Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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7-9-2upwell.tex

7.9 Coastal upwelling in a two-layered sea

When a steady wind blows along the shore, an Ekman drift is induced that leads to mass flux perpendicular to the wind. Consider the eastern coast in the northern hemisphere. If the wind blows to the north, so that the coast is on its left (west), there is an Ekman drift with mass moving away from the coast to the east. Fluid must be replenished from below, so that the interface must rise, see figure 7.9.1. This is called *coastal upwelling*, first analyzed theoretically by Kozo Yoshida (1959). This phenomenon is important to life in the ocean. Small organisms such as phytoplanktons need nutrient and sun light to prosper. If upwelling occurs near a coastal water, nutrients can be tranported from great depth to near the sea surface where sun light is rich. Fishes are therefore more bountiful.

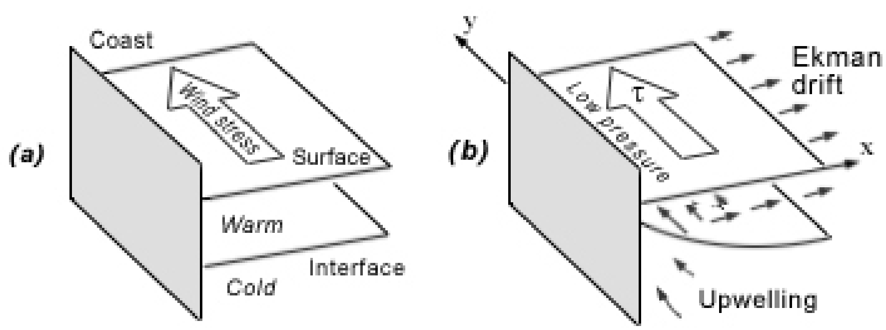

Figure 7.9.1: Physical mechanism of coastal upwelling. From Cushman-Roisin

We consider a spatially uniform wind blowing along a coastline $x = 0$. According to the normal mode formulation the equations for modes $k = 1, 2$ are

$$\frac{\partial \bar{U}_k}{\partial t} - f\bar{V}_k = -g\beta_k h \frac{\partial \bar{\zeta}_k}{\partial x} \qquad (7.9.1)$$

$$\frac{\partial \bar{V}_k}{\partial t} + f\bar{U}_k = \frac{\tau_k}{\rho} \qquad (7.9.2)$$

$$\frac{\partial \bar{\zeta}_k}{\partial t} + \frac{\partial \bar{U}_k}{\partial x} = 0 \qquad (7.9.3)$$

where $\bar{U}_k, \bar{V}_k$ and $\bar{\zeta}_k$ are related to the depth-integrated fluxes U, V, ζ in the upper layer and U', V', ζ' in the lower layer by

$$\bar{U}_k = a_k U + b_k U', \quad \bar{V}_k = a_k V + b_k V', \quad \bar{\zeta}_k = a_k \zeta + (b_k - a_k)\zeta' \tag{7.9.4}$$

and the normal form of wind forcing is

$$\tau_k = a_k \tau_y^S \tag{7.9.5}$$

Assume that the wind oscillates in time at the frequency ω so that

$$\tau_y^S = \Re\left(i\tau_o e^{-i\omega t}\right) = \tau_o \sin\omega t \tag{7.9.6}$$

Let us look for the response that is also sinusoidal in time, and write the solutions as

$$(\bar{U}_k, \bar{V}_k, \bar{\zeta}_k) = \Re\left\{(U_k, V_k, \zeta_k)e^{-\imath\omega t}\right\}$$

then

$$-i\omega U_k - fV_k = -g\beta_k h \frac{\partial \zeta_k}{\partial x} \tag{7.9.7}$$

