8

Displacement Method— Ideal Truss

8–1. GENERAL

The basic equations defining the behavior of an ideal truss consist of force-equilibrium equations and force-displacement relations. One can reduce the system to a set of equations involving only the unknown joint displacements by substituting the force-displacement relations into the force-equilibrium equations. This particular method of solution is called the *displacement* or *stiffness* method. Alternatively, one can, by eliminating the displacements, reduce the governing equations to a set of equations involving certain bar forces. The latter procedure is referred to as the *force* or *flexibility* method. We emphasize that these two methods are *just* alternate procedures for solving the same basic equations. The displacement method is easier to automate than the force method and has a wider range of application. However, it is a computer-based method, i.e., it is not suited for hand computation. In contrast, the force method is more suited to hand computation than to machine computation.

In what follows, we first develop the equations for the displacement method by operating on the governing equations expressed in partitioned form. We then describe a procedure for assembling the necessary system matrices using only the connectivity table. This procedure follows naturally if one first operates on the unpartitioned equations and then introduces the displacement restraints. The remaining portion of the chapter is devoted to the treatment of nonlinear behavior. We outline an incremental analysis procedure, apply the classical stability criterion, and finally, discuss linearized stability analysis.

8–2. OPERATION ON THE PARTITIONED EQUATIONS

The governing partitioned equations for an ideal truss are developed in Sec. 6–7. For convenience, we summarize these equations below.

$$\bar{\mathbf{P}}_1 = \mathbf{B}_1\mathbf{F} \qquad (n_d \text{ eqs.}) \qquad \text{(a)}$$
$$\mathbf{P}_2 = \mathbf{B}_2\mathbf{F} \qquad (r \text{ eqs.}) \qquad \text{(b)}$$
$$\mathbf{F} = \mathbf{F}_i + \mathbf{k}\mathbf{A}_1\mathbf{U}_1 \qquad (m \text{ eqs.}) \qquad \text{(c)}$$
$$\mathbf{F}_i = \mathbf{k}(-\mathbf{e}_0 + \mathbf{A}_2\bar{\mathbf{U}}_2) \qquad \text{(d)}$$

The unknowns are the m bar forces ($\mathbf{F}$), the r reactions ($\mathbf{P}_2$), and the n_d joint displacements ($\mathbf{U}_1$). One can consider $\mathbf{F}_i$ to represent the initial bar forces, that is, the bar forces due to the initial elongations and support movements with $\mathbf{U}_1 = 0$. The term $\mathbf{k}\mathbf{A}_1\mathbf{U}_1$ represents the bar forces due to $\mathbf{U}_1$. When the material is *linear elastic*, $\mathbf{k}$ and $\mathbf{e}_0$ are constant. Also, $\mathbf{A}_j = \mathbf{B}_j^T$ when the geometry is *linear*.

We obtain a set of n_d equations relating the n_d displacement unknowns, $\mathbf{U}_1$, by substituting for $\mathbf{F}$ in (a). The resulting matrix equation has the form

$$(\mathbf{B}_1\mathbf{k}\mathbf{A}_1)\mathbf{U}_1 = \bar{\mathbf{P}}_1 - \mathbf{B}_1\mathbf{F}_i \qquad (8\text{–}1)$$

We solve (8–1) for $\mathbf{U}_1$, determine $\mathbf{F}$ from (c), and $\mathbf{P}_2$ from (b). The coefficient matrix for $\mathbf{U}_1$ is called the system *stiffness* matrix and written as

$$\mathbf{K}_{11} = \mathbf{B}_1\mathbf{k}\mathbf{A}_1 \qquad (8\text{–}2)$$

One can interpret $\mathbf{B}_1\mathbf{F}_i$ as representing the initial joint forces due to the initial elongations and support movements with $\mathbf{U}_1 = 0$. Then $\bar{\mathbf{P}}_1 - \mathbf{B}_1\mathbf{F}_i$ represents the net unbalanced joint forces.

When the geometry is linear, $\mathbf{K}_{11}$ reduces to

$$\mathbf{K}_{11} = \mathbf{B}_1\mathbf{k}\mathbf{B}_1^T = \mathbf{A}_1^T\mathbf{k}\mathbf{A}_1 \qquad (8\text{–}3)$$

If the material is linear, $\mathbf{k}$ is constant and positive definite for real materials. Then, the stiffness matrix for the linear case is *positive definite* when the system is initially stable, that is, when $r(\mathbf{B}_1) = n_d$.† Conversely, if it is not positive definite, the system is initially unstable.

If the material is nonlinear, $\mathbf{k}$ and $\mathbf{e}_0$ depend on $\mathbf{e}$. We have employed a piecewise linear representation for the force-elongation curve which results in linear relations. However, one has to iterate when the limiting elongation for a segment is exceeded.

The geometrically nonlinear case is more difficult since both $\mathbf{A}$ and $\mathbf{B}$ depend on $\mathbf{U}_1$. One can iterate on (8–1), but this requires solving a nonsymmetrical system of equations. It is more efficient to transform (8–1) to a symmetrical system by transferring some nonlinear terms to the right-hand side. Nonlinear analysis procedures are treated in Sec. 8–4.

Even when the behavior is completely linear, the procedure outlined above for generating the system matrices is not efficient for a large structure, since

† See Prob. 2–14.

it requires the multiplication of large sparse matrices. For example, one obtains the system stiffness matrix by evaluating the triple matrix product,

$$\mathbf{K}_{11} = \mathbf{A}_1^T \mathbf{k} \mathbf{A}_1 \tag{a}$$

One can take account of symmetry and the fact that $\mathbf{k}$ is diagonal, but $\mathbf{A}_1$ is generally quite sparse. Therefore, what is needed is a method of generating $\mathbf{K}$ which does not involve multiplication of large sparse matrices. A method which has proven to be extremely efficient is described in the next section.

8–3. THE DIRECT STIFFNESS METHOD

We start with (6–37), the force-displacement relation for bar n:

$$\begin{aligned} F_n &= F_{0,n} + k_n \boldsymbol{\gamma}_n \mathbf{u}_{n_+} - k_n \boldsymbol{\gamma}_n \mathbf{u}_{n_-} \\ F_{0,n} &= -k_n e_{0,n} \end{aligned} \tag{a}$$

where n_+, n_- denote the joints at the positive and negative ends of bar n. One can consider $F_{0,n}$ as the bar force due to the initial elongation with the ends fixed ($\mathbf{u}_{n_+} = \mathbf{u}_{n_-} = \mathbf{0}$). Now, we let $\mathbf{p}_{n_+}$, $\mathbf{p}_{n_-}$ be the *external joint force matrices required to equilibrate* the action of F_n. Noting (6–43), we see that

$$\begin{aligned} \mathbf{p}_{n_+} &= +F_n \boldsymbol{\beta}_n^T \\ \mathbf{p}_{n_-} &= -\mathbf{p}_{n_+} \end{aligned} \tag{8–4}$$

Substituting for F_n, (8–4) expands to

$$\begin{aligned} \mathbf{p}_{n_+} &= \boldsymbol{\beta}_n^T F_{0,n} + k_n \boldsymbol{\beta}_n^T \boldsymbol{\gamma}_n \mathbf{u}_{n_+} - k_n \boldsymbol{\beta}_n^T \boldsymbol{\gamma}_n \mathbf{u}_{n_-} \\ \mathbf{p}_{n_-} &= -\mathbf{p}_{n_+} \end{aligned} \tag{b}$$

One can interpret (b) as end action–joint displacement relations since the elements of $\pm F_n \boldsymbol{\beta}_n^T$ are the components of the bar force with respect to the basic frame.

