Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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6.3 Saffman-Taylor instability in porous layer- Viscous fingering

Refs:
P. G. Saffman & G. I. Taylor, 1958, The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more visous fluid. *Proc. Royal Society*, 245, 312-329.
G. Homsy; 1987. Viscous fingering in porous media. *Annual Rev of Fluid Mech.* 19, 271 - 314.

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In petroleum recovery water is often used to drive oil from the reservoir. An oil reservoir can also be covered by a layer of water from above. Phenomenon of fingering often occurs when oil is extracted frome beneath the water layer. Although known to mining engineers, Saffman & Taylor (1958) gave the first theory and performed simulated experiments in a Hele-Shaw cell.

Consider a moving interface in a stationary coordinate system. Let the initial seepage velocity V be vertical and the interface be a plane, then

$$y = \frac{Vt}{n} \tag{6.3.1}$$

where n is the porosity. If the interface is disturbed then its position is at

$$y = \frac{Vt}{n} + \eta(x,t) \tag{6.3.2}$$

At any interior point, ϕ is the velocity potential,

$$\phi = -\frac{k}{\mu}(p + \rho g y) \tag{6.3.3}$$

where k is the permeability related to conductivity K by

$$K = \frac{\rho g k}{\mu} \tag{6.3.4}$$

The pressure is

$$p = -\frac{\mu}{k}\phi - \rho g y \tag{6.3.5}$$

Thus in fluid 1(upper fluid)

$$p_1 = -\frac{\mu_1}{k_1} g\phi_1 - \rho_1 g y \tag{6.3.6}$$

By continuity,

$$\nabla^2\phi_1 = 0, \quad y > \frac{Vt}{n} + \eta(x,t), \tag{6.3.7}$$

In the lower fluid (2),

$$p_2 = -\frac{\mu_2}{k_2} g\phi_2 - \rho_2 g y \tag{6.3.8}$$

and

$$\nabla^2\phi_2 = 0, \quad y < \frac{Vt}{n} + \eta(x,t) \tag{6.3.9}$$

Let us first examine the basic uniform flow where the interface is plane ($\eta = 0$). The potentials are

$$\phi_1^o = Vy + f_1(t) = -\frac{k_1}{\mu_1}\left(p_1^o + \rho_1 g y\right) \tag{6.3.10}$$

$$\phi_2 = Vy + f_2(t) = -\frac{k_2}{\mu_2}\left(p_2^o + \rho_2 g y\right) \tag{6.3.11}$$

Note that an arbitrary function of $f(t)$ is added to the potential without affecting the velocity field. The pressures are

$$p_1^o = -\left(\frac{\mu_1 V}{k_1} + \rho_1 g\right) y - \frac{\mu_1 f_1(t)}{k_1}, \quad y > \frac{Vt}{n} \tag{6.3.12}$$

and in the lower fluid (2),

$$p_2^o = -\left(\frac{\mu_2 V}{k_2} + \rho_2 g\right) y - \frac{\mu_2 f_2(t)}{k_2}, \quad y < \frac{Vt}{n} \tag{6.3.13}$$

In order that pressure is continuous at $y = Vt/n$ for all t, we must have

$$f_1(t) = F_1 t, \quad f_2(t) = F_2 t \tag{6.3.14}$$

where F_1, F_2 are constants and

$$-\left(\frac{\mu_1 V}{k_1} + \rho_1 g\right)\frac{V}{n} - \frac{\mu_1 F_1}{k_1} = -\left(\frac{\mu_2 V}{k_2} + \rho_2 g\right)\frac{V}{n} + \frac{\mu_2 F_2}{k_2}$$

Thus

$$\frac{\mu_2 F_2}{k_2} - \frac{\mu_1 F_1}{k_1} = -\frac{V}{n}\left\{\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right) V + (\rho_2 - \rho_1 g)\right\} \tag{6.3.15}$$

Note that it is only the difference that matters.

We now consider a small disturbance on the interface .

$$y = \frac{Vt}{n} + \eta(x,t) \tag{6.3.16}$$

where

$$\eta = ae^{i\alpha x - i\omega t} \tag{6.3.17}$$

is small. The total solution is

$$\phi_1 = Vy + F_1 t + B_1 e^{i\alpha x - \alpha(y - Vt/n) - i\omega t}, \quad y > \frac{Vt}{n} + \eta(x,t) \tag{6.3.18}$$

$$\phi_2 = Vy + F_2 t + B_2 e^{i\alpha x + \alpha(y - Vt/n) - i\omega t}, \quad y < \frac{Vt}{n} + \eta(x,t) \tag{6.3.19}$$

The linearized kinematic boundary condition is that velocities must be continuous.

$$n\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y}|_{y=Vt/n} = \frac{\partial \phi_2}{\partial y}|_{y=Vt/n} \tag{6.3.20}$$

Thus

$$-i\omega n a e^{i\alpha x - i\omega t} = -\alpha a B_1 e^{i\alpha x - i\omega t} = \alpha a B_2 e^{i\alpha x - i\omega t} \tag{6.3.21}$$

hence,

$$B_1 = -B_2 = \frac{i\omega n a}{\alpha} \tag{6.3.22}$$

Now we require continuity of pressure on $y = Vt/n + \eta$,

$$-\frac{\mu_1}{k_1}\left(V\eta + \frac{i\omega n \eta}{\alpha}\right) - \rho_1 g \eta = -\frac{\mu_2}{k_2}\left(V\eta - \frac{i\omega n \eta}{\alpha}\right) - \rho_2 g \eta \tag{6.3.23}$$

Eliminating η we get

$$i\omega = \frac{\alpha}{n}\frac{(\rho_2 - \rho_1)g + V\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)}{\frac{\mu_2}{k_2} + \frac{\mu_1}{k_1}} \tag{6.3.24}$$

Clearly $i\omega$ is real. If $i\omega > 0$, or

$$(\rho_2 - \rho_1)g + V\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right) > 0 \tag{6.3.25}$$

the flow is stable. If $i\omega < 0$, or

$$(\rho_2 - \rho_1)g + V\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right) < 0, \tag{6.3.26}$$

the flow is unstable.

From the simple model of a tubular porous medum,

$$K = \frac{n\rho g R^2}{8\mu} = \frac{\rho g k}{\mu} \tag{6.3.27}$$

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$$k = \frac{nR^2}{8} \tag{6.3.28}$$

is independent of viscosity and depends only on n and the pore size. Assume therefore $k_1 = k_2$ and that oil (lighter more viscous) lies above water $\rho_1 < \rho_2$ and $\mu_1 > \mu_2$. If $V < 0$ (water pushed downward by oil) then the flow is always stable. Consider $V > 0$. The flow is unstable if

$$V > V_c = \frac{(\rho_2 - \rho_1)g}{\left(\frac{\mu_1}{k_1} - \frac{\mu_2}{k_2}\right)} \tag{6.3.29}$$

Too high an extraction rate causes instability which marks the onset of fingers.

If the water layer is on top of the oil layer, then $\rho_2 - \rho_1 < 0$; the flow is unstable even if $V = 0$. Since $\mu_2/k_2 - \mu_1/k_1 > 0$ a downward flow (water toward oil) is always unstable. A upward flow can be unstable if

$$0 < V < V_c = \frac{(\rho_1 - \rho_2)g}{\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)} \tag{6.3.30}$$

Note also that the growth(decay) rate is higher for shorter waves.

A gallary of beautiful photographs of fingering taken fromn Hele-Shaw erperiments can be found in the survey by Homsy.