$$fU_k - i\omega V_k = \frac{\tau_k}{\rho} \tag{7.9.8}$$

$$i\omega\zeta_k + \frac{\partial U_k}{\partial x} = 0 \tag{7.9.9}$$

From Eqns. (7.9.7) and (7.9.8) we solve for U_k and V_k,

$$U_k = \frac{\begin{vmatrix} -g\beta_k h \frac{\partial \bar{\zeta}_k}{\partial x} & -f \\ \frac{\tau_k}{\rho} & -i\omega \end{vmatrix}}{\begin{vmatrix} -i\omega & -f \\ f & -i\omega \end{vmatrix}} = \frac{i\omega g\beta_k h \frac{\partial \zeta_k}{\partial x} + f\frac{\tau_k}{\rho}}{f^2 - \omega^2} \tag{7.9.10}$$

and

$$V_k = \frac{\begin{vmatrix} -i\omega & -g\beta_k h \frac{\partial \bar{\zeta}_k}{\partial x} \\ f & \frac{\tau_k}{\rho} \end{vmatrix}}{\begin{vmatrix} -i\omega & -f \\ f & -i\omega \end{vmatrix}} = \frac{fg\beta_k h \frac{\partial \zeta_k}{\partial x} - i\omega\frac{\tau_k}{\rho}}{f^2 - \omega^2} \tag{7.9.11}$$

Substituting Eqn. (7.9.10) into (7.9.9), we get

$$-i\omega\zeta_k + \frac{1}{f^2 - \omega^2}\left(i\omega g\beta_k h \frac{\partial^2 \zeta_k}{\partial x^2}\right) = 0$$

or,

$$\frac{\partial^2 \zeta_k}{\partial x^2} - \left(\frac{f^2 - \omega^2}{g\beta_k h}\right)\zeta_k = 0. \tag{7.9.12}$$

Let us limit our attention to low freuencies so that so that $f^2 > \omega^2$. The solution bounded at $x \sim \infty$ is,

$$\zeta_k = A_k e^{-x/R_k}, \tag{7.9.13}$$

where

$$R_k = \frac{\sqrt{g\beta_k h}}{\sqrt{f^2 - \omega^2}}. \tag{7.9.14}$$

is the modified Rossby radius of deformation. As $b.c$ at $x = 0$, we require $U_k = 0$, hence from (7.9.10)

$$i\omega g\beta_k h \frac{\partial \zeta_k}{\partial x} = -\frac{f}{\rho\tau_k}$$

$$i\omega g\beta_k h A_k \left(-\frac{1}{R_k}\right) = -\frac{f\tau_k}{\rho}$$

The coefficient is

$$A_k = \frac{\frac{f\tau_k}{\rho} R_k}{i\omega g\beta_k h} \tag{7.9.15}$$

therefore

$$\zeta_k = \frac{\frac{f\tau_k}{\rho} R_k}{i\omega g\beta_k h} e^{-x/R_k}. \tag{7.9.16}$$

Recall for Mode 1 (barotropic or surface mode):

$$\beta_1 = \frac{h + h'}{h}, \quad a_1 = 1, \quad \bar{\tau}_1 = \tau_1 e^{-i\omega t} = \tau_y^S = i\tau_o e^{-i\omega t} \tag{7.9.17}$$

then

$$\tau_1 = i\tau_o, \quad R_1 = \frac{\sqrt{g\frac{h+h'}{h} h}}{\sqrt{f^2 - \omega^2}} = \frac{\sqrt{g(h + h')}}{\sqrt{f^2 - \omega^2}} \tag{7.9.18}$$

hence we have

$$\zeta_1 = \frac{\frac{f\tau_o}{\rho} R_1}{\omega g(h + h')} e^{-x/R_1} \tag{7.9.19}$$

For Mode 2 (baroclinic or internal mode):

$$\beta_2 = \frac{\epsilon h'}{h + h'}, \quad \tau_2 = i\tau_0 \left(\frac{-h'}{h}\right) \tag{7.9.20}$$

$$R_2 = \frac{\sqrt{g\epsilon \frac{hh'}{h+h'}}}{\sqrt{f^2 - \omega^2}} = \frac{\sqrt{g\epsilon \left(\frac{1}{1/h + 1/h'}\right)}}{\sqrt{f^2 - \omega^2}} \tag{7.9.21}$$

hence

$$\zeta_2 = \frac{-\frac{h'}{h}\frac{\tau_0}{\rho}\frac{f}{\omega} R_2}{\frac{g\epsilon}{1/h + 1/h'}} e^{-x/R_2}$$

Note, $R_2 = O\sqrt{\epsilon}R_1 \ll R_1$.