Continuing, we let

$$\mathbf{k}_n = k_n \boldsymbol{\beta}_n^T \boldsymbol{\gamma}_n \tag{8–5}$$

Note that $\mathbf{k}_n$ is of order $i \times i$ where $i = 2$ or 3 for a two or three-dimensional truss, respectively. When the geometry is linear, $\boldsymbol{\beta}_n = \boldsymbol{\gamma}_n = \boldsymbol{\alpha}_n$ and $\mathbf{k}_n$ is symmetrical. With this notation, (b) takes a more compact form,

$$\begin{aligned} \mathbf{p}_{n_+} &= \boldsymbol{\beta}_n^T F_{0,n} + \mathbf{k}_n \mathbf{u}_{n_+} - \mathbf{k}_n \mathbf{u}_{n_-} \\ \mathbf{p}_{n_-} &= -\boldsymbol{\beta}_n^T F_{0,n} - \mathbf{k}_n \mathbf{u}_{n_+} + \mathbf{k}_n \mathbf{u}_{n_-} \end{aligned} \tag{8–6}$$

We refer to $\mathbf{k}_n$ as the bar stiffness matrix. Equation (8–6) defines the joint forces required for bar n. The total joint forces required are obtained by summing over the *bars*.

We have defined

$$\begin{aligned} \mathscr{P} &= \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_j\} \qquad (ij \times 1) \\ \mathscr{U} &= \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\} \qquad (ij \times 1) \end{aligned}$$

as the general external joint force and joint displacement matrices. Now, we write the complete system of ij joint force-equilibrium equations, expressed in terms of the displacements, as

$$\mathscr{P} = \mathscr{K}\mathscr{U} + \mathscr{P}_0 \tag{8–7}$$

We refer to $\mathscr{K}$, which is of order $ij \times ij$, as the *unrestrained* system stiffness matrix. The elements of $\mathscr{P}_0$ are the required joint forces due to the initial elongations and $\mathscr{K}\mathscr{U}$ represents the required joint forces due to the joint displacements.

We assemble $\mathscr{K}$ and $\mathscr{P}_0$ in partitioned form, working with successive members. The contributions for member n follow directly from (8–6).

$\mathscr{P}_0$ (Partitioned Form Is $j \times 1$)

$+F_{0,n}\boldsymbol{\beta}_n^T$	in row n_+	(8–8)
$-F_{0,n}\boldsymbol{\beta}_n^T$	in row n_-	

$\mathscr{K}$ (Partitioned Form Is $j \times j$)

$+\mathbf{k}_n$	in row n_+, column n_+	(8–9)
$-\mathbf{k}_n$	in row n_+, column n_-	
$-\mathbf{k}_n$	in row n_-, column n_+	
$+\mathbf{k}_n$	in row n_-, column n_-	

Example 8–1

The connectivity table and general form of $\mathscr{K}$ and $\mathscr{P}_0$ for the numbering shown in Fig. E8–1 are presented below:

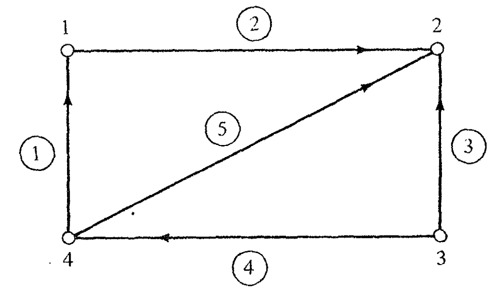


Fig. E8–1

Bar	1	2	3	4	5
+joint	1	2	2	4	2
−joint	4	1	3	3	4

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$
$\mathbf{p}_1$	$\mathbf{k}_1 + \mathbf{k}_2$	$-\mathbf{k}_2$		$-\mathbf{k}_1$
$\mathbf{p}_2$	$-\mathbf{k}_2$	$\mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_5$	$-\mathbf{k}_3$	$-\mathbf{k}_5$
$\mathbf{p}_3$		$-\mathbf{k}_3$	$\mathbf{k}_3 + \mathbf{k}_4$	$-\mathbf{k}_4$
$\mathbf{p}_4$	$-\mathbf{k}_1$	$-\mathbf{k}_5$	$-\mathbf{k}_4$	$\mathbf{k}_1 + \mathbf{k}_4 + \mathbf{k}_5$

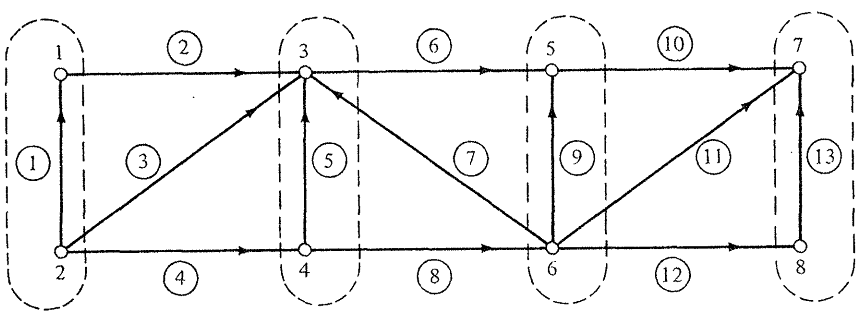
$$\mathscr{P}_0 = \begin{Bmatrix} \mathbf{p}_{0,1} \\ \mathbf{p}_{0,2} \\ \mathbf{p}_{0,3} \\ \mathbf{p}_{0,4} \end{Bmatrix} = \begin{Bmatrix} F_{0,1}\boldsymbol{\beta}_1^T - F_{0,2}\boldsymbol{\beta}_2^T \\ F_{0,2}\boldsymbol{\beta}_2^T + F_{0,3}\boldsymbol{\beta}_3^T + F_{0,5}\boldsymbol{\beta}_5^T \\ -F_{0,3}\boldsymbol{\beta}_3^T - F_{0,4}\boldsymbol{\beta}_4^T \\ -F_{0,1}\boldsymbol{\beta}_1^T + F_{0,4}\boldsymbol{\beta}_4^T - F_{0,5}\boldsymbol{\beta}_5^T \end{Bmatrix}$$

Example 8–2

The external force matrix, $\mathbf{p}_j$, involves $\mathbf{u}_j$ and the displacement matrices for those joints connected to joint j by bars. Now, $\mathbf{p}_j$ corresponds to row j and $\mathbf{u}_j$ to column j of $\mathscr{K}$. By suitably numbering the *joints*, one can restrict the finite elements of $\mathscr{K}$ to a zone about the diagonal. This is quite desirable from a computational point of view.

