Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
by Chiang C. Mei, 2002

5-1KHinstab.tex

5.2 Kelvin-Helmholz Instability for continuous shear and stratification

5.2.1 Heuristic reasoning

Due to viscosity, shear flow exists along the boundary of a jet, a wake or a plume . On the interface of salt and fresh water, density stratification further comes into play. When will dynamic instability occur?

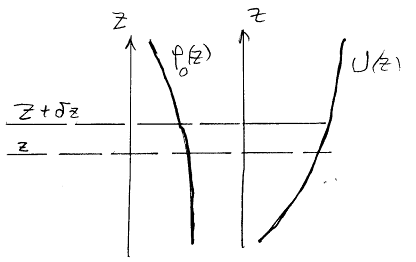

Figure 5.2.1: Exchanging fluid parcels in a stratified shear flow

Refering to Figure 5.2.1, Consider two fluid parcels, each of unit volume, at levels z and $z+dz$. Let their positions be interchanged. To overcome gravity, the force needed to lift the heavier fluid parcel by η is

$$g\left[\overline{\rho}(z)-\overline{\rho}(z+\eta)\right]=-g\frac{d\overline{\rho}}{dz}\eta.$$

Work needed to lift the heavier parcel by dz is

$$-g\frac{d\overline{\rho}}{dz}\int_z^{z+dz}\eta d\eta=-\frac{1}{2}d\overline{\rho}\,dz.$$

Similarly, the work needed to push the light parcel down by dz is $-\frac{1}{2}gd\overline{\rho}dz$. Therefore the total work needed is

$$-gd\overline{\rho}\,dz.$$

Before the exchange, the total kinetic energy is

$$\frac{1}{2}\overline{\rho}[U^2+(U+dU)^2]$$

where Boussinesq approximation is used. After the exchange, the parcels mix with the surrounding fluid and attain the average velocity

$$(U + U + dU)/2 = U + dU/2$$

Therefore the total kinetic energy is

$$\overline{\rho}(U + dU/2)^2$$

The available kinetic energy is the difference between the kinetic energies before and after the exchange.

$$\frac{\overline{\rho}}{2}\left\{U^2 + (U + dU)^2 - 2(U + dU/2)^2\right\} = \frac{\overline{\rho}}{4}dU^2.$$

If the net available kinetic energy exceeds the work needed for the exchange, the disturbance will grow and the flow will become unstable, i.e.,

$$\frac{\overline{\rho}dU^2}{4} > -gd\overline{\rho}dz$$

Let the Richardson number be defined by

$$R_i \equiv \frac{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2} \tag{5.2.1}$$

Instabilty occurs if

$$\frac{1}{4} > R_i \equiv \frac{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2} \tag{5.2.2}$$

(Chandrasekar, 1961).

Remark: A slightly more accurate estimate can be made without Boussinesq approximation. Before the exchange, the total kinetic energy is

$$\frac{1}{2}\left\{\overline{\rho}U^2 + (\overline{\rho} + d\overline{\rho})(U + dU)^2\right\}.$$

After the exchange, the parcels mix with the surrounding fluid and attain the average velocity

$$(U + U + dU)/2 = U + dU/2$$

but their densities are preserved. Therefore the total kinetic energy is

$$\frac{1}{2}(\overline{\rho} + \overline{\rho} + d\overline{\rho})(U + dU/2)^2$$

The available kinetic energy is the difference between the kinetic energies before and after the exchange.

$$\frac{\overline{\rho}}{4}dU^2 - UdUd\overline{\rho} + \frac{1}{4}d\overline{\rho}dU^2$$

Ignoring the last term, the necessary condition for instability is

$$\frac{\overline{\rho}}{4}dU^2 - UdUd\overline{\rho} + \frac{1}{4}d\overline{\rho}dU^2 > -gd\overline{\rho}dz$$

or

$$\frac{1}{4} - \frac{\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\frac{1}{U}\frac{dU}{dz}} + \frac{1}{4}\frac{d\overline{\rho}}{\overline{\rho}} > \frac{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2}$$

On the left-hand side, the third term is negligible compared to the first. The ratioi of the second term on the left to the term on the right is

$$\frac{U}{g}\frac{dU}{dz} \sim \frac{U^2}{gL}$$

where L is the length scale of stratification. As long as the last ratio is very small, the criterion $R_i < 1/4$ still holds.

Let us confirm the heuristic result but the linearize theory.

5.2.2 Linearized instability theory for continuous shear and stratification.

Let the total flow field be $(U+u, w, P+p, \bar{\rho}+\tilde{\rho})$ where $U, P, \bar{\rho}$ represent the backgraound flow $(u, w, p, \tilde{\rho})$ the dynamical perturbations of infinitesimal magnitude. The linearized governing equations are: continuity:

$$u_x + w_z = 0 \tag{5.2.3}$$

incompressiblity:

$$\tilde{\rho}_t + U\tilde{\rho}_x + w\overline{\rho}' = 0 \tag{5.2.4}$$

where

$$\overline{\rho}' \equiv \frac{d\overline{\rho}}{dz}$$

and momentum conservation:

$$\overline{\rho}\left(u_t + Uu_x + wU_z\right) = -p_x \tag{5.2.5}$$

$$\overline{\rho}\left(w_t + Uw_x\right) = -p_z - \tilde{\rho}g. \tag{5.2.6}$$

where $\tilde{\rho}$ denotes the perturbation of density from $\bar{\rho}$.

