Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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2-5Stokes.tex

2.5 Stokes flow past a sphere

[Refs]
Lamb: *Hydrodynamics*
Acheson : *Elementary Fluid Dynamics*, p. 223 ff

One of the fundamental results in low Reynolds hydrodynamics is the Stokes solution for steady flow past a small sphere. The apllicatiuon range widely form the determination of electron charges to the physics of aerosols.

The continuity equation reads

$$\nabla \cdot \vec{q} = 0 \tag{2.5.1}$$

With inertia neglected, the approximate momentum equation is

$$0 = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q} \tag{2.5.2}$$

Physically, the presssure gradient drives the flow by overcoming viscous resistence, but does affect the fluid inertia significantly.

Refering to Figure 2.5 for the spherical coordinate system (r, θ, ϕ). Let the ambient velocity be upward and along the polar (z) axis: $(u, v, w) = (0, 0, W)$. Axial symmetry demands

$$\frac{\partial}{\partial \phi} = 0, \quad \text{and} \quad \vec{q} = (q_r(r,\theta), q_\theta(r,\theta), 0)$$

Eq. (2.5.1) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta \sin\theta) = 0 \tag{2.5.3}$$

As in the case of rectangular coordinates, we define the stream function ψ to satisify the continuity equation (2.5.3) identically

$$q_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad q_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} \tag{2.5.4}$$

At infinity, the uniform velocity W along z axis can be decomposed into radial and polar components

$$q_r = W\cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad q_\theta = -W\sin\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \quad r \sim \infty \tag{2.5.5}$$

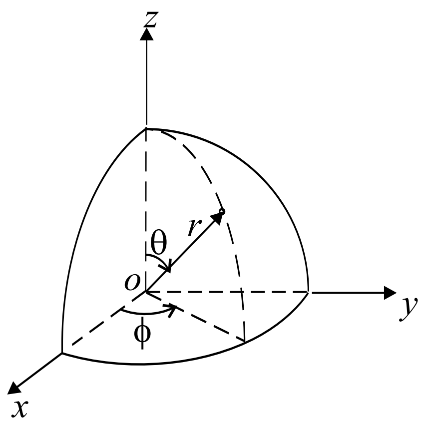

Figure 2.5.1: The spherical coordinates

The corresponding stream function at infinity follows by integration

$$\psi = \frac{W}{2} r^2 \sin^2\theta, \quad r \sim \infty \tag{2.5.6}$$

Using the vector identity

$$\nabla \times (\nabla \times \vec{q}) = \nabla(\nabla \cdot \vec{q}) - \nabla^2 \vec{q} \tag{2.5.7}$$

and (2.5.1), we get

$$\nabla^2 \vec{q} = -\nabla \times (\nabla \times \vec{q}) = -\nabla \times \vec{\zeta} \tag{2.5.8}$$

Taking the curl of (2.5.2) and using (2.5.8) we get

$$\nabla \times (\nabla \times \vec{\zeta}) = 0 \tag{2.5.9}$$

After some straightforward algebra given in the Appendix, we can show that

$$\vec{q} = \nabla \times \left(\frac{\psi \vec{e}_\phi}{r \sin\theta} \right) \tag{2.5.10}$$

and

$$\vec{\zeta} = \nabla \times \vec{q} = \nabla \times \nabla \times \left(\frac{\psi \vec{e}_\phi}{r \sin\theta} \right) = -\frac{\vec{e}_\phi}{r \sin\theta} \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial\psi}{\partial\theta} \right) \right) \tag{2.5.11}$$

Now from (2.5.9)

$$\nabla \times \nabla \times (\nabla \times \vec{q}) = \nabla \times \nabla \times \left[\nabla \times \left(\nabla \times \frac{\psi \vec{e}_\phi}{r \sin\theta} \right) \right] = 0$$

hence, the momentum equation (2.5.9) becomes a scalar equation for ψ.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right)^2 \psi = 0 \tag{2.5.12}$$

The boundary conditions on the sphere are

$$q_r = 0 \quad q_\theta = 0 \quad \text{on} \quad r = a \tag{2.5.13}$$

The boundary conditions at ∞ is

$$\psi \to \frac{W}{2} r^2 \sin^2\theta \tag{2.5.14}$$

Let us try a solution of the form:

$$\psi(r,\theta) = f(r)\sin^2\theta \tag{2.5.15}$$

then f is governed by the equi-dimensional differential equation:

$$\left[\frac{d^2}{dr^2} - \frac{2}{r^2}\right]^2 f = 0 \tag{2.5.16}$$

whose solutions are of the form $f(r) \propto r^n$, It is easy to verify that $n = -1, 1, 2, 4$ so that

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4$$

or

$$\psi = \sin^2\theta\left[\frac{A}{r} + Br + Cr^2 + Dr^4\right]$$

To satisfy (2.5.14) we set $D = 0, C = W/2$. To satisfy (2.5.13) we use (2.5.4) to get

$$q_r = 0 = \frac{W}{2} + \frac{A}{a^3} + \frac{B}{a} = 0, \quad q_\theta = 0 = W - \frac{A}{a^3} + \frac{B}{a} = 0$$

Hence

$$A = \frac{1}{4}Wa^3, \qquad B = -\frac{3}{4}Wa$$

Finally the stream function is

$$\psi = \frac{W}{2}\left[r^2 + \frac{a^3}{2r} - \frac{3ar}{2}\right]\sin^2\theta \tag{2.5.17}$$

Inside the parentheses, the first term corresponds to the uniform flow, and the second term to the doublet; together they represent an inviscid flow past a sphere. The third term is called the Stokeslet, representing the viscous correction.