Clearly

$$\zeta_2 = O\left(\frac{\zeta_1}{\epsilon}\right) \gg \zeta_1 \tag{7.9.22}$$

Recalling from (7.7.51) and (7.7.56) of the last section,

$$\bar{\zeta}_1 \cong \zeta, \quad \bar{\zeta}_2 = -\frac{h'}{h}\zeta + \frac{h+h'}{h}\zeta' \tag{7.9.23}$$

hence the free surface and interface displacments can be solved,

$$\zeta \cong \bar{\zeta}_1 = \Re\left\{\zeta_1 e^{-i\omega t}\right\} = \Re\left\{\frac{\frac{f\tau_o}{\rho}R_1}{\omega g(h+h')}e^{-x/R_1}e^{-i\omega t}\right\} \tag{7.9.24}$$

$$\begin{aligned}\zeta' \quad &\cong \quad \frac{\frac{h'}{h}\bar{\zeta}_1 + \bar{\zeta}_2}{1+\frac{h'}{h}} = \Re\left\{\frac{1}{1+\frac{h'}{h}}\frac{h'}{h}\frac{\frac{f\tau_o}{\rho}R_1}{\omega g(h+h')}e^{-x/R_1}e^{-i\omega t}\right\} \\ &+ \quad \Re\left\{\frac{1}{1+\frac{h'}{h}}\left[\frac{-\frac{h'}{h}\frac{\tau_0}{\rho}\frac{f}{\omega}R_2}{\frac{g\epsilon}{1/h+1/h'}}e^{-x/R_2}\right]e^{-i\omega t}\right\}\end{aligned} \tag{7.9.25}$$

Very close to the coast, $x/R_2 = O(1)$, the internal wave mode dominates.

$$\begin{aligned}\zeta' \approx \frac{h\bar{\zeta}_2}{h+h'} &= \frac{h}{h+h'}\frac{-\frac{h'}{h}\frac{\tau_o}{\rho}\frac{f}{\omega}R_2}{\frac{g\epsilon}{1/h+1/h'}}e^{-x/R_2}\cos\omega t \\ &= -\frac{h'}{h+h'}\frac{\frac{\tau_o}{\rho}\frac{f}{\omega}R_2}{\frac{g\epsilon}{1/h+1/h'}}e^{-x/R_2}\cos\omega t\end{aligned} \tag{7.9.26}$$

Thus as the wind stress is from south to north, $0 < \omega t < \pi$, the interface rises fro its lowest (negative) to the highest (positive) level; this is **upwelling**. As the wind reverses direction and blows southward, the interfaces sinks; this is **downwelling**. For $\tau_0 = 0.1Pa$, the upwelling can be several meters.

Farther away from the coast, $x/R_1 = O(1)$, the barotropic (surface wave) mode dominates.

$$\zeta' = \frac{\frac{h'}{h}\bar{\zeta}_1}{\frac{h+h'}{h}} = \frac{h'}{h+h'}\bar{\zeta}_1 = \frac{\frac{f\tau_o}{\rho}R_1}{\omega g(h+h')}e^{-x/R_1}\cos\omega t. \tag{7.9.27}$$

The free surface and the inteface rises together when the wind blows northward, and falls together when the wind blows southward. See sketch in Figure 7.9.2.

Figure 7.9.2: Possible scenarios of coastal upwelling. From Cushman-Roisin.