Fig. E8–2

Sect. 1



Consider the structure shown. We group the vertical joints into sections. The equilibrium equations for section k involve only the joints in section k and the adjacent sections. For example, the equations for section 3 (which correspond to $\mathbf{p}_5$, $\mathbf{p}_6$) will involve only the displacement matrices for sections 2, 3, 4. This suggests that we number the joints by *section*. The unpartitioned stiffness matrix corresponding to the above numbering scheme

is listed below. Note that $\mathscr{K}$ has the form of a quasi-tridiagonal band matrix when it is partitioned according to *sections* rather than individual joints. The submatrices for this truss are of order 4 × 4.

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$	$\mathbf{u}_6$	$\mathbf{u}_7$	$\mathbf{u}_8$
$\mathbf{p}_1$	$\mathbf{k}_1+\mathbf{k}_2$	$-\mathbf{k}_1$	$-\mathbf{k}_2$					
$\mathbf{p}_2$	$-\mathbf{k}_1$	$\mathbf{k}_1+\mathbf{k}_3+\mathbf{k}_4$	$-\mathbf{k}_3$	$-\mathbf{k}_4$				
$\mathbf{p}_3$	$-\mathbf{k}_2$	$-\mathbf{k}_3$	$\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_5+\mathbf{k}_7+\mathbf{k}_6$	$-\mathbf{k}_5$	$-\mathbf{k}_6$	$-\mathbf{k}_7$		
$\mathbf{p}_4$		$-\mathbf{k}_4$	$-\mathbf{k}_5$	$\mathbf{k}_4+\mathbf{k}_5+\mathbf{k}_8$		$-\mathbf{k}_8$		
$\mathbf{p}_5$			$-\mathbf{k}_6$		$\mathbf{k}_6+\mathbf{k}_9+\mathbf{k}_{10}$	$-\mathbf{k}_9$	$-\mathbf{k}_{10}$	
$\mathbf{p}_6$			$-\mathbf{k}_7$	$-\mathbf{k}_8$	$-\mathbf{k}_9$	$\mathbf{k}_8+\mathbf{k}_7+\mathbf{k}_9+\mathbf{k}_{11}+\mathbf{k}_{12}$	$-\mathbf{k}_{11}$	$-\mathbf{k}_{12}$
$\mathbf{p}_7$					$-\mathbf{k}_{10}$	$-\mathbf{k}_{11}$	$\mathbf{k}_{10}+\mathbf{k}_{11}+\mathbf{k}_{13}$	$-\mathbf{k}_{13}$
$\mathbf{p}_8$						$-\mathbf{k}_{12}$	$-\mathbf{k}_{13}$	$\mathbf{k}_{12}+\mathbf{k}_{13}$

The introduction of displacement restraints involves first transforming the partitioned elements of $\mathscr{P}_0$ and $\mathscr{K}$ to local frames associated with the restraints, permuting the actual rows, and finally partitioning the actual rows. The steps are indicated below.

$$\mathscr{P} \to \mathscr{P}^J \to \mathbf{P} \to \begin{Bmatrix} \mathbf{P}_1 \\ \hline \mathbf{P}_2 \end{Bmatrix}$$

$$\mathscr{U} \to \mathscr{U}^J \to \mathbf{U} \to \begin{Bmatrix} \mathbf{U}_1 \\ \hline \mathbf{U}_2 \end{Bmatrix}$$

We write the system of joint force-equilibrium equations referred to the local joint frames as

$$\mathscr{P}^J = \mathscr{K}^J\mathscr{U}^J + \mathscr{P}_0^J \qquad (8\text{–}10)$$

The transformation laws for the submatrices of $\mathscr{K}$ and $\mathscr{P}_0$ follow from (6–57).

$$\begin{aligned} \mathscr{P}_{0,n}^n &= \mathbf{R}^{0n}\mathscr{P}_{0,n} \\ \mathscr{K}_{\ell n}^J &= \mathbf{R}^{0\ell}\mathscr{K}_{\ell n}\mathbf{R}^{0n,T} \\ \ell, n &= 1, 2, \ldots, j \end{aligned} \qquad (8\text{–}11)$$

The step, $\mathscr{P}^J \rightarrow \mathbf{P}$, involves only a rearrangement of the rows of $\mathscr{P}^J$. We obtain the corresponding stiffness matrix, $\mathbf{K}$, by performing the *same* operations on both the *rows* and *columns* of $\mathscr{K}^J$. The rearranged system of equations is written as

$$\mathbf{P} = \mathbf{K}\mathbf{U} + \mathbf{P}_0 \tag{8-12}$$

Finally, we express (8–12) in partitioned form:

$$\begin{aligned} \bar{\mathbf{P}}_1 &= \mathbf{K}_{11}\mathbf{U}_1 + \mathbf{K}_{12}\bar{\mathbf{U}}_2 + \mathbf{P}_{0,1} \\ \mathbf{P}_2 &= \mathbf{K}_{21}\mathbf{U}_1 + \mathbf{K}_{22}\bar{\mathbf{U}}_2 + \mathbf{P}_{0,2} \end{aligned} \tag{8-13}$$

The first equation in (8–13) is identical to (8–1).

Example 8–3

It is of interest to express the partitioned elements of $\mathbf{K}$ in terms of the geometrical, connectivity, and displacement transformation matrices. We start with the general unpartitioned equations (6–28), (6–40), and (6–44), (6–50):

$$\mathscr{P} = \mathscr{B}\mathbf{F} = \mathbf{C}^T\boldsymbol{\beta}^T\mathbf{F} \tag{a}$$

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{k}\mathscr{A}\mathscr{U} = \mathbf{F}_0 + \mathbf{k}\boldsymbol{\gamma}\mathbf{C}\mathscr{U} \tag{b}$$

