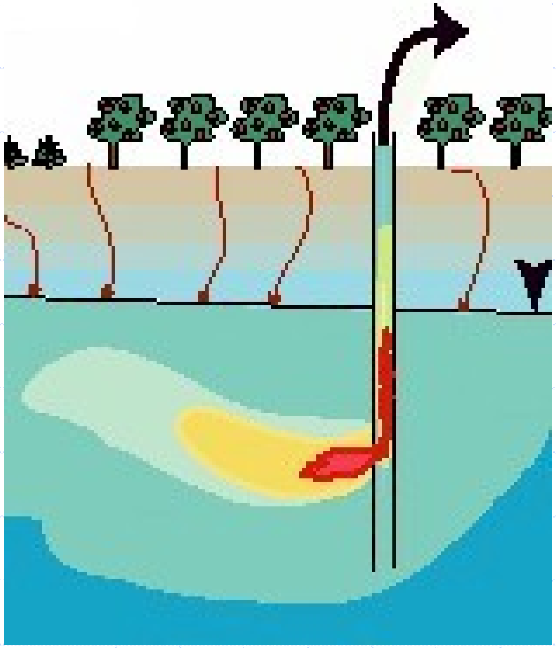
STABILITY CLASSIFICATIONS*

Figure by MIT OCW.

Groundwater Plumes

- No (dynamic) NF
- Distributed, poorly characterized sources
- Multiple phases (contaminant and medium)
- Laminar (turbulent fluctuations replaced by heterogeneity)
- Time scales: months to decades





Heteorogeneity

- Causes non-uniform flow => macro-dispersion
- Often poorly resolved: handled stochastically
- Plumes often (very) non-Gaussian

MADE experiments at CAFB

Please see:

http://repositories.cdlib.org/cgi/viewcontent.cgi?article=1408&context=lbnl

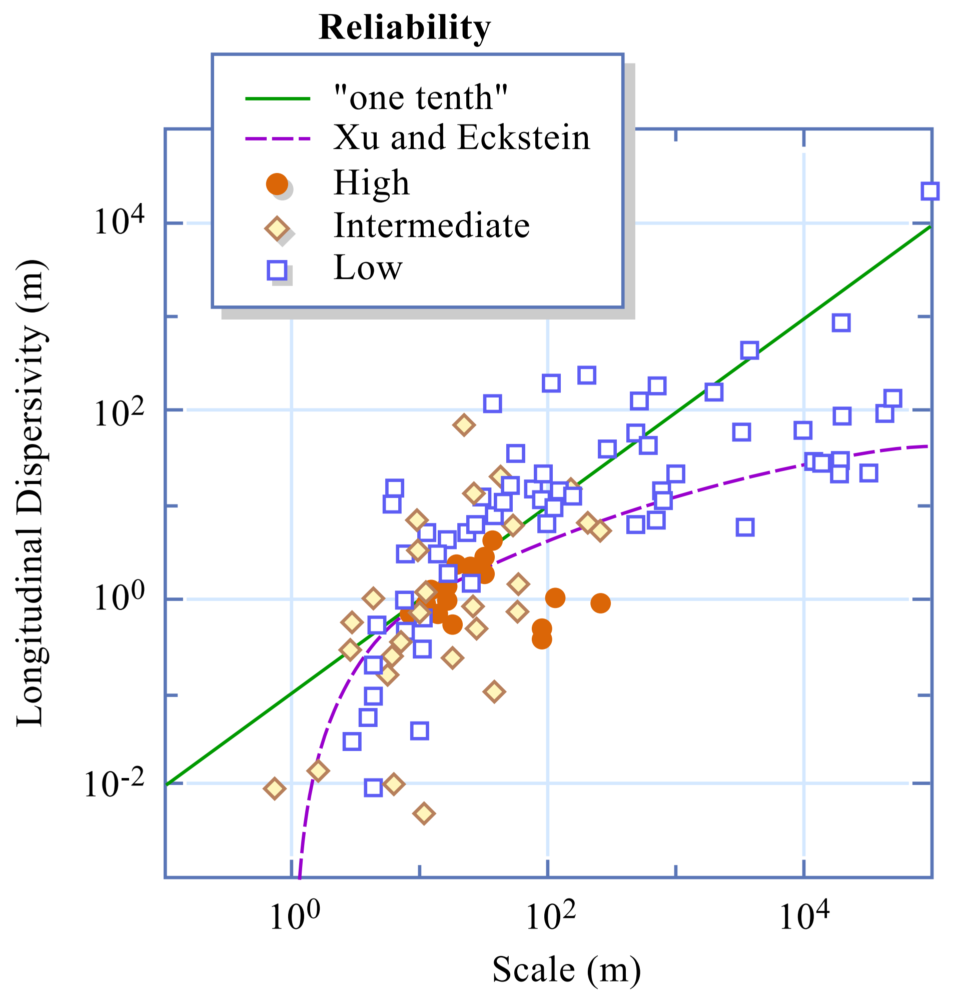
Dispersivity, α [L]

$$E = \alpha u$$

$$\sim \overline{u^2}\tau \sim \overline{u^2}\frac{\lambda}{u} \sim u\lambda$$

$$\alpha \sim \lambda$$

Length scale of hydraulic conductivity correlation



Gelhar, Welty and Rehfeldt (1992) Dispersivity Data

Figure by MIT OCW.