The velocity components in the fluid are: (cf. (2.5.4) :

$$q_r = W\cos\theta\left[1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right] \tag{2.5.18}$$

$$q_\theta = -W\sin\theta\left[1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right] \tag{2.5.19}$$

2.5.1 Physical Deductions

1. *Streamlines*: With respect to the the equator along $\theta = \pi/2$, $\cos\theta$ and q_r are odd while $\sin\theta$ and q_θ are even. Hence the streamlines (velocity vectors) are symmetric fore and aft.

2. *Vorticity*:
$$\vec{\zeta} = \zeta_\phi \vec{e}_\phi \left(\frac{1}{r}\frac{\partial(rq_\theta)}{\partial r} - \frac{1}{r}\frac{\partial q_r}{\partial \theta} \right) \vec{e}_\phi = -\frac{3}{2} W a \frac{\sin\theta}{r^2} \vec{e}_\phi$$

3. *Pressure* : From the r-component of momentum equation
$$\frac{\partial p}{\partial r} = \frac{\mu W a}{r^3} \cos\theta (= -\mu \nabla \times (\nabla \times \vec{q}))$$
Integrating with respect to r from r to ∞, we get
$$p = p_\infty - \frac{3}{2}\frac{\mu W a}{r^3} \cos\theta \tag{2.5.20}$$

4. *Stresses and strains*:
$$\frac{1}{2} e_{rr} = \frac{\partial q_r}{\partial r} = W \cos\theta \left(\frac{3a}{2r^2} - \frac{3a^3}{2r^4} \right)$$
On the sphere, $r = a$, $e_{rr} = 0$ hence $\sigma_{rr} = 0$ and
$$\tau_{rr} = -p + \sigma_{rr} = -p_\infty + \frac{3}{2}\frac{\mu W}{a} \cos\theta \tag{2.5.21}$$
On the other hand
$$e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial \theta} = -\frac{3}{2}\frac{W a^3}{r^4} \sin\theta$$
Hence at $r = a$:
$$\tau_{r\theta} = \sigma_{r\theta} = \mu e_{r\theta} = -\frac{3}{2}\frac{\mu W}{a} \sin\theta \tag{2.5.22}$$
The resultant stress on the sphere is parallel to the z axis.
$$\Sigma_z = \tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta = -p_\infty \cos\theta + \frac{3}{2}\frac{\mu W}{a}$$
The constant part exerts a net drag in z direction
$$D = \int_o^{2\pi} a d\phi \int_o^{\pi} d\theta \sin\theta \Sigma_z == \frac{3}{2}\frac{\mu W}{a} 4\pi a^2 = 6\pi\mu W a \tag{2.5.23}$$
This is the celebrated Stokes formula.

A drag coefficient can be defined as
$$C_D = \frac{D}{\frac{1}{2}\rho W^2 \pi a^2} = \frac{6\pi\mu W a}{\frac{1}{2}\rho W^2 \pi a^2} = \frac{24}{\frac{\rho W(2a)}{\mu}} = \frac{24}{Re_d} \tag{2.5.24}$$

5. *Fall velocity* of a particle through a fluid. Equating the drag and the buoyant weight of the eparticle

$$6\pi\mu W_o a = \frac{4\pi}{3} a^3(\rho_s - \rho_f) g$$

hence

$$W_o = \frac{2}{9} g \left(\frac{a^2}{\nu}\frac{\Delta\rho}{\rho_f}\right) = 217.8 \left(\frac{a^2}{\nu}\frac{\Delta\rho}{\rho_f}\right)$$

in cgs units. For a sand grain in water,

$$\frac{\Delta\rho}{\rho_f} = \frac{2.5-1}{1} = 1.5, \quad \nu = 10^{-2}\text{cm}^2/s$$

$$W_o = 32,670\ a^2 \text{cm}/s \tag{2.5.25}$$

To have some quantitative ideas, let us consider two sand of two sizes :

$$a = 10^{-2}\text{cm} = 10^{-4}\text{m}: \quad W_o = 3.27\text{cm}/s;$$
$$a = 10^{-3}\text{cm} = 10^{-5} = 10\mu m, \quad W_o = 0.0327\text{cm}/s = 117\text{cm/hr}$$

For a water droplet in air,

$$\frac{\Delta\rho}{\rho_f} = \frac{1}{10^{-3}} = 10^3, \quad \nu = 0.15\ \text{cm}^2/\text{sec}$$

then

$$W_o = \frac{(217.8)10^3}{0.15} a^2 \tag{2.5.26}$$

in cgs units. If $a = 10^{-3}$ cm $= 10\mu$m, then $W_o = 1.452$ cm/sec.

Details of derivation

Details of (2.5.10).

$$\nabla \times \left(\frac{\psi}{r\sin\theta}\vec{e}_\phi\right) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & \vec{e}_\theta & r\sin\theta\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \psi \end{vmatrix}$$

$$= \vec{e}_r\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right) - \vec{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)$$

Details of (2.5.11).

$$
\begin{aligned}
\nabla \times \nabla \times \frac{\psi \vec{e}_\phi}{r \sin\theta} &= \nabla \times \vec{q} \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta \vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} & \frac{-1}{\sin\theta}\frac{\partial \psi}{\partial r} & 0 \end{vmatrix} \\
&= \frac{\vec{e}_\theta}{r \sin\theta} \left[\frac{\partial^2 \phi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial \theta} \right) \right]
\end{aligned}
$$