Then, substituting for $\mathbf{F}$ in (a) and equating the result to (8–7) leads to

$$\mathscr{K} = \mathbf{C}^T(\boldsymbol{\beta}^T\mathbf{k}\boldsymbol{\gamma})\mathbf{C} \tag{c}$$

$$\mathscr{P}_0 = \mathbf{C}^T(\boldsymbol{\beta}^T\mathbf{F}_0) \tag{d}$$

The matrix, $\boldsymbol{\beta}^T\mathbf{k}\boldsymbol{\gamma}$, is a quasi-diagonal matrix of order im. The diagonal submatrices are of order i, and the submatrix at location n has the form, $k_n\boldsymbol{\beta}_n^T\boldsymbol{\gamma}_n$. We have defined this product as $\mathbf{k}_n$. Then, if we let

$$\mathbf{k}_B = \begin{bmatrix} \mathbf{k}_1 & & & \\ & \mathbf{k}_2 & & \\ & & \ddots & \\ & & & \mathbf{k}_m \end{bmatrix} = \begin{bmatrix} k_1\boldsymbol{\beta}_1^T\boldsymbol{\gamma}_1 & & & \\ & k_2\boldsymbol{\beta}_2^T\boldsymbol{\gamma}_2 & & \\ & & \ddots & \\ & & & k_m\boldsymbol{\beta}_m^T\boldsymbol{\gamma}_m \end{bmatrix} \tag{e}$$

we can express $\mathscr{K}$ as

$$\mathscr{K} = \mathbf{C}^T\mathbf{k}_B\mathbf{C} \tag{f}$$

Carrying out (8–9) for $n = 1, 2, \ldots, m$ is the same as evaluating the triple matrix product. Obviously, (8–9) is more efficient than (f).

The introduction of displacement restraints can be represented as

$$\begin{aligned} \mathbf{P} &= \mathbf{D}\mathscr{P} \\ &\Downarrow \\ \bar{\mathbf{P}}_1 &= \mathbf{D}_1\mathscr{P} \\ \mathbf{P}_2 &= \mathbf{D}_2\mathscr{P} \end{aligned} \tag{g}$$

and

$$\mathscr{U} = \mathbf{D}^T\mathbf{U} = \mathbf{D}_1^T\mathbf{U}_1 + \mathbf{D}_2^T\bar{\mathbf{U}}_2 \tag{h}$$

Substituting (g) and (h) in (8–7) and equating the result to (8–13), we obtain

$$\begin{aligned} \mathbf{K}_{st} &= \mathbf{D}_s\mathscr{K}\mathbf{D}_t^T = \mathbf{D}_s\mathbf{C}^T\boldsymbol{\beta}^T\mathbf{k}\boldsymbol{\gamma}\mathbf{C}\mathbf{D}_t^T \\ \mathbf{P}_{0,s} &= \mathbf{D}_s\mathscr{P}_0 = \mathbf{D}_s\mathbf{C}^T\boldsymbol{\beta}\mathbf{k}\mathbf{e}_0 \end{aligned} \qquad s, t = 1, 2 \tag{i}$$

In order to obtain (8–13), we must rearrange the rows and columns of $\mathscr{K}$ and then partition. This operation is quite time-consuming. Also, it leads to rectangular submatrices. In what follows, we describe a procedure for introducing displacement restraints which avoids these difficulties.

We start with the complete system of equations referred to the basic frame,

$$\mathscr{K}\mathscr{U} = \mathscr{P} - \mathscr{P}_0 = \mathscr{P}_N \tag{8-14}$$

We assemble $-\mathscr{P}_0$ and $\mathscr{K}$, using (8–8) and (8–9). Then, we add to $-\mathscr{P}_0$ the *external* force matrices for those joints which are unrestrained. It remains to modify the rows and columns corresponding to joints which are either fully or partially restrained.

Case A: Full Restraint

Suppose $\mathbf{u}_q = \bar{\mathbf{u}}_q$. Then $\mathbf{p}_q$ is unknown. We replace the equation for $\mathbf{p}_q$ by

$$\mathbf{u}_q = \bar{\mathbf{u}}_q$$

This involves the following operations on the submatrices of $\mathscr{K}$ and $\mathscr{P}_N$.

1. On $\mathscr{K}$. Set off diagonal matrix elements in row q and column q equal to $\mathbf{0}$ and the diagonal matrix element equal to $\mathbf{I}_i$.

$$\begin{aligned} \mathscr{K}_{q\ell} &= \mathbf{0} \\ \mathscr{K}_{\ell q} &= \mathbf{0} \\ \mathscr{K}_{qq} &= \mathbf{I}_i \end{aligned} \qquad \begin{aligned} \ell &\neq q \\ \ell &= 1, 2, \ldots, j \end{aligned} \tag{8-15}$$

2. On $\mathscr{P}_N$. Add terms in $\mathscr{P}_N$ due to $\bar{u}_q$:

$$\begin{aligned} \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - \mathscr{K}_{\ell q}\bar{\mathbf{u}}_q \\ \mathscr{P}_{N,q} &= \bar{\mathbf{u}}_q \\ \ell \neq q \qquad & \ell = 1, 2, \ldots, j \end{aligned} \tag{8-16}$$

Case B: Partial Restraint—Local Frame

We suppose the rth element in $\mathbf{u}_q^q$ is prescribed.

$$\begin{aligned} u_{qr}^q &= \bar{u}_{qr}^q = \text{prescribed} \\ p_{qr}^q &= \text{unknown} \end{aligned}$$

We have to delete the equation corresponding to p^q_{qr} and replace it with

$$u^q_{qr} = -\bar{u}^q_{qr}$$

Step 1—Assemblage of Basic Matrices

We assemble $\mathbf{E}_q$, $\mathbf{G}_q$, $\mathbf{u}^*_q$, $\mathbf{p}^*_q$ according to the following:

1. $\mathbf{E}_q$ and $\mathbf{G}_q$. We start with

$$\mathbf{E} = \mathbf{I}_i \qquad \mathbf{G} = \mathbf{0}_i$$

and we set

$$E_{rr} = 0 \qquad G_{rr} = +1$$

2. $\mathbf{u}^*_q$. We start with an ith-order column vector having zero elements and we set the element in the rth row equal to $\bar{\mathbf{u}}^q_{qr}$. Note that this matrix involves only the *prescribed displacements* (local frame) in their natural locations.

3. $\mathbf{p}^*_q$. We start with an ith-order column vector having zero elements and we insert the values of the *prescribed* joint forces (local frame) in their natural locations. Note that the elements corresponding to the reactions are zero.

