Lecture Notes on Fluid Dynmics
(1.63J/2.21J)
by Chiang C. Mei, MIT

3-3-lamjet.tex,

3.3 Two dimensional laminar jet

If new fluid is injected at high momentum into a stationary fluid of the same density, a jet is formed. If viscosity were absent only the layer as thin as the jet nozzle would be moved. Due to viscosity, the surounding fluid is dragged along in the forward direction. When viscosity is low the jet is a thin boundary layer within which the viscous stress is as important as the fluid inertia.

Consider a two dimensional jet from a steady source of momentum:

$$u_x + w_z = 0 \tag{3.3.1}$$

$$uu_x + wu_z = -p_x + \nu(u_{xx} + u_{zz}) \tag{3.3.2}$$

$$uw_x + ww_z = -p_z + \nu(w_{xx} + w_{zz}) \tag{3.3.3}$$

The inital momtum is given,

$$\int_{-\infty}^{\infty} \rho u^2 dy = \rho M \tag{3.3.4}$$

In addition, we impose

$$u, v, \to 0, \quad y \to \pm\infty \tag{3.3.5}$$

Anticipating the moving layer to be thin relative to the length of the jet, we introduce the two sharply contrasting scales to normalize the spatial coordinates and change to normalized variables

$$u \to Uu, \; v \to \frac{\delta}{L} Uv, \; x \to Lx, \; z = \delta z \tag{3.3.6}$$

where U can be the centerline jet velocity yet unknown and L the distance from the nozzle. To be brief, the normalized variables are without primes. Then

$$\frac{U}{L}(u_x + w_z) = 0 \tag{3.3.7}$$

$$\frac{U^2}{L}\{uu_x + wu_z\} = -\frac{P}{\rho L} p_x + \frac{\nu U}{\delta^2}\left(\frac{\delta^2}{L^2} u_{xx} + u_{zz}\right) \tag{3.3.8}$$

Equivalently, we have

$$uu_x + wu_z = -\frac{P}{\rho U^2} p_x + \frac{\nu}{UL}\frac{L^2}{\delta^2}\left(\frac{\delta^2}{L^2} u_{xx} + u_{zz}\right) \tag{3.3.9}$$

and

$$\frac{\delta}{L}\frac{U^2}{L}\left\{uw_x + ww_z\right\} = -\frac{P}{\rho\delta}p_z + \frac{\nu U}{\delta^2}\frac{\delta}{L}\left(\frac{\delta^2}{L^2}w_{xx} + w_{zz}\right) \tag{3.3.10}$$

or

$$\frac{\delta^2}{L^2}\left\{uw_x + ww_z\right\} = -\frac{P}{\rho U^2}p_z + \frac{\nu}{UL}\left(\frac{\delta^2}{L^2}w_{xx} + w_{zz}\right) \tag{3.3.11}$$

For high Reynolds numbers,

$$R = \frac{UL}{\nu} \gg 1 \tag{3.3.12}$$

we must have

$$\frac{\delta}{L} = O\left(\frac{1}{\sqrt{R}}\right) \tag{3.3.13}$$

so that the most important viscous stress is retained to balance the effects of inertia.

From (3.3.10),

$$p_z = 0 \tag{3.3.14}$$

That the pressure is constant across the boundary layer is the signature property of the boundary approximation at high Reynolds number flows.

Since the pressure gradient is zero outside the jet, we have simply

$$p_x = p_x(x, z = \pm\infty) = 0 \tag{3.3.15}$$

Eq (3.3.8) becomes approximately

$$uu_x + wu_z = \frac{1}{R}u_{zz} \tag{3.3.16}$$

This is the boundary layer approximation for the jet mementum. Mission of approximation now accompolshed, we return to the physical variables :

$$u_x + w_z = 0 \tag{3.3.17}$$

$$uu_x + wu_z = \nu u_{zz} \tag{3.3.18}$$

Integrating (3.3.18)

$$\int_{-\infty}^{\infty}(uu_x + wu_z)dz = \nu\int_{-\infty}^{\infty}u_{zz}dz \tag{3.3.19}$$

By continuity

$$wu_z = (wu)_z - uw_z = (wu)_z + uu_x$$

After integration and using the boundary conditions that

$$u \to 0, \quad z = \pm\infty \tag{3.3.20}$$

the left hand side becomes

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty}u^2dz = 0 \tag{3.3.21}$$

thus the momentum flux is constant in x,

$$\rho \int_{-\infty}^{\infty} u^2 dz = M \tag{3.3.22}$$

Comment: At the nozzle $\delta(0) \to 0$,

$$\rho u(0)^2 \delta(0) = M$$

hence

$$u(0) = \delta(0)^{-1/2}$$

Mass flux from the nozzle is

$$\rho u(0)\delta(0) \propto \delta(0)^{1/2} \to 0$$

hence a jet is defined by its initial moemtum; the mass discharge is unimportant. A jet is the result of a momentum source, not a mass source.

