

Notes on
1.63 Advanced Environmental Fluid Mechanics
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4-4 buoyplum.tex

4.4 Buoyant plume from a steady heat source

[Reference]:

Gebhart, et. al. (Jalluria, Maharjan, Saammakia),
Buoyancy-induced Flows and Transport, 1988,
 Hemisphere Publishing Corporation

Let $\tilde{T} = T - T_\infty =$ temperature variation where T_{infty} is a constant (no ambient stratification). For a strong enough heat source, we expect boundary layer behavior,

$$\frac{\partial}{\partial r} \gg \frac{\partial}{\partial x}, \quad u \gg v, \quad \frac{\partial p}{\partial r} \cong 0$$

The boundary layer equations are

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 \quad (4.4.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = g\beta(T - T_\infty) + \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \quad (4.4.2)$$

$$u \frac{\partial \tilde{T}}{\partial x} + v \frac{\partial \tilde{T}}{\partial r} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{T}}{\partial r} \right) \quad (4.4.3)$$

The centerline $r = 0$ is an axis of symmetry,

$$v = \frac{\partial u}{\partial r} = \frac{\partial \tilde{T}}{\partial r} = 0 \quad (4.4.4)$$

Far outside the plume $r \rightarrow \infty$

$$u \rightarrow 0 \text{ and } T \rightarrow T_\infty, (\tilde{T} \rightarrow 0) \quad (4.4.5)$$

Rewrite (4.4.3) as

$$\begin{aligned} & \frac{\partial(ru\tilde{T})}{\partial x} + \frac{\partial(rv\tilde{T})}{\partial r} - \tilde{T} \left(\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} \right) \\ &= \frac{\partial(ru\tilde{T})}{\partial x} + \frac{\partial(rv\tilde{T})}{\partial r} = k \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{T}}{\partial r} \right) \end{aligned} \quad (4.4.6)$$

after using continuity. Now integrating the last equation from $r = 0$ to $r = \infty$

$$\begin{aligned} \frac{\partial}{\partial x} \int_0^\infty 2\pi r u \tilde{T} dr + 2\pi \int_0^\infty \frac{\partial(rv\tilde{T})}{\partial r} dr \\ = k2\pi \int_0^\infty dr \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{T}}{\partial r} \right) \end{aligned}$$

Therefore

$$\frac{\partial}{\partial x} \int_0^\infty 2\pi r u \tilde{T} dr + 2\pi r v \tilde{T} \Big|_0^\infty = 2\pi k \left(r \frac{\partial \tilde{T}}{\partial r} \right)_{r=0}^{r=\infty} \quad (4.4.7)$$

Using the boundary conditions, we get or

$$\int_0^\infty 2\pi r u \tilde{T} dr = \text{constant}$$

Note that

$$\begin{aligned} \int_0^\infty 2\pi r dr u \rho C \tilde{T} &= \text{rate of buoyancy flux} \\ &= \text{rate of heat flux} \\ &= Q (\text{given rate of heat release at } x = 0) \end{aligned}$$

Therefore,

$$Q = \int_0^\infty 2\pi r dr \rho u C \tilde{T} \quad (4.4.8)$$

This is a boundary condition.

Let the stream function ψ be defined by

$$ru = \frac{\partial \psi}{\partial r}, \quad rv = -\frac{\partial \psi}{\partial x} \quad (4.4.9)$$

(4.4.1) is automatically satisfied. From the momentum equation:

$$\left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \frac{1}{r} \frac{\partial^2 \psi}{\partial x \partial r} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = g\beta \tilde{T} + \frac{\nu}{r} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right] \quad (4.4.10)$$

From the energy equation

$$\frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial \tilde{T}}{\partial x} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial \tilde{T}}{\partial r} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{T}}{\partial r} \right) \quad (4.4.11)$$

and from the buoyancy flux condition

$$Q = 2\pi \rho C \int_0^\infty r dr \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \tilde{T} \quad (4.4.12)$$

Try a similarity solution with the one-parameter transformation

$$x = \lambda^a x^*, \quad r = \lambda^b r^*, \quad \psi = \lambda^c \psi^*, \quad \tilde{T} = \lambda^d T^*$$

From (4.4.10),

$$\lambda^{2c-4b-a} = \lambda^{2c-4b-a} = \lambda^d = \lambda^{c-4b} \quad (4.4.13)$$

from (4.4.11)

$$\lambda^{c+d-2b-a} = \lambda^{d-2b} \quad (4.4.14)$$

and from (4.4.12)

$$\lambda^{c+d} = 1 \quad (4.4.15)$$

From these three equations we get

$$\frac{c}{a} = 1, \quad \frac{b}{a} = \frac{1}{2}, \quad \frac{d}{a} = -1.$$

The similarity variable can be taken to be

$$\eta = \frac{r}{x^{1/2}} \quad (4.4.16)$$

and the similarity solution to be

$$\psi = xF(\eta) \quad (4.4.17)$$

and

$$\tilde{T} = x^{-1}G(\eta) \quad (4.4.18)$$

After a lot of algebra, and noting

$$\frac{\partial \eta}{\partial r} = \frac{1}{x^{1/2}}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{2} \frac{r}{x^{3/2}} = -\frac{1}{2} \frac{r}{x^{1/2}} \frac{1}{x} = -\frac{\eta}{2x}$$

we get from (4.4.10)

$$\nu F'''' + \left(\frac{F'}{\eta} \right)' (F - \nu) + g\beta\eta G = 0 \quad (4.4.19)$$

and from (4.4.11)

$$k(\eta G')' + (FG)' = 0 \quad (4.4.20)$$

Before integrating, let us normalize :

$$\eta = \alpha \bar{\eta}, \quad F = \gamma \bar{F}, \quad G = \sigma \bar{G}. \quad (4.4.21)$$

It follows from (4.4.19) that

$$\frac{\nu\gamma}{\alpha^3} \bar{F}'''' + \frac{\gamma}{\alpha^3} \left(\frac{\bar{F}'}{\bar{\eta}} \right)' (\gamma \bar{F} - \nu) + g\beta\alpha\sigma\bar{\eta}\bar{G} = 0 \quad (4.4.22)$$

where prime denotes $d/d\bar{\eta}$. Setting $\gamma = \nu$ and

$$\frac{\nu^2}{\alpha^3} = g\beta\alpha\sigma$$

which relates σ and α ,

$$\sigma = \frac{\nu^2}{g\beta\alpha^4} \quad (4.4.23)$$

we get

$$\bar{F}''' + \left(\frac{\bar{F}'}{\bar{\eta}}\right)' (\bar{F} - 1) + \bar{\eta}\bar{G} = 0 \quad (4.4.24)$$

Similar normalization of (4.4.20) gives

$$\frac{k\alpha\sigma}{\alpha^2}(\bar{\eta}\bar{G}')' + \frac{\gamma\sigma}{\alpha}(\bar{F}\bar{G})' = 0 \quad (4.4.25)$$

. which can be simplified to

$$(\bar{\eta}\bar{G}')' + Pr(\bar{F}\bar{G})' = 0 \quad (4.4.26)$$

. where

$$Pr = \frac{\nu}{k} = \text{Prandtl Number} \quad (4.4.27)$$

For water $\nu = 10^{-2}cm^2/s, k = 1.42cm^2/s$, hence $Pr = 7$. For air $\nu = 0.145cm^2/s, k = 0.202cm^2/s$, hence $Pr = 0.75$.

We now integrate (4.4.26) to give

$$\bar{\eta}\bar{G}' + Pr\bar{F}\bar{G} = \text{constant}$$

Since $\psi(x, 0) = 0$, we must have $\bar{F}(0) = 0$; the constant above is zero.

$$\bar{\eta}\bar{G}' + Pr\bar{F}\bar{G} = 0 \quad (4.4.28)$$

Equation (4.4.28) can be written

$$\frac{\bar{G}'}{\bar{G}} = -Pr\frac{\bar{F}}{\bar{\eta}}, \quad \text{or} \quad \frac{d \ln \bar{G}}{d\bar{\eta}} = -Pr\frac{\bar{F}}{\bar{\eta}}$$

$$\ln \bar{G} = -Pr \int_0^{\bar{\eta}} \frac{\bar{F}}{\bar{\eta}} d\bar{\eta} + \text{constant}$$

$$\bar{G}(\bar{\eta}) = \bar{G}(0) \exp\left(-Pr \int_0^{\bar{\eta}} \frac{\bar{F}}{\bar{\eta}} d\bar{\eta}\right) \quad (4.4.29)$$

Substituting Eqn. (4.4.29) into Eqn. (4.4.24), the resulting equation for \bar{F} must be integrated numerically.

Now let us find the boundary conditions for F or \bar{F} .

Eqn. (4.4.8) becomes

$$\frac{Q}{2\pi\rho C} = \int_0^\infty dr r \left(\frac{1}{r} \frac{\partial\psi}{\partial r} \right) \frac{G(\eta)}{x} = \int_0^\infty dr \frac{r}{r} x^{1/2} F' \frac{G}{x} = \int_0^\infty d\eta (F'G) = \nu\sigma \int_0^\infty d\bar{\eta} (\bar{F}'\bar{G}) \quad (4.4.30)$$

Therefore,

$$\int_0^\infty d\bar{\eta} \bar{F}'\bar{G} = \frac{Q}{2\pi\rho C\nu\sigma} \quad (4.4.31)$$

Let us choose

$$\frac{Q}{2\pi\rho C\nu\sigma} = 1 \quad (4.4.32)$$

so that

$$\int_0^\infty d\bar{\eta} \bar{F}'\bar{G} = 1 \quad (4.4.33)$$

is the boundary condition for \bar{F} and \bar{G} . Now (4.4.32) defines σ , the scale of G . Note that larger Q implies larger σ and smaller α . Thus a stronger heat source leads to a greater centerline temperature and a thinner plume. Also,

$$u \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

hence

$$u = \frac{1}{r} \psi_r = \frac{F'}{\eta} = \frac{\nu}{\alpha^2} \frac{\bar{F}'}{\bar{\eta}} \rightarrow 0, \quad \text{as } \eta \sim \bar{\eta} \rightarrow \infty$$

The radial velocity is, in general

$$v = \frac{1}{r} \psi_x = \frac{1}{r} \left(F - \eta \frac{F'}{2} \right)$$

Since

$$v \rightarrow 0 \quad \text{as } \eta \rightarrow 0,$$

we must have,

$$F(0) = 0.$$

Clearly

$$\frac{\bar{F}'}{\bar{\eta}} \rightarrow 0, \quad \text{and } \bar{F}(\bar{\eta}) = 0 \quad \text{as } \bar{\eta} \rightarrow 0 \quad (4.4.34)$$

The numerical results by Mollendorf & Gelhart, 1974, are shown in Figs. 4.4.1, for various Prandtl numbers. A schlieren photograph due to Gebhart (copied from Van Dyke **An Album of Fluid Motion**) is shown in Figure fig:plumeVD.

Remark:

$$u = \frac{1}{r} \frac{\partial\psi}{\partial r} = \frac{F'}{\eta} \left(= \frac{x}{r} F' \frac{1}{x^{1/2}} \right)$$

Along the centerline $u(x, 0) = \left(\frac{F'}{\eta}\right)_0 = \text{constant}$ depending on P_r . Why? Buoyancy acceleration is counteracted by entrainment.

Remark: Let the radius of the plume be a which varies as

$$a \sim x^{1/2}$$

This is consistent with the behavior that $u \sim x^0$, and $\tilde{T} \sim x^{-1}$, since

$$a^2 u \tilde{T} = Q$$

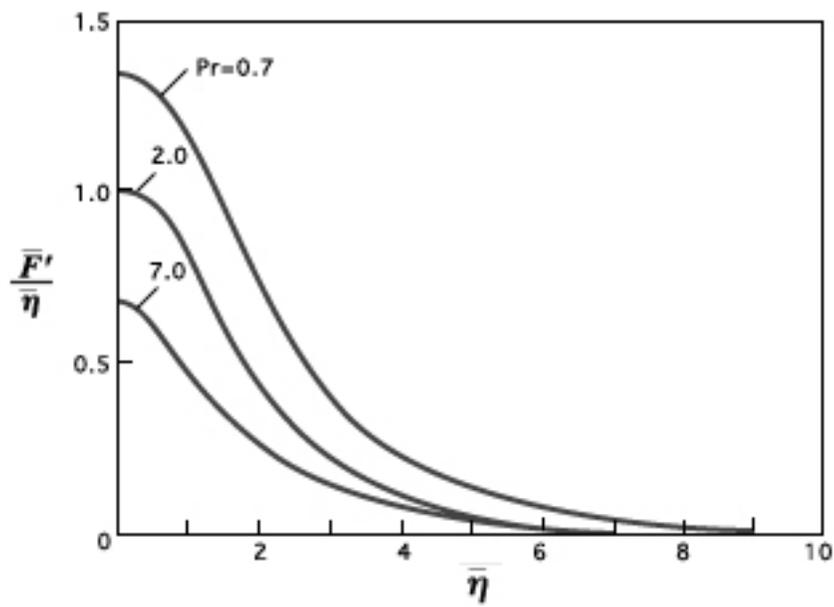
On the other hand the mass flux rate is

$$u a^2 \sim x$$

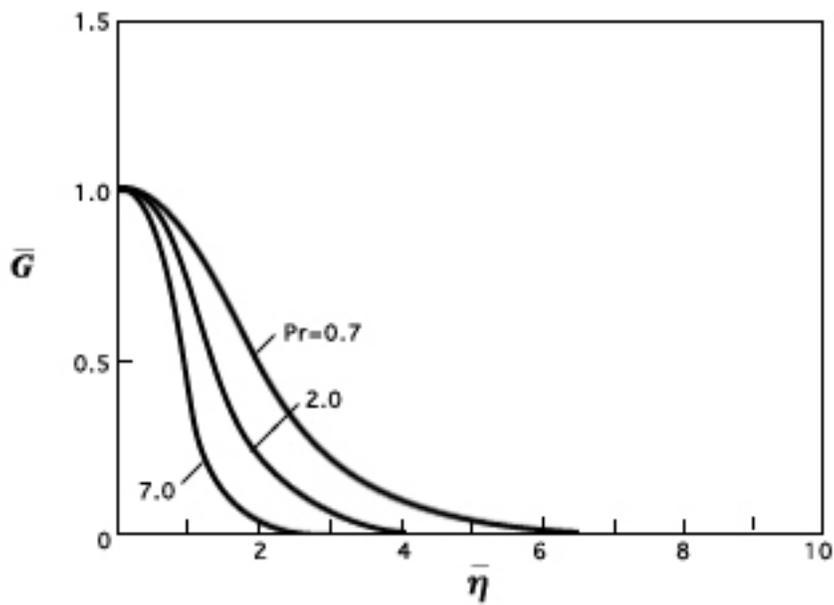
and the momentum flux rate is

$$u^2 a^2 \sim x$$

hence both approach zero at the source. Thus a plume is the result of energy source, not of mass or momentum.



Velocity profiles in an axisymmetric plume.
 (Adapted from Mollendorf and Gebhart, 1974).



Temperature profiles in an axisymmetric plume.
 (Adapted from Mollendorf and Gebhart, 1974).