**Lecture Notes on Fluid Dynamics**
(1.63J/2.21J)
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7-5cyclone.tex
Ref: D. J. Acheson: Elementary Fluid Mechanics, §8.5

## 7.5 Cyclonic current forced by a swirling wind

Of practical interest is the case of nonuniform wind stress on the surface. As an extremely simplified model we consider a vortical wind stress over a large sea[1]. See Figure 7.5.1.

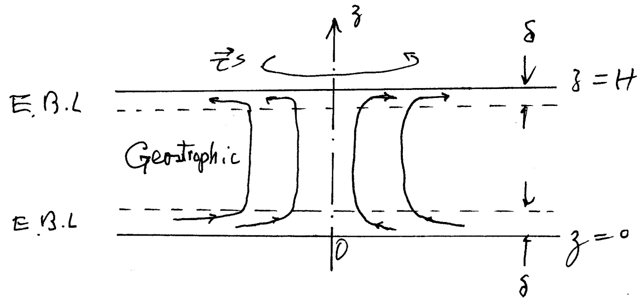


Figure 7.5.1: Steady cyclonic flow in a shallow sea forced by swirling wind

Let us restricting to a low Rossby number flow for simplicity. Continuity requires:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{7.5.1}$$

The momentum equations are

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{7.5.2}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\,\nabla^2 v \tag{7.5.3}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\nabla^2 w \tag{7.5.4}$$

[1]Acheson demonstrated a very similar problem of a circular layer of water bounded above and below by two horizontal planes. While the bottom plane rotates about the vertical axis at the rate $\Omega$ the top cover rotates steadily at a different rate $(1+\epsilon)\Omega$.

The boundary conditions are : no slip on the bottom:

$$u = v = w = 0, \quad z = 0 \tag{7.5.5}$$

and given wind stress on the top:

$$\tau^S_{\theta z} = \rho T r/2, \quad \tau^S_{rz} = 0, \quad z = H. \tag{7.5.6}$$

The wind stress is cyclonic, where $T$ is the curl of the wind sress vector:

$$\nabla \times \vec{\tau}^S = \vec{k}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau^S_{\theta z} - \frac{1}{r}\frac{\partial \tau^S_{rz}}{\partial \theta}\right) = \rho T\vec{k}. \tag{7.5.7}$$

In cartesian coordinates the wind stress components are:

$$\tau^S_{xz} = -\tau^S_{\theta z}\sin\theta = -\frac{\rho T}{2} r\sin\theta = -\frac{\rho T}{2} y, \tag{7.5.8}$$

$$\tau^S_{yz} = \tau^S_{\theta z}\cos\theta = \frac{\rho T}{2} r\cos\theta = \frac{\rho T}{2} x, \tag{7.5.9}$$

Kinematically we assume that

$$w = 0, \quad z = H. \tag{7.5.10}$$

### 7.5.1 Inviscid core

Outside the surface an bottom boundary layers, we have

$$-fv_I = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{7.5.11}$$

$$fu_I = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{7.5.12}$$

This is clearly the state of geostrophyic balance. Momentum balence in the vertical direction is trivial,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Consequently $u_I$ and $v_I$ must be independent of $z$. in accordance with the Taylor-Proudman theorem. Note that conservation of mass is automatically satisfied,

$$\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} = 0$$

and the vorticity is

$$\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} = -\frac{1}{f}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right)$$

The horizontal components $u_I(x, y), v_I(x, y)$ are not determined yet. The vertical velocity $w_I$ can at best be a contant in $z$.

### 7.5.2 Bottom boundary layer

Let us keep the dominant viscous stress terms in the momentum equations,

$$-f\,(v - v_I) = \nu\,\frac{\partial^2\,(u-u_I)}{\partial z^2} \tag{7.5.13}$$

$$f\,(u - u_I) = \nu\,\frac{\partial^2\,(v-v_I)}{\partial z^2} \tag{7.5.14}$$

The boundary conditions are

$$\begin{array}{lll} u-u_I = -u_I & v - v_I = -v_I & z = 0 \\ u - u_I \to 0 & v - v_I \to 0 & z \gg \delta \end{array}$$

where

$$\delta = \sqrt{\frac{2\nu}{f}} \tag{7.5.15}$$

is the Ekman boundary layer thickness.