Superposition: Puff models

MIT Transient Plume Model

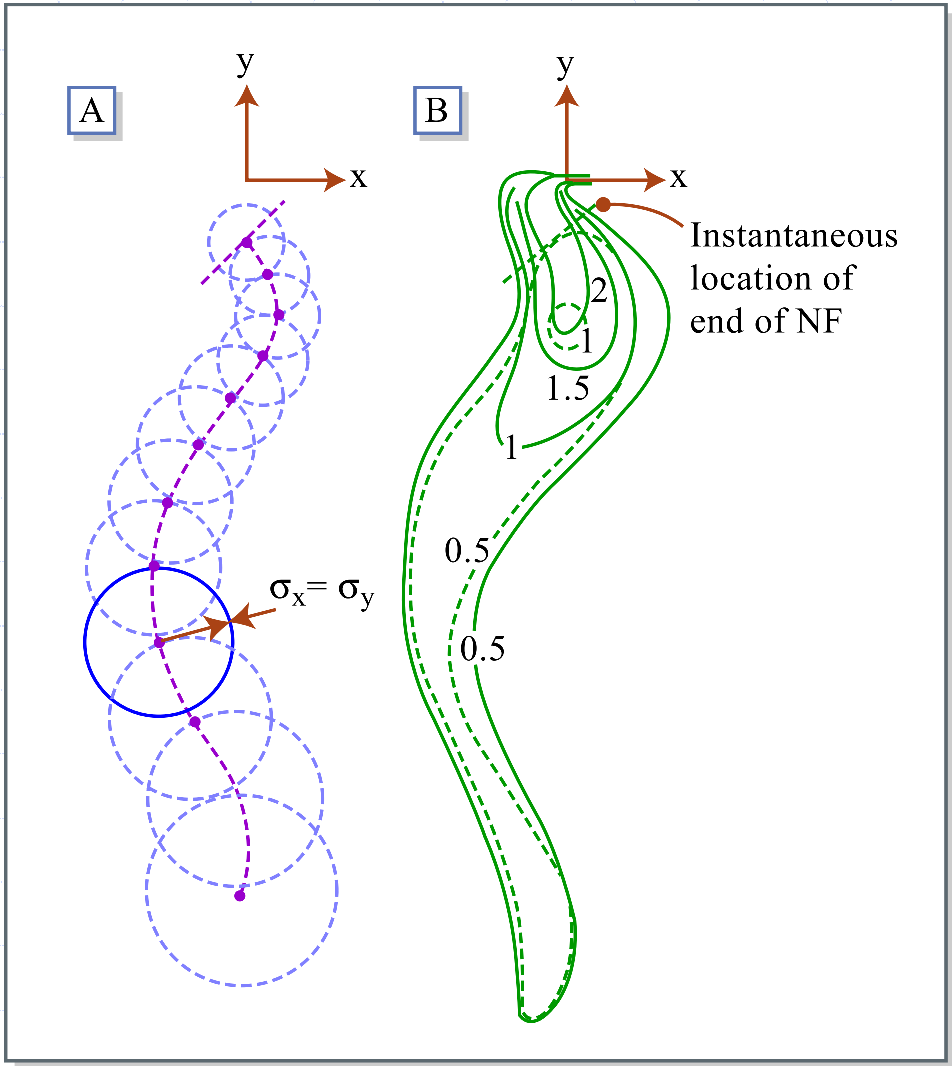


Figure by MIT OCW.

$$u = 0.34\cos(\omega t + \pi/2)$$

$$v = 0.18\cos(\omega t + \pi/2) - 0.5$$

$$c(x,y,z,t) = \sum_k \frac{m_o(z,t,t_k)}{\pi\sigma_r^2(t,t_k)}$$

$$\exp - \left\{ \frac{[x - x_c(t,t_k)]^2}{\sigma_r^2(t,t_k)} + \frac{[y - y_c(t,t_k)]^2}{\sigma_r^2(t,t_k)} \right\}$$

Adams (1995)