Lecture Notes on Fluid Dynmics
(1.63J/2.21J)
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3-5Karman.tex

3.5 Karman's momentum integral approach

Ref: Schlichting: **Boundary layer theory**

With a general pressure gradient the boundary layer equations can be solved by a variety of modern numerical means. An alternative which can still be employed to simplify calculations is the momentum integral method of Karman. We explain this method for a transient boundary layer along the x-axis forced by an unsteady pressure gradient outside. This pressure gradient can be due to some unsteady and nonuniform flow such as waves or gust.

3.5.1 General ideas

Let us limit our discussoin to a two dimentional flow. Recall the continuity

$$u_x + w_z = 0 \tag{3.5.1}$$

and boundary layer equation:

$$\rho(u_t + uu_x + wu_z) = -p_x + \mu u_{zz} \tag{3.5.2}$$

$$0 = -p_z \tag{3.5.3}$$

The boundary conditions are

$$u = w = 0, \qquad z = 0; \ \ u \to U, \ w \to 0, \qquad z \to \infty. \tag{3.5.4}$$

For general $U(x,t)$ the longitudinal pressure gradient is

$$-p_x = \rho(U_t + UU_x) \tag{3.5.5}$$

so that

$$\rho(u_t + uu_x + wu_z) = \rho(U_t + UU_x) + \mu u_{zz} \tag{3.5.6}$$

Instead of solving the initial-boundary-value problem accurately we introduce a moment method by integrating the momentum equation across the boundary layer, then make a reasonable assumption on the velocity profile to get the longitudinal variation of the boundary layer thickness. This is called the Karman's momentum integral approximation in boundary layer theory.

Let us add the following two equations

$$-\mu u_{zz} = \rho \left\{ U_t + UU_x - (u_t + uu_x + wu_z) \right\}$$

and

$$0 = (U - u)u_x + (U - u)w_z$$

to get,

$$-\mu u_{zz} = \rho \left\{ \frac{\partial}{\partial t}(U - u) + \frac{\partial}{\partial x}[u(U - u)] + (U - u)\frac{\partial U}{\partial x} + \frac{\partial}{\partial z}[(w(U - u)] \right\}$$

Now let us integrate across the boundary layer,

$$\mu u_z|_0 = \rho \left\{ \frac{\partial}{\partial t} \int_0^\infty (U - u)dz + \frac{\partial}{\partial x} \int_0^\infty u(U - u)dz + \frac{\partial U}{\partial x} \int_0^\infty (U - u)dz \right\} + \rho[w(U - u)]_0^\infty$$

The boundary terms again vanish. Finally, we have,

$$\frac{\partial}{\partial t} \int_0^\infty \rho(U - u)dz + \frac{\partial}{\partial x} \int_0^\infty \rho\left(Uu - u^2\right)dz + \frac{\partial U}{\partial x} \int_0^\infty (U - u)dz = \mu\frac{\partial u}{\partial z}\bigg|_0 \qquad (3.5.7)$$

This is the Kármán momentum integral equation, representing the momentum balance across the thickness of the boundary layer.
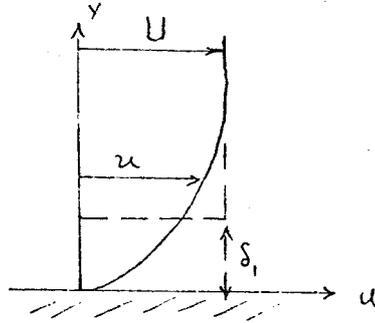
Figure 3.5.1: Displacement thickness

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Karman also introduces the displacement thickness as a measure of the lost volume

$$U\delta_1 = \int_0^\infty (U - u)\,dz, \quad \text{or} \quad \delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right)dz, \qquad (3.5.8)$$

and the momentum thickness as a measure of the lost momentum

$$U^2 \delta_2 = \int_0^\infty u(U - u)dz, \quad \text{or} \quad \delta_2 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dz, \tag{3.5.9}$$

both due to the slowing down of the fluid in the boundary layer. See Figure 3.5.1.