When the joint is fully restrained,

$$\mathbf{E} = \mathbf{0}_i \qquad \mathbf{G} = \mathbf{I}_i$$
$$\mathbf{u}^*_q = \mathbf{u}_q \qquad \mathbf{p}^*_q = \mathbf{0}_i$$

Suppose joint 5 is partially restrained. The data consist of:

(a) The rotation matrix, $\mathbf{R}^{05}$, defining the direction of the local frame at 5 with respect to the basic frame.
(b) The direction (or directions) of the displacement restraint and the value (or values) of the prescribed displacement.

$$\text{direction } r, \bar{u}^5_{5r}$$

(c) The values of the prescribed joint forces:

$$\bar{p}^5_{5j} \qquad j \neq r \qquad j = 1, \ldots, i$$

As an illustration, suppose $r = 2$. Then, in (b), we read in

$$r = 2 \qquad \bar{u}^5_{52}$$

In (c), we read in

$$\bar{p}^5_{51} \qquad \bar{p}^5_{53}$$

The four basic matrices are (for $r = 2$)

$$\mathbf{E}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{G}_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{u}^*_5 = \begin{Bmatrix} 0 \\ \bar{u}^5_{52} \\ 0 \end{Bmatrix} \qquad \mathbf{p}^*_5 = \begin{Bmatrix} \bar{p}^5_{51} \\ 0 \\ \bar{p}^5_{53} \end{Bmatrix}$$

Step 2—Operation on $\mathscr{K}$ and $\mathscr{P}_N$

1. Premultiply row q of $\mathscr{K}$ and $\mathscr{P}_N$ by $\mathbf{E}_q\mathbf{R}^{0q}$.

$$\begin{aligned} \mathscr{K}_{q\ell} &= \mathbf{E}_q\mathbf{R}^{0q}\mathscr{K}_{q\ell} \\ \mathscr{P}_{N,q} &= \mathbf{E}_q\mathbf{R}^{0q}\mathscr{P}_{N,q} \end{aligned} \qquad \ell = 1, 2, \ldots, j \tag{a}$$

2. Postmultiply column q of $\mathscr{K}$ by $-\mathbf{R}^{0q,T}\mathbf{u}^*_q$ and add to $\mathscr{P}_N$.

$$\begin{aligned} \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - \mathscr{K}_{\ell q}\mathbf{R}^{0q,T}\mathbf{u}^*_q \\ \ell &= 1, 2, \ldots, j \end{aligned} \tag{b}$$

3. Postmultiply column q of $\mathscr{K}$ by $(\mathbf{E}_q\mathbf{R}^{0q})^T$

$$\mathscr{K}_{\ell q} = \mathscr{K}_{\ell q}(\mathbf{E}_q\mathbf{R}^{0q})^T \qquad \ell = 1, 2, \ldots, j \tag{c}$$

4. Add $\mathbf{G}_q$ to $\mathscr{K}_{qq}$

$$\mathscr{K}_{qq} = \mathscr{K}_{qq} + \mathbf{G}_q \tag{d}$$

5. Add $+\mathbf{p}^*_q$ and $+\mathbf{u}^*_q$ to $\mathscr{P}_{N,q}$

$$\mathscr{P}_{N,q} = \mathscr{P}_{N,q} + \mathbf{u}^*_q + \mathbf{p}^*_q \tag{e}$$

The operation on row q and column q are summarized below.

On $\mathscr{K}$

$$\begin{aligned} \mathscr{K}_{q\ell} &= \mathbf{E}_q\mathbf{R}^{0q}\mathscr{K}_{q\ell} \\ \mathscr{K}_{\ell q} &= \mathscr{K}_{\ell q}(\mathbf{E}_q\mathbf{R}^{0q})^T \\ \mathscr{K}_{qq} &= (\mathbf{E}_q\mathbf{R}^{0q})\mathscr{K}_{qq}(\mathbf{E}_q\mathbf{R}^{0q})^T + \mathbf{G}_q \end{aligned} \qquad \begin{aligned} \ell &\neq q \\ \ell &= 1, 2, \ldots, j \end{aligned} \tag{8–17}$$

On $\mathscr{P}_N$

$$\begin{aligned} \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - \mathscr{K}_{\ell q}\mathbf{R}^{0q,T}\mathbf{u}^*_q \\ \ell &\neq q \qquad \ell = 1, 2, \ldots, j \\ \mathscr{P}_{N,q} &= (\mathbf{E}_q\mathbf{R}^{0q})\mathscr{P}_{N,q} - (\mathbf{E}_q\mathbf{R}^{0q})\mathscr{K}_{qq}\mathbf{R}^{0q,T}\mathbf{u}^*_q + \mathbf{p}^*_q + \mathbf{u}^*_q \end{aligned} \tag{8–18}$$

When $\mathscr{K}$ is symmetrical (this will be the case when the system is geometrically linear), we can work only with the submatrices *on* and *above* the diagonal. The contracted operations for the symmetrical case are threefold:

$$\begin{aligned} \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - \mathscr{K}_{\ell q}\mathbf{R}^{0q,T}\mathbf{u}^*_q \\ \mathscr{K}_{\ell q} &= \mathscr{K}_{\ell q}(\mathbf{E}_q\mathbf{R}^{0q})^T \\ \ell &= 1, 2, \ldots, q-1 \end{aligned} \tag{8–19}$$

$$\begin{aligned}\mathscr{P}_{N,q} &= (\mathbf{E}_q\mathbf{R}^{0q})\mathscr{P}_{N,q} - (\mathbf{E}_q\mathbf{R}^{0q})\mathscr{K}_{qq}\mathbf{R}^{0q,T}\mathbf{u}_q^* + \mathbf{p}_q^* + \mathbf{u}_q^* \\ \mathscr{K}_{qq} &= (\mathbf{E}_q\mathbf{R}^{0q})\mathscr{K}_{qq}(\mathbf{E}_q\mathbf{R}^{0q})^T + \mathbf{G}_q\end{aligned} \qquad (8\text{–}20)$$

$$\begin{aligned}\mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - \mathscr{K}_{q\ell}^T\mathbf{R}^{0q,T}\mathbf{u}_q^* \\ \mathscr{K}_{q\ell} &= (\mathbf{E}_q\mathbf{R}^{0q})\mathscr{K}_{q\ell} \\ \ell &= q+1, q+2, \ldots, j\end{aligned} \qquad (8\text{–}21)$$

The operations outlined above are carried out for each restrained joint. Note that the modifications for joint q involve only row q and column q. We denote the modified system of equations by

$$\mathscr{K}^*\mathscr{U}^J = \mathscr{P}_N^* \qquad (8\text{–}22)$$

Equation (8–22) represents ij equations. The coefficient matrix $\mathscr{K}^*$ will be *nonsingular* when $\mathbf{K}_{11}$ is nonsingular. To show this, we start with the first equation in (8–13) and an additional set of r dummy equations:

$$\begin{bmatrix}\mathbf{K}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r\end{bmatrix}\begin{Bmatrix}\mathbf{U}_1 \\ \mathbf{U}_2\end{Bmatrix} = \begin{Bmatrix}\bar{\mathbf{P}}_1 \\ \mathbf{0}\end{Bmatrix} + \begin{Bmatrix}-P_{0,1} - \mathbf{K}_{12}\bar{\mathbf{U}}_2 \\ \bar{\mathbf{U}}_2\end{Bmatrix} = \mathbf{P}_N^* \qquad (a)$$