The solution is left to the reader as an exercise

$$u - u_I = -e^{-z/\delta}\left(u_I\,\cos\frac{z}{\delta} + v_I\,\sin\frac{z}{\delta}\right) \tag{7.5.16}$$

$$v - v_I = -e^{-z/\delta}\left(v_I\,\cos\frac{z}{\delta} - u_I\,\cos\frac{z}{\delta}\right). \tag{7.5.17}$$

From continuity, the vertical component can be computed. Let $\zeta = z/\delta$,

$$\begin{aligned} \frac{\partial w}{\partial z} &= \frac{1}{\delta}\frac{\partial w}{\partial \zeta} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ &= \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) e^{-\zeta}\,\sin\zeta + \left(\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y}\right)\left(e^{-\zeta}\,\cos\zeta\right). \end{aligned} \tag{7.5.18}$$

The second term vanishes, hence,

$$\begin{aligned} w &= \delta \int_0^\zeta d\zeta \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) e^{-\zeta}\,\sin\zeta \\ &= \delta\left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right)\frac{e^{-\zeta}}{2}\,(-\sin\zeta - \cos\zeta)\Big|_0^\zeta \\ &= \frac{\delta}{2}\left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right)\left[1 - e^{-\zeta}\,(\cos\zeta + \sin\zeta)\right]. \end{aligned}$$

At the outer edge of the bottom boundary layer, $\zeta = z/\delta \gg 1$

$$w(\infty) \equiv \frac{\delta}{2}\left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) = \frac{\delta}{2}\,\omega_I \tag{7.5.19}$$

where $\omega_I$ is the vorticity in the geostrophic interior. Thus there is vertical flux from the bottom boundary layer when the interior flow is horizontally nonuniform; this is called the Ekman pumping!

We still don't know the geostrophic flow field.

### 7.5.3 Surface boundary layer

The momentum equations are

$$\begin{aligned} -f\left(v-v_I\right) &= \nu\frac{\partial^2\left(u-u_I\right)}{\partial z^2} \\ f\left(u-u_I\right) &= \nu\frac{\partial^2\left(v-v_I\right)}{\partial z^2}. \end{aligned} \tag{7.5.20}$$

On $z=H$ the boundary conditions are

$$\nu\frac{\partial u}{\partial z}=-\frac{T}{2}y, \quad \nu\frac{\partial v}{\partial z}=\frac{T}{2}x, \quad z=H \tag{7.5.21}$$

Far beneath the surface

$$u\to u_I, \quad v\to v_I; \quad (H-z)\gg\delta \tag{7.5.22}$$

Let us introduce the boundary-layer coordinate

$$\eta=\frac{H-z}{\delta} \qquad 0<\eta<\infty. \tag{7.5.23}$$

so that

$$\frac{\partial}{\partial z}\to-\frac{1}{\delta}\frac{\partial}{\partial\eta} \tag{7.5.24}$$

The solution satisfies the momentum equations and (7.5.22) is of the form

$$u-u_I = e^{-\eta}\left(A\cos\eta+B\sin\eta\right) \tag{7.5.25}$$

$$v-v_I = e^{-\eta}\left(B\cos\eta-A\sin\eta\right). \tag{7.5.26}$$

In order to satisfy (7.5.21), we first note that

$$\frac{\partial u}{\partial\eta} = e^{-\eta}\left((-A+B)\cos\eta+(-A-B)\sin\eta\right) \tag{7.5.27}$$

$$\frac{\partial v}{\partial\eta} = e^{-\eta}\left((-A-B)\cos\eta+(A-B)\sin\eta\right). \tag{7.5.28}$$

Applying (7.5.21), we get

$$-\frac{\nu}{\delta}(-A+B)=-\frac{Ty}{2}, \quad -\frac{\nu}{\delta}(-A-B)=\frac{Tx}{2} \tag{7.5.29}$$

with the results,

$$A=\frac{T\delta}{4\nu}(x-y), \quad B=\frac{T\delta}{4\nu}(x+y) \tag{7.5.30}$$

Hence the horizontal velocities are

$$u-u_I = \frac{T\delta}{4\nu}e^{-\eta}\left((x-y)\cos\eta+(x+y)\sin\eta\right) \tag{7.5.31}$$

$$v-v_I = \frac{T\delta}{4\nu}e^{-\eta}\left((x+y)\cos\eta-(x-y)\sin\eta\right). \tag{7.5.32}$$