In certain cases when the velocity profile in the boundary layer can be reasonably guessed in advance, the Karman momentum intergal equation can be the basis of obtaining an approximate solution. The procedure is to assume a reasonable profile :

$$u = Uf\left(\frac{y}{\delta}\right) \tag{3.5.10}$$

and substitute (3.5.10) into (3.5.7) equation. After evaluating the integrals a differential equation is obtained for the boundary layer thickness $\delta(x,t)$, which can be more easily solved for certain initial and boundary conditions.

3.5.2   Application to a transient boundary layer along a flat plate

Consider the two dimensional boundary layer near edge of half infinite plane along the x axis due to the impulsive start along its own plane. There is no motion anywhere for $t < 0$. At $t = 0$ the plane suddenly advances from right to left perpendicular to its leading edge. What is the boundary layer flow? Referring to Figure 3.5.2 where the coordinate system is
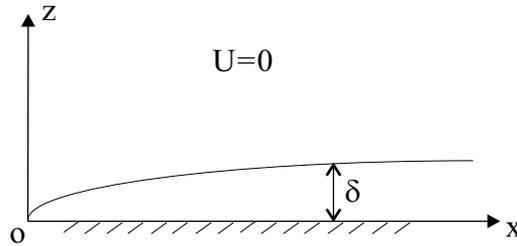
Figure 3.5.2: Front of boundary layer

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fixed on the plane. For all $t > 0$,

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} = 0$$

The boundary layer equation reads:

$$\rho(u_t + uu_x + wu_z) = \mu u_{zz} \tag{3.5.11}$$

and the Karman momentum inetgral equation reads

$$\frac{\partial}{\partial t}\int_0^\infty \rho(U - u)dz + \frac{\partial}{\partial x}\int_0^\infty \rho\left(Uu - u^2\right)dz = \mu\frac{\partial u}{\partial z}\bigg|_0 \tag{3.5.12}$$

Let us assume

$$\frac{u}{U} = f(\eta), \qquad \eta \equiv \frac{z}{\delta(x,t)} \tag{3.5.13}$$

then

$$\delta_1 = \delta \int_0^\infty (1 - f(\eta))d\eta = \alpha_1 \delta \tag{3.5.14}$$

$$\delta_2 = \delta \int_0^\infty f(\eta)(1 - f(\eta))d\eta = \alpha_2 \delta \tag{3.5.15}$$

and

$$u_z = \frac{U}{\delta} f'(0). \tag{3.5.16}$$

Substituting into Eq. (3.5.12)

$$\alpha_1 U \frac{\partial \delta}{\partial t} + \alpha_2 U^2 \frac{\partial \delta}{\partial x} = \nu \frac{U}{\delta} f'(0)$$

Therefore,

$$(\alpha_1) \frac{\partial (\delta^2/2)}{\partial t} + (\alpha_2 U) \frac{\partial (\delta^2/2)}{\partial x} = \nu f'(0) \tag{3.5.17}$$

which is a first order wave (hyperbolic) equation for $\delta^2/2$. We must add the boundary and initial conditions:

$$\delta(0,t) = 0, \quad \forall t > 0 \tag{3.5.18}$$

$$\delta(x,0) = 0, \quad \forall x > 0 \tag{3.5.19}$$

Heuristic solution

The fact that the boundary value in (3.5.18) is independent of t for all t suggests that the solution is independent t for sufficiently large x, i.e.,

$$\frac{\partial}{\partial x}\left(\frac{\delta^2}{2}\right) = \frac{2\nu f'(0)}{\alpha_2 U} \tag{3.5.20}$$

subject to the boundary condition (3.5.18). The solution is simply

$$\delta = \sqrt{\frac{2\nu f'(0)x}{\alpha_2 U}} \tag{3.5.21}$$

This is the approximate version of the solution by Blasius who solved the steady boundary layer equation

$$u u_x + v u_y = \nu u_{yy} \quad \text{for} \ \ x > 0, \ \ y > 0 \tag{3.5.22}$$

by the method of similarity.