Equation (a) represents ij equations. This system is transformed to (8–22) when we permute $\mathbf{U}$, $\mathbf{P}_N$ to $\mathscr{U}^J$, $\mathscr{P}_N^J$. They are related by (see (6–63))

$$\begin{aligned}\mathbf{U} &= \mathbf{\Pi}\mathscr{U}^J \\ \mathscr{P}^J &= \mathbf{\Pi}^T\mathbf{P}\end{aligned} \qquad (b)$$

where $\mathbf{\Pi}$ is a permutation matrix. It follows that

$$\begin{aligned}\mathscr{K}^* &= \mathbf{\Pi}^T\begin{bmatrix}\mathbf{K}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_R\end{bmatrix}\mathbf{\Pi} \\ \mathscr{P}_N^* &= \mathbf{\Pi}^T\mathbf{P}_N^*\end{aligned} \qquad (c)$$

and, since $\mathbf{\Pi}$ is an orthogonal matrix,

$$|\mathscr{K}^*| = |\mathbf{K}_{11}| \qquad (8\text{–}23)$$

It is more convenient to work with (8–22) rather than (a) since the solution of (8–22) yields the joint displacement matrices listed in their natural order, that is, according to increasing joint number. Once $\mathscr{U}^J$ is known, we convert the joint displacement matrices to the basic frame, using

$$\mathbf{u}_q = \mathbf{R}^{0q,T}\mathbf{u}_q^q$$

The bar forces are determined from

$$F_n = F_{0,n} + k_n\gamma_n(\mathbf{u}_{n+} - \mathbf{u}_{n-})$$

Next, we calculate $F_n\,\boldsymbol{\beta}_n^T$ and assemble $\mathscr{P}$ in partitioned form by summing the

contribution for each member. For member n, we put (see (8–4))

$$\begin{aligned}&+F_n\boldsymbol{\beta}_n^T \quad \text{in row } n_+ \\ &-F_n\boldsymbol{\beta}_n^T \quad \text{in row } n_-\end{aligned}$$

Once $\mathscr{P}$ is known, we convert the force matrix for each partially restrained joint to the local joint reference frame, using

$$\mathbf{p}_q^q = \mathbf{R}^{0q}\mathbf{p}_q$$

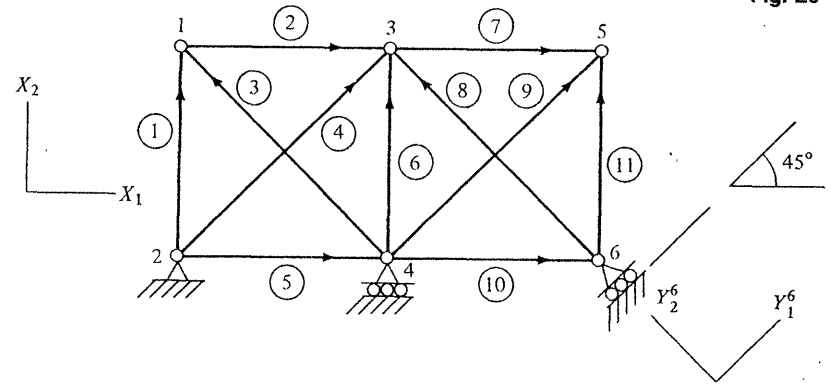
The final result is $\mathscr{P}^J$ required to equilibrate the bar forces. This operation provides a *static* check on the solution in addition to furnishing the *reactions*.

When the problem is geometrically nonlinear, γ_n, $\boldsymbol{\beta}_n$, and $\mathbf{k}_n$ depend on the joint displacements. In this case, it is generally more efficient to apply an incremental formulation rather than iterate on (8–22).

Example 8–4

We illustrate these operations for the truss shown in Fig. E8–4.

Fig. E8–4



1. ***Member-Joint Connectivity Table***

Bar (n)	1	2	3	4	5	6	7	8	9	10	11
+joint (n_+)	1	3	1	3	4	3	5	3	5	6	5
−joint (n_-)	2	1	4	2	2	4	3	6	4	4	6

2. ***Assemblage of*** $\mathscr{K}$

We consider the geometry to be linear. Then, $\boldsymbol{\beta}_n = \boldsymbol{\alpha}_n$ and $\mathbf{k}_n = k_n\boldsymbol{\alpha}_n^T\boldsymbol{\alpha}_n$. Applying (8–9) results in $\mathscr{K}$ listed below.

	1	2	3	4	5	6
1	$k_1+k_2+k_3$	$-k_1$	$-k_2$	$-k_3$	$\mathbf{0}$	
2	$-k_1$	$k_1+k_4+k_5$	$-k_4$	$-k_5$		
3	$-k_2$	$-k_4$	$k_2+k_4+k_6$ $+k_7+k_8$	$-k_6$	$-k_7$	$-k_8$
4	$-k_3$	$-k_5$	$-k_6$	$k_3+k_5+k_6$ $+k_9+k_{10}$	$-k_9$	$-k_{10}$
5	$\mathbf{0}$		$-k_7$	$-k_9$	$+k_7+k_9$ $+k_{11}$	$-k_{11}$
6			$-k_8$	$-k_{10}$	$-k_{11}$	$+k_8+k_{10}$ $+k_{11}$

Note that $\mathscr{K}$ is symmetrical and quasi-tridiagonal, with submatrices of order 4 × 4.

3. *Introduction of Joint Displacement Restraints*

The original equations are

$$\mathscr{K}\mathscr{U} = \mathscr{P} - \mathscr{P}_0 = \mathscr{P}_N$$

where $\mathscr{P}$ contains the *external* joint forces. We start with $\mathscr{P}_N = -\mathscr{P}_0$. If joint q is unrestrained, we put $\mathbf{p}_q$ in row q of $\mathscr{P}_N$. If joint q is fully restrained, we modify $\mathscr{K}$ and $\mathscr{P}_N$ according to (8–15) and (8–16). Finally, if joint q is partially restrained, we use (8–19) through (8–21). Since $\mathscr{K}$ is symmetrical, we have to list only the submatrices on and above the diagonal. It is convenient to work with successive joint numbers. For this system, joint 2 is fully restrained and joints 4, 6 are partially restrained. The basic matrices for joints 4, 6 and the initial and final forms of $\mathscr{K}$, $\mathscr{P}_N$ are listed below. Note that this procedure does not destroy the banding of the stiffness matrix.