Let us follow Miles and introduce a new unknown η by enoting $\tilde{\rho} = -\overline{\rho}'\eta$, then Eqn. (5.2.4) gives

$$\eta_t + U\eta_x = w \tag{5.2.7}$$

Consider

$$\eta = F(z)e^{ik(x-ct)}, \tag{5.2.8}$$

where

$$c = \omega/k = c_r + ic_i.$$

For fixed k the flow is unstable if $c_i > 0$, since

$$e^{-ikct} = e^{-ikc_rt}e^{kc_it}.$$

Let

$$\{u, w, p, \tilde{\rho}\} = \{\hat{u}(z), \hat{w}(z), \hat{p}(z), -\overline{\rho}' F(z)\}\, e^{ik(x-ct)} \tag{5.2.9}$$

We get from Eqn. (5.2.7)

$$\hat{w} = ik(U - c)F, , \tag{5.2.10}$$

from Eqn. (5.2.3)

$$\hat{u} = -[(U - c)F]', \tag{5.2.11}$$

and from Eqn. (5.2.5)

$$\overline{\rho}\,(ik(U - c)\hat{u} + U'[ik(U - c)F]) = \hat{p}ik$$

or

$$\overline{\rho}[(U - c)(-)[(U - c)F]' + U'(U - c)F] = \hat{\rho},$$

hence

$$\hat{p} = \overline{\rho}(U - c)^2 F'. \tag{5.2.12}$$

Substituting Eqns. (5.2.9), (5.2.10), (5.2.11) and (5.2.12) into Eqn. (5.2.6), we get

$$\left[\overline{\rho}(U - c)^2 F'\right]' + \overline{\rho}\left[N^2 - k^2(U - c)^2\right] F = 0, \tag{5.2.13}$$

where N is the Brunt-Väisälä frequency defined by

$$N^2 = -\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}. \tag{5.2.14}$$

Let the top and bottom be rigid walls, then $w = 0$. Hence,

$$\eta = 0 \;\text{ i.e., }\; F = 0, \qquad z = 0, d. \tag{5.2.15}$$

The argument is unchanged if the top and bottom are at $z = \infty$ and $z = -\infty$. Equations (5.2.13) and (5.2.15) consititute an eigenvalue problem where $c = c_r + ic_i$ is the eigenvalue. If $c_i > 0$, instability occurs.

5.2.3 A necessary condition for instability (J.W. Miles, L. N. Howard).

For brevity we set $W = U - c$. Miles further introduce $G = \sqrt{W}F$, so that Eqn. (5.2.13) becomes

$$(\overline{\rho}WG')' - \left[\frac{1}{2}(\overline{\rho}U')' + k^2\overline{\rho}W + \frac{\overline{\rho}}{W}\left(\frac{1}{4}U'^2 - N^2\right)\right]G = 0. \tag{5.2.16}$$

The boundary conditions are

$$G(0) = G(d) = 0. \tag{5.2.17}$$

Multiplying Eqn. (5.2.16) by G^* and integrating by parts

$$\int_0^d \left\{\overline{\rho}W\left(|\ G_1'\ |^2 + k^2\ |\ G_1\ |^2\right) + \frac{1}{2}(\overline{\rho}U')'\,|G|^2 + \overline{\rho}\left(\frac{1}{4}U'^2 - N^2\right)W^*\ |\ \frac{G}{W}\ |^2\right\}dz = 0. \tag{5.2.18}$$

We now seek the necessary condition for instability, i.e., $c_i \neq 0$. Writing

$$W = (U - c_r) - ic_i \qquad W^* = (U - c_r) + ic_i$$

and substituting these in (5.2.18), we get

$$\int_0^d \left\{\overline{\rho}(U - c_r - ic_i)\left(|\ G'\ |^2 + k^2\ |\ G\ |^2\right)\right.$$

$$\left.+\frac{1}{2}(\overline{\rho}U')'\,|G|^2 + \overline{\rho}\left(\frac{1}{4}U'^2 - N^2\right)(U - c_r + ic_i)\ |\ \frac{G}{W}\ |^2\right\}dz = 0.$$

Separating the imaginary part, we get, if $c_i \neq 0$,

$$\int_0^d \overline{\rho}\left((|\ G'\ |^2 + k^2\ |\ G\ |^2\right)dz + \int_0^d \overline{\rho}\left(g\beta - \frac{1}{4}(U')^2\right)\ |\ \frac{G}{W}\ |^2\ dz = 0.$$

For this to be true it is necessary that $N^2 < \frac{1}{4}(U')^2$ or

$$R_i = \frac{N^2}{(U')^2} = \frac{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2} < \frac{1}{4}. \tag{5.2.19}$$

This confirms the heurisic result as the necessary (but not sufficient) condition for instabililty (J.W. Miles, L. N. Howard).