On the other hand, the fact that the initial value in (3.5.18) is independent of x for all x suggests that the solution is independent of x for sufficiently large t, i.e.,

$$\frac{\partial}{\partial t}\left(\frac{\delta^2}{2}\right) = \frac{2\nu f'(0)}{\alpha_1} \tag{3.5.23}$$

subject to the initial condition (3.5.19). The solution is simply

$$\delta = \sqrt{\frac{2\nu f'(0)t}{\alpha_1}} \tag{3.5.24}$$

This is the approximate version of the solution by Rayleigh's problem which is known to be governed by

$$u_t = \nu u_{yy} \quad \text{for} \quad t > 0, \quad y > 0 \tag{3.5.25}$$

To find the range of each solution in the $x \sim t$ plane, we equate the two δ's and get

$$x = \frac{\alpha_2 U t}{\alpha_1} \tag{3.5.26}$$

Thus (3.5.21) holds in the wedge $x > \frac{\alpha_2 U t}{\alpha_1} > 0$ and (3.5.24) holds in the wedge $0 < x < \frac{\alpha_2 U t}{\alpha_1}$. See Figure 3.5.3.

Physically: if $t > \alpha_1 x / \alpha_2 U$, (3.5.21) applies and one has the steady Blasius flow past a semi-infinite plate; the boundary layer is already in the steady state. See Figure 3.5.4, If $t < \alpha_1 x / \alpha_2 U$, (3.5.24) applies and one has Rayleigh's problem of impulsively started plane infinite in both $x < 0$ and $x > 0$. The boundary layer is still in the initial stage and the effect of the leading edge is not felt elsewhere. The result is summarized in Figure 3.5.5.

To calculate α_1 and α_2 let us make a special choice of the velocity profile so that $f(0) = 0, f'(1) = f''(1) = 0$ (due to Karman and Polhausen)

$$f(\eta) = 2\eta - 2\eta^3 + \eta^4, \quad 0 < \eta < 1 \tag{3.5.27}$$

Note that

$$f(0) = 0, \quad f'(\eta) = 2 - 6\eta^2 + 4\eta^3, \quad f''(\eta) = -12\eta + 12\eta \tag{3.5.28}$$

then

$$\alpha_1 = \int_0^1 \left(1 - 2\eta + 2\eta^3 - \eta^4\right) d\eta = 3/10 = 0.3$$

$$\alpha_2 = \int_0^1 \left(2\eta - 2\eta^3 + \eta^4\right)\left(1 - 2\eta + 2\eta^3 - \eta^4\right) d\eta = 37/315 \cong 0.117$$

and

$$f'(0) = 2, \quad \text{or} \quad \left.\frac{\partial u}{\partial z}\right|_{z=0} = \frac{2U}{\delta}$$

Hence the displacement thickness of the Blasius boundary layer is

$$\delta_1 = \alpha_1 \delta = \alpha_1 \sqrt{\frac{2\nu f'(0)x}{\alpha_2 U}} = 1.754\sqrt{\frac{\nu x}{U}} \tag{3.5.29}$$

The momentum integral method is the special case of the moment method, since the Karman equation is the zeroth moment of the boundary layer equation. For the classical steady boundary layer problem solved exactly by Blasius using the similarity method, the momentum integral approximation gives fairly good results, even with various crude profiles, see Table 3.1.
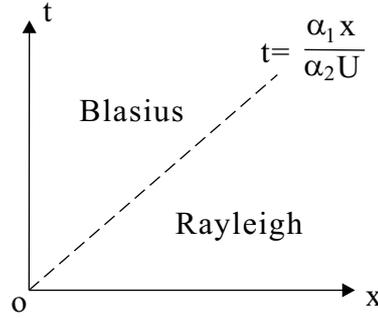
Figure 3.5.3: Evolution of a transient boundary layer. (a) Blasius region and Rayleigh region
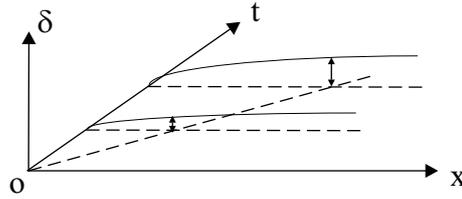
Figure 3.5.4: Evolution of a transient boundary layer. (b) Blasius region