Joint 4 (u_{42} is prescribed)

$$\mathbf{R}^{04} = \mathbf{I}_2$$

$$\mathbf{E}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{G}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{u}_4^* = \{0, \bar{u}_{42}\} \qquad \mathbf{p}_4^* = \{\bar{p}_{41}, 0\}$$

Joint 6 (u_{62}^6 is prescribed)

$$\mathbf{R}^{06} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{E}_6 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{G}_6 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{u}_6^* = \{0, \bar{u}_{62}^6\} \qquad \mathbf{p}_6^* = \{\bar{p}_{61}^6, 0\}$$

Initial matrices ($\mathscr{K}$ and $\mathscr{P}_N = -\mathscr{P}_0$)

$(\mathbf{u}_1)$	$(\mathbf{u}_2)$	$(\mathbf{u}_3)$	$(\mathbf{u}_4)$	$(\mathbf{u}_5)$	$(\mathbf{u}_6)$	
$\mathscr{K}_{11}$	$\mathscr{K}_{12}$	$\mathscr{K}_{13}$	$\mathscr{K}_{14}$	$\mathbf{0}$		$-\mathscr{P}_{0,1}$
	$\mathscr{K}_{22}$	$\mathscr{K}_{23}$	$\mathscr{K}_{24}$			$-\mathscr{P}_{0,2}$
		$\mathscr{K}_{33}$	$\mathscr{K}_{34}$	$\mathscr{K}_{35}$	$\mathscr{K}_{36}$	$-\mathscr{P}_{0,3}$
			$\mathscr{K}_{44}$	$\mathscr{K}_{45}$	$\mathscr{K}_{46}$	$-\mathscr{P}_{0,4}$
Sym				$\mathscr{K}_{55}$	$\mathscr{K}_{56}$	$-\mathscr{P}_{0,5}$
					$\mathscr{K}_{66}$	$-\mathscr{P}_{0,6}$

Final matrices ($\mathscr{K}^*$ and $\mathscr{P}_N^*$)

$(\mathbf{u}_1)$	$(\mathbf{u}_2)$	$(\mathbf{u}_3)$	$(\mathbf{u}_4)$	$(\mathbf{u}_5)$	$(\mathbf{u}_6^6)$	
$\mathscr{K}_{11}$	$\mathbf{0}$	$\mathscr{K}_{13}$	$\mathscr{K}_{14}\mathbf{E}_4$	$\mathbf{0}$		$-\mathscr{P}_{0,1}+\mathbf{p}_1-\mathscr{K}_{12}\bar{\mathbf{u}}_2-\mathscr{K}_{14}\mathbf{u}_4^*$
	$\mathbf{I}_2$	$\mathbf{0}$	$\mathbf{0}$			$\bar{\mathbf{u}}_2$
		$\mathscr{K}_{33}$	$\mathscr{K}_{34}\mathbf{E}_4$	$\mathscr{K}_{35}$	$\mathscr{K}_{36}(\mathbf{E}_6\mathbf{R}^{06})^T$	$-\mathscr{P}_{0,3}-\mathscr{K}_{23}^T\bar{\mathbf{u}}_2-\mathscr{K}_{34}\mathbf{u}_4^*$ $-\mathscr{K}_{36}\mathbf{R}^{06,T}\mathbf{u}_6^*+\mathbf{p}_3$
			$\mathbf{E}_4\mathscr{K}_{44}\mathbf{E}_4$ $+\mathbf{G}_4$	$\mathbf{E}_4\mathscr{K}_{45}$	$\mathbf{E}_4\mathscr{K}_{46}(\mathbf{E}_6\mathbf{R}^{06})^T$	$\mathbf{E}_4(-\mathscr{P}_{0,4}-\mathscr{K}_{24}^T\bar{\mathbf{u}}_2)-\mathbf{E}_4\mathscr{K}_{44}\mathbf{u}_4^*$ $+\mathbf{p}_4^*+\mathbf{u}_4^*-\mathbf{E}_4\mathscr{K}_{46}\mathbf{R}^{06,T}\mathbf{u}_6^*$
Sym.				$\mathscr{K}_{55}$	$\mathscr{K}_{56}(\mathbf{E}_6\mathbf{R}^{06})^T$	$-\mathscr{P}_{0,5}-\mathscr{K}_{45}^T\mathbf{u}_4^*$ $+\mathbf{p}_5-\mathscr{K}_{56}\mathbf{R}^{06,T}\mathbf{u}_6^*$
					$(\mathbf{E}_6\mathbf{R}^{06})\mathscr{K}_{66}(\mathbf{E}_6\mathbf{R}^{06})^T$ $+\mathbf{G}_6$	$\mathbf{E}_6\mathbf{R}^{06}(-\mathscr{P}_{0,6}-\mathscr{K}_{46}^T\mathbf{u}_4^*$ $-\mathscr{K}_{66}\mathbf{R}^{06,T}\mathbf{u}_6^*)+\mathbf{p}_6^*+\mathbf{u}_6^*$

8–4. INCREMENTAL FORMULATION; CLASSICAL STABILITY CRITERION

Equations (8–13), (8–22) are valid for both linear and nonlinear behavior. However, it is more efficient with respect to computational effort to employ an incremental formulation when the system is nonlinear. With an incremental formulation, one applies the load in increments and determines the corresponding *incremental* displacements. The total displacement is obtained by summing the displacement increments. An incremental loading procedure can also be used with (8–13) but, in this case, one is working with total displacement rather than with incremental displacement. In this section, we develop a set of equations relating the incremental external load and the resulting incremental displacements. These equations are also nonlinear, but if one works with small load increments, the equations can be *linearized.* Our approach will be similar to that followed previously. We first establish incremental member force-displacement relations and then apply the direct stiffness method to generate the incremental system equations. We complete the section with a discussion of the classical stability criterion.

We start with (8–4), which defines the external joint forces required to equilibrate the action of the force for bar n,

$$\mathbf{p}_{n_+} = \mathbf{F}_n \boldsymbol{\beta}_n^T \qquad \mathbf{p}_{n_-} = -\mathbf{p}_{n_+} \tag{a}$$

Equations (a) are satisfied at an equilibrium position. We suppose an incremental external load $\Delta\mathbf{P}$ is applied and define $\Delta\mathbf{U}$ as the resulting incremental displacement for the new equilibrium position. Since F and $\boldsymbol{\beta}$ depend on $\mathbf{U}$, their values will change. Letting $\Delta\mathbf{F}$, $\Delta\boldsymbol{\beta}$ be the total increments in F, $\boldsymbol{\beta}$ due to $\Delta\mathbf{U}$, and requiring (a) to be satisfied at both positions, leads to the following incremental force-equilibrium equations:

$$\begin{aligned}\Delta\mathbf{p}_{n_+} &= F_n\,\Delta\boldsymbol{\beta}_n^T + \Delta F_n \boldsymbol{\beta}_n^T + \Delta F_n\,\Delta\boldsymbol{\beta}_n^T \\ \Delta\mathbf{p}_{n_-} &= -\Delta\mathbf{p}_{n_+}\end{aligned} \tag{8–24}$$

To proceed further, we need to evaluate the increments in e and $\boldsymbol{\beta}$. The exact relations are given by (6–22):

$$\begin{aligned}e_n &= \boldsymbol{\alpha}_n(\mathbf{u}_{n_+} - \mathbf{u}_{n_-}) + \tfrac{1}{2}(\boldsymbol{\beta}_n - \boldsymbol{\alpha}_n)(\mathbf{u}_{n_+} - \mathbf{u}_{n_-}) \\ \boldsymbol{\beta}_n - \boldsymbol{\alpha}_n &= \frac{1}{L_n}(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})^T\end{aligned} \tag{a}$$

To allow for the possibility of retaining only certain nonlinear terms, we write (a) as

$$\begin{aligned}\beta_n - \boldsymbol{\alpha}_n &= (\mathbf{u}_{n_+} - \mathbf{u}_-)^T\mathbf{g}_n \\ e_n &= \boldsymbol{\alpha}_n(\mathbf{u}_{n_+} - \mathbf{u}_{n_-}) + \tfrac{1}{2}(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})^T\mathbf{g}_n(\mathbf{u}_{n_+} - \mathbf{u}_{n_-}) \\ &= \boldsymbol{\gamma}_n(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})\end{aligned} \tag{8–25}$$

If all the nonlinear terms are retained,

$$\mathbf{g}_n = \frac{1}{L_n}\mathbf{I}_i$$

To neglect a particular displacement component, we delete the corresponding element in $\mathbf{I}_i$. For geometrically linear behavior, $\mathbf{g}_n = \mathbf{0}$. Operating on (8–25), we obtain

$$\Delta\boldsymbol{\beta}_n \equiv d\boldsymbol{\beta}_n = (\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-})^T\mathbf{g}_n \tag{8–26}$$

and

$$\begin{aligned}\Delta e_n &= de_n + \tfrac{1}{2}d^2e_n \\ de_n &= \boldsymbol{\beta}_n(\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-}) \\ d^2e_n &= d\boldsymbol{\beta}_n(\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-})\end{aligned} \tag{8–27}$$

It remains to evaluate ΔF_n. We allow for a piecewise linear material and employ the relations† developed in Sec. 6–4. For convenience, we drop all the

† See (6–31), (6–32), and (6–33).

notation pertaining to a segment and write the "generalized" incremental expression in the simple form

$$\Delta F = k(\Delta e - \Delta e_0) \tag{8–28}$$

where k, Δe_0 are constant for a segment. They have to be changed if the limit of the segment is exceeded or the bar is unloading. Since Δe is unknown, one has to iterate, taking the values of k, Δe_0 corresponding to the initial equilibrium position as the first estimate. This is equivalent to using the tangent stiffness. The initial elongation, Δe_0, is included to allow for an incremental temperature change. Substituting for Δe, (8–28) takes the form

$$\begin{aligned}\Delta F_n &= \Delta F_{0,n} + k_n\,de_n + \tfrac{1}{2}k_n\,d^2e_n \\ \Delta F_{0,n} &= -k_n\,\Delta e_{0,n}\end{aligned} \tag{8–29}$$

Finally, we substitute for ΔF_n, $\Delta\boldsymbol{\beta}_n$ in (8–24) and group the terms as follows:

$$\begin{aligned}\Delta\mathbf{p}_{n_+} &= \mathbf{k}_{t,n}(\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-}) + \Delta\mathbf{p}_{0,n} + \Delta\mathbf{p}_{g,n} \\ \Delta\mathbf{p}_{n_-} &= -\Delta\mathbf{p}_{n_+}\end{aligned} \tag{8–30}$$

where

$$\begin{aligned}\mathbf{k}_{t,n} &= F_n\mathbf{g}_n + k_n\boldsymbol{\beta}_n^T\boldsymbol{\beta}_n \\ \Delta\mathbf{p}_{0,n} &= \Delta F_{0,n}\boldsymbol{\beta}_n^T \\ \Delta\mathbf{p}_{g,n} &= k_n(-\Delta e_{0,n}\,d\boldsymbol{\beta}_n^T + \tfrac{1}{2}d^2e_n\boldsymbol{\beta}_n^T + de_n\,d\boldsymbol{\beta}_n^T + \tfrac{1}{2}d^2e_n\,d\boldsymbol{\beta}_n^T)\end{aligned} \tag{8–31}$$

We interpret $\mathbf{k}_t$ as the tangent stiffness matrix. The vector, $\Delta\mathbf{p}_g$, contains linear, quadratic, and cubic terms in $\Delta\mathbf{u}$. We have included the subscript g to indicate that it is a nonlinear geometric term.

We write the total set of incremental joint equilibrium equations as

$$\Delta\mathscr{P} = \mathscr{K}_t\Delta\mathscr{U} + \Delta\mathscr{P}_0 + \Delta\mathscr{P}_g \tag{8–32}$$

where $\mathscr{K}_t$ is assembled using (8–9) and $\Delta\mathscr{P}_0 + \Delta\mathscr{P}_g$ with (8–8). Note that $\mathscr{K}_t$ is symmetrical. Finally, we introduce the displacement restraints by applying (8–19)–(8–21). The modified equations are

$$\mathscr{K}_t^*\,\Delta\mathscr{U}^J = \Delta\mathscr{P}^* - \Delta\mathscr{P}_0^* - \Delta\mathscr{P}_g^* \tag{8–33}$$

It is convenient to include the prescribed incremental support displacement terms in $\Delta\mathscr{P}^*$ so that $\Delta\mathscr{P}_0^*$ involves only the incremental temperature and $\Delta\mathscr{P}_g^*$ the variable displacement increments. The contracted equations are

$$\mathbf{K}_{t,11}\,\Delta\mathbf{U}_1 = \Delta\bar{\mathbf{P}}_1 - \Delta\mathbf{P}_{0,1} - \Delta\mathbf{P}_{g,1} - \mathbf{K}_{t,12}\,\Delta\bar{\mathbf{U}}_2 \tag{8–34}$$

where $\mathbf{K}_{t,11}$ is symmetrical.

We cannot solve (8–33) directly for $\Delta\mathscr{U}$ since $\Delta\mathscr{P}_g^*$ contains quadratic and cubic terms in $\Delta\mathscr{U}$. There are a number of techniques for solving nonlinear algebraic equations.† We describe here the method of successive substitutions,

† See Ref. 12.

which is the easiest to implement, but its convergence rate is slower in comparison to most of the other methods.†

First, we note that $\mathcal{K}_t^*$, $\Delta\mathcal{P}^*$, and $\Delta\mathcal{P}_0^*$ are independent of $\Delta\mathcal{U}$. They depend only on the initial equilibrium position and the incremental loading. We combine $\Delta\mathcal{P}^*$ and $\Delta\mathcal{P}_0^*$ and write (8–33) as

$$\mathcal{K}_t^* \,\Delta\mathcal{U}^J = \Delta\mathcal{P}_N^* - \Delta\mathcal{P}_g^* \tag{8–35}$$

Now, we let $\Delta\mathcal{U}^{(n)}$ represent the nth estimate for $\Delta\mathcal{U}^J$ and determine the $(n+1)$th estimate by solving

$$\mathcal{K}_t^* \,\Delta\mathcal{U}^{(n+1)} = \Delta\mathcal{P}_N^* - \Delta\mathcal{P}_g^*\big|_{\Delta\mathcal{U}