3.3.1 Similarity solution

Introducing the stream function ψ so that

$$u = \psi_z, \quad w = -\psi_x \tag{3.3.23}$$

The x-momentum equation becomes

$$\psi_z \psi_{xz} - \psi_x \psi_{zz} = \nu \psi_{zzz} \tag{3.3.24}$$

with the boundary conditons that

$$\psi_z \downarrow 0, \quad z \to \pm\infty \tag{3.3.25}$$

and

$$\rho \int_{-\infty}^{\infty} \psi_z^2 dz = M \tag{3.3.26}$$

Try the transformation:

$$x = \lambda^a x', \quad z = \lambda^b z', \quad \psi = \lambda^c \psi' \tag{3.3.27}$$

Requiring invariance we get from (3.3.24)

$$2c - 2b - a = c - 3b, \quad \text{or} \quad \frac{c}{a} = 1 - \frac{b}{a}$$

No information is gained from (3.3.24). From (3.3.25) we get

$$2(c - b) + b = 0$$

hence

$$c = \frac{b}{2}, \quad \frac{c}{a} = 1 - \frac{2c}{a} \quad \text{or} \quad \frac{3c}{a} = 1$$

implying

$$c = a/3, \quad b = 2a/3 \tag{3.3.28}$$

The final tranformation is

$$x = \lambda^a x', \quad z = \lambda^{2a/3} z', \quad \psi = \lambda^{a/3} \psi' \tag{3.3.29}$$

This suggests that we take

$$\frac{\psi}{Bx^{1/3}} = f\left(\frac{Cz}{x^{2/3}}\right) \tag{3.3.30}$$

The coefficients B and C are chosen to simplify the appearance of the final equation. Let us take

$$\eta = \left(\frac{M}{\rho\nu^2 x^2}\right)^{1/3} z, \quad \psi = \left(\frac{M\nu x}{\rho}\right)^{1/3} f(\eta) \tag{3.3.31}$$

then

$$u = \psi_z = \left(\frac{M^2}{\rho\nu^2 x}\right)^{1/3} f'(\eta), \tag{3.3.32}$$

$$w = -\psi_x = \frac{1}{3}\left(\frac{M\nu}{\rho x^2}\right)^{1/3} (2\eta f' - f) \tag{3.3.33}$$

From (3.3.24)

$$3f''' + (f')^2 + ff'' = 0 \tag{3.3.34}$$

The boundary conditions become

$$f'(\pm\infty) = 0, \quad f(0) = f''(0) = 0 \;\; \text{symmetry} \tag{3.3.35}$$

and

$$M = \rho\left(\frac{M^2}{\rho^2 \nu x}\right)^{2/3} \frac{1}{\left(\frac{M}{\rho\nu^2 x^2}\right)^{1/3}} \int_{-\infty}^{\infty} [f'(\eta)]^2 d\eta = M \int_{-\infty}^{\infty} [f'(\eta)]^2 d\eta$$

or

$$1 = \int_{-\infty}^{\infty} [f'(\eta)]^2 d\eta \tag{3.3.36}$$

Integrating once

$$3f'' + ff' = \text{constant} = 0$$

Integrating again

$$3f' + \frac{1}{2} f^2 = c^2$$

Let

$$f = F\sqrt{2}, \; \eta = 3\sqrt{2}\zeta \tag{3.3.37}$$

then

$$\frac{dF}{d\zeta} + F^2 = c^2, \quad \to \quad \frac{dF/c}{1 - F^2/c^2} = c\,d\zeta$$

which can be integrated:

$$c\zeta = \tanh^{-1}\frac{F}{c}$$

since $F(0) = 0$. Thus

$$f = \sqrt{2}F = \sqrt{2}c\tanh\left(\frac{c\eta}{3\sqrt{2}}\right)$$

Using (3.3.36),

$$1 = \frac{c^3\sqrt{2}}{3}\int_{-\infty}^{\infty}\mathrm{sech}^4 c\zeta\, dc\zeta = \frac{4\sqrt{2}c^3}{9}$$

hence

$$c^3 = \frac{9}{4\sqrt{2}}; \tag{3.3.38}$$

$$f(\eta) = \left(\frac{9}{2}\right)^{1/3}\tanh\left[\left(\frac{1}{48}\right)^{1/3}\eta\right] \tag{3.3.39}$$

Finally let

$$\xi = \left(\frac{M}{48\rho\nu^2}\right)^{1/3}\frac{z}{x^{2/3}} \tag{3.3.40}$$

the stream function is

$$\psi = \left(\frac{9M\nu x}{2\rho}\right)^{1/3}\tanh\xi \tag{3.3.41}$$

The jet velocty components are:

$$u = \left(\frac{3M^2}{32\rho^2\nu x}\right)^{1/3}\mathrm{sech}^2\xi \tag{3.3.42}$$

$$v = \left(\frac{M\nu}{6\rho x^2}\right)^{1/3}(2\xi\mathrm{sech}^2\xi - \tanh\xi) \tag{3.3.43}$$

See Figure (3.3.1).

3.3.2 Physcial implications

The jet width can be defined by $\xi = \pm\xi_0$ so that $u \downarrow 0$. Then

1. Jet width $\delta \propto x^{2/3}$
2. Centerline velocity : $U = u_{max} \propto x^{-1/3}$,
3. $v \to \mp\left(\frac{M\nu}{6\rho x^2}\right)^{1/3}$, $\xi \to \pm\infty$. There is entrainment from the jet edges.
4. $R = u_{max}\delta/\nu \propto x^{1/3}$.

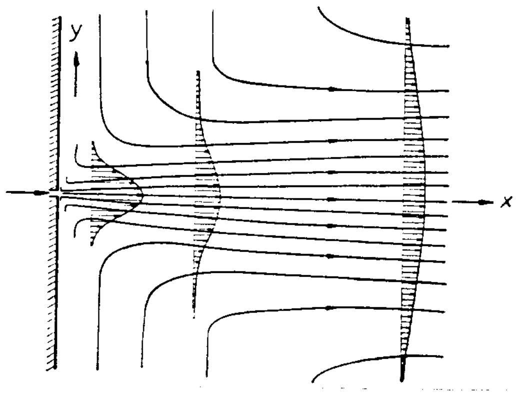

Figure 3.3.1: The laminar jet