Formal solution

A more formal solution can be obtained as follows: Let

$$a = \alpha_1 \qquad b = \alpha_2 U \qquad c = \nu f'(0)$$

then

$$a\frac{\partial}{\partial t}\left(\frac{\delta^2}{2}\right) + b\frac{\partial}{\partial x}\left(\frac{\delta^2}{2}\right) = c \qquad (3.5.30)$$

The solution can be facilitated by a change of coordinates from (x, t) to (ξ, ζ) where

$$\xi = ax + bt \qquad \zeta = ax - bt \qquad (3.5.31)$$

then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial t}; \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

$$\left(\delta^2\right)_t = \left(\delta^2\right)_\xi b + \left(\delta^2\right)_\zeta (-b)$$
$$\left(\delta^2\right)_x = \left(\delta^2\right)_\xi a + \left(\delta^2\right)_\zeta a$$

Therefore, from (3.5.17),

$$\frac{1}{2}\left[ab\left(\delta^2\right)_\xi - ab\left(\delta^2\right)_\zeta + ab\left(\delta^2\right)_\xi + ab\left(\delta^2\right)_\zeta\right] = c$$
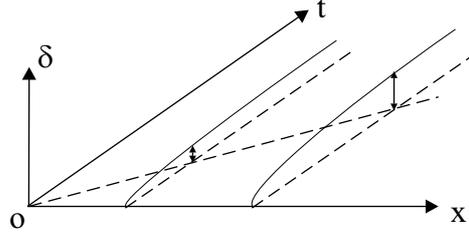
Figure 3.5.5: Evolution of a transient boundary layer. (c) Rayleigh region

Velocty profile	Displacment boundary layer thickness
$\frac{u}{U} = f(\eta), \quad \eta = \frac{y}{\delta(x)}$	$\delta_1 \sqrt{\frac{U}{\nu x}}$
η	1.732
$\sin \frac{\pi \eta}{2}$	1.741
$2\eta - 2\,\eta^3 + \eta^4$	1.754
Blausius	1.721

Table 3.1: Comparson of approximate solution by momentum integral method with the exact solution of Blasius

$$\frac{1}{2}\left(\delta^2\right)_\xi = \frac{c}{2ab} = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}$$

Integrate once

$$\frac{\delta^2}{2} = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\xi + G(\zeta) \tag{3.5.32}$$

where G is an arbitrary function of ζ.

To determine $G(\zeta)$ we first use the initial condition that $\delta = 0$ at $t = 0$ <u>for all $x > 0$</u>.

$$0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(\xi)_{t=0} + G(\zeta)_{t=0}$$

or

$$0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(ax) + G(ax) \quad \text{for} \quad x > 0.$$

Therefore

$$G(\zeta) = -\frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta, \qquad \zeta > 0 \tag{3.5.33}$$

What is $G(\zeta)$ for $\zeta < 0$? Let us use the boundary condition at $x = 0$ that $\delta = 0$ for all $t > 0$.

$$0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(\xi)_{x=0} + G(\zeta)_{x=0}$$

or

$$0 = -\frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(-bt) + G(-bt), \qquad t > 0$$

Note that for $t > 0$, $-bt < 0$. Therefore,

$$G(\zeta) = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta \qquad \text{for all} \quad \zeta < 0. \tag{3.5.34}$$

Eqs. (3.5.33) and (3.5.34) complete the solution for all $\zeta > 0$ and $\zeta < 0$.

Let us return to x and t

$$\begin{aligned}
\frac{\delta^2}{2} &= \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\xi - \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}2bt \qquad \text{for} \quad ax - bt > 0 \\
&= \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\xi + \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}2ax \qquad \text{for} \quad ax - bt < 0
\end{aligned}$$

Therefore,

$$\delta = \sqrt{\frac{2\nu f'(0)}{\alpha_1}t} \qquad \text{for} \quad x > \frac{\alpha_2 U}{\alpha_1}t \tag{3.5.35}$$

and

$$\delta = \sqrt{\frac{2\nu f'(0)}{\alpha_2 U}x} \qquad \text{for} \quad x < \frac{\alpha_2 U}{\alpha_1}t \tag{3.5.36}$$

These are the same as (3.5.24) and (3.5.21).