By continuity

$$\frac{\partial w}{\partial z} = -\frac{1}{\delta}\frac{\partial w}{\partial \eta} = -\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)$$
$$= \frac{T\delta}{4\nu} e^{-\eta}(2\cos\,\eta + 2\sin\,\eta)$$

the vertrical velocity can be found,

$$\begin{aligned} w(\eta) &= \frac{T\delta}{2\nu}\int_0^\eta d\eta\, e^{-\eta}\,(\cos\,\eta + \sin\,\eta) \\ &= \frac{T\delta}{2\nu}\left[e^{-\eta}\,(-\cos\,\eta + \sin\,\eta) + e^{-\eta}\,(-\cos\,\eta - \sin\,\eta)\right] \\ &= \frac{T\delta}{2\nu}\left[(1 - e^{-\eta}\cos\,\eta)\right] \end{aligned} \qquad (7.5.33)$$

At the outer edge of the surface boundary layer $\eta \gg 1$

$$w(\infty) = w_T = \frac{T\delta}{2\nu} \qquad (7.5.34)$$

By Taylor-Proudman theorem, $w(z) = w_B = w_T$. Therefore

$$w_B = \frac{\delta}{2}\,\omega_I = \frac{T\delta}{2\nu} = w_T \qquad (7.5.35)$$

and the interior vorticity is

$$\omega_I = \frac{T}{\nu}. \qquad (7.5.36)$$

What are $u_I$ and $v_I$? In cylindrical polar coordinates

$$\omega_I = \frac{1}{r}\frac{\partial}{\partial r}\,(r\,u_{I_\theta}) - \frac{1}{r}\frac{\partial u_{I_r}}{\partial \theta} == \frac{1}{r}\frac{\partial}{\partial r}\,(r\,u_{I_\theta})\,.$$

Since $\partial/\partial\theta = 0$, we have ,

$$\omega_I = \frac{1}{r}\frac{d}{dr}\,(r\,u_{I_\theta})$$
$$\frac{d}{dr}\,(r\,u_{I_\theta}) = \frac{T}{\nu}r$$

which implies

$$u_{I_\theta} = \frac{T}{2\nu}\,r.$$

Since

$$\frac{1}{r}\frac{\partial}{\partial r}\,(r\,u_{I_r}) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

which leads to

$$u_{I_r} = 0.$$

The interior flow is geostrophic and cyclonic.

In cartesian form we have

$$u_I = -u_{I_\theta} \sin\theta = -\frac{T}{2\nu} r \sin\theta, \tag{7.5.37}$$

$$v_I = u_{I_\theta} \cos\theta = \frac{T}{2\nu} r \cos\theta \tag{7.5.38}$$

Now the radial component inside the bottom boundary layer is

$$u_r = u_r - u_{I_r}$$

since $u_{I_r} = 0$. The latter is

$$\begin{aligned} u_r - u_{I_r} &= -e^{-\zeta} \left[ (u_I \cos\zeta + v_I \sin\zeta) \cos\theta + (v_I \cos\zeta - u_I \sin\zeta) \sin\theta \right] \\ &= -e^{-\zeta} \left[ \cos\zeta (u_I \cos\theta + v_I \sin\theta) + \sin\zeta (v_I \cos\theta - u_I \sin\theta) \right] \\ &= -e^{-\zeta} \sin\zeta (v_I \cos\theta - u_I \sin\theta) \\ &= -\frac{Tr}{2\nu} e^{-\zeta} \sin\zeta (\cos^2\theta + \sin^2\theta) \\ &= -\frac{Tr}{2\nu} e^{-\zeta} \sin\zeta \end{aligned}$$

and is negative in most of the boundary layer. Hence the flow spirals inward towards the $z$ axis in the bottom boundary layer. Similarly one can show that the flow in the surface boundary layer has an outward radial component.

In summary, the swirling wind induces a vorticity $T/\nu$ in the geostrophic interior. The flow in the bottom Ekman layer spirals inward, rises vertically at a uniform velocity while spiralling at the angular velocity $T/\nu$ and maintaining a constant vorticity in the geotrophic interior, then spirals outward in the surface Ekman layer. The flow is therefore cyclonic.