Lecture notes in Fluid Dynamics
(1.63J/2.01J)
by Chiang C. Mei, MIT

4- 2approx.tex

4.2 Approximations for small temperature variation

4.2.1 Mass conservation and almost incompressibilty

Recall the law of mass conservation:

$$-\frac{1}{\rho}\frac{D\rho}{Dt} = \nabla \cdot \vec{q}$$

Let the time scale be L/U. The left-hand-side is of the order $\frac{U}{L}\frac{\Delta\rho}{\rho}$ while the right-hand-side is $\frac{U}{L}$. For $\Delta T = O(10^o C)$, their ratio is

$$\frac{\Delta\rho}{\rho} \sim \frac{\Delta T}{T} \sim \frac{10^o K}{300^o K} \ll 1$$

Therefore,

$$\nabla \cdot \vec{q} = 0. \qquad (4.2.1)$$

The fluid is approximately incompressible even if $\Delta T \neq 0$.

4.2.2 Momentum conservation and Boussinesq approximation

In static equilibirum $\vec{q_o} \equiv 0$. Therefore,

$$-\nabla p_o + \vec{f}\rho_o = 0. \qquad (4.2.2)$$

Let $p = p_d + p_o$ where p_d is the dynamic pressure

$$\rho = \rho_d + \rho_o$$

$$-\nabla p + \rho\vec{f} = -\nabla p_o + \rho_o\vec{f} - \nabla p_d + (\rho - \rho_o)\,\vec{f}$$

Therefore,

$$\rho\frac{D\vec{q}}{Dt} = -\nabla p_d + \nabla\cdot\,\bar{\bar{\tau}} + \underbrace{(\rho - \rho_o)\,\vec{f}}_{\text{buoyancy force}} \qquad (4.2.3)$$

Now

$$\rho = \bar{\rho}_o[1 - \beta(\Delta T_o + \Delta T_d)]) \qquad (4.2.4)$$

Hence

$$\rho_o = \bar{\rho}_o(1 - \beta\Delta T_o), \quad \rho_d = -\bar{\rho}_o\beta\Delta T_d,$$

and

$$(\rho - \rho_o)\vec{f} = -\bar{\rho}_o(-g)\beta\Delta T_d\vec{k} = \bar{\rho}_o g\beta\Delta T_d\vec{k} \tag{4.2.5}$$

For mildly varying ρ_o and small $\rho - \rho_o$, we ignore the variation of density and approximate ρ_o by a constant everywhere, except in the body force. This is called the **Boussinesq approximation**. Thus

$$\bar{\rho}_o\frac{D\vec{q}}{Dt} = -\nabla p_d + \nabla\cdot \bar{\bar{\tau}} + \bar{\rho}_o g\beta\Delta T_d\,\vec{k} \tag{4.2.6}$$

where

$$\bar{\rho}_o = \rho_o(z = 0)$$

4.2.3 Total energy

Using Eqn. (4.2.1) in Eqn. (**??**) and the Boussinesq approximation

$$\bar{\rho}_o C\frac{DT}{Dt} = \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i} + \Phi \tag{4.2.7}$$

Here T is the total temperature (static + dynamic).

Now

$$\frac{\Phi}{\bar{\rho}_o C\frac{DT}{Dt}} \sim \frac{\mu U^2/L^2}{\bar{\rho}_o C\frac{U\Delta T}{L}} \sim \frac{\mu}{\bar{\rho}_o UL}\frac{U^2}{C\Delta T} = \frac{E}{Re}$$

where

$$E = \frac{U^2}{C\Delta T} = \text{Eckart No.}, \quad Re = \frac{\rho UL}{\mu} = \text{Reynolds No.}$$

In environmental problems, $\Delta T \sim 10\ {}^oK, L \sim 10\ m, U \sim 1m/sec$, the last two columns of

Table 4.1: Typical values E/Re for air and water

	Water	Air
C (erg/s-gr-$^\circ K$)	4×10^7	10^7
K (ergs-cm-$^\circ K$)	0.6×10^5	0.3×10^5
$\nu(cm^2/s)$	10^2	2×10^{-2}
$\beta(1/^\circ K)$	10^{-3}	$1/300$
E	0.25×10^{-2}	10^{-4}
Re	10^5	0.5×10^5

Table 4.1 is obtained. Hence Φ is negligible, and

$$\bar{\rho}_o C\frac{DT}{Dt} = \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i} \tag{4.2.8}$$

Only convection and diffusion are dominant. This is typical in natural convectin problems.

Remark 1. In many engineering problems (aerodynamics, rocket reentry, etc.), heat is caused by frictional dissipation in the flow, therefore, Φ is important. These are called *forced convection* problems. In environmental problems, flow is often the result of heat addition. Here the flow problems are referred to as the *natural convection.*

Remark 2: Since $\bar{T}$ appears as a derivative only, only the variation of T, i.e., the difference $T - \bar{T}_o$ matters, where $\bar{T}_o$ is a reference temperature.

Remark 3: In turbulent natural convection

$$u = \bar{u} + u' \qquad T = \bar{T} + T' \tag{4.2.9}$$

Averaging Eqn. (4.2.8)

$$\bar{\rho}_o c \frac{D\bar{T}}{Dt} = - \underbrace{\bar{\rho}_o c \frac{\partial}{\partial x_i} \overline{u_i' T'}}_{\text{heat flux by turbulence}} + \frac{\partial}{\partial x_i} K \frac{\partial \bar{T}}{\partial x_i} \tag{4.2.10}$$

If the the correlation term is modeled as eddy diffusion, the form would be similar to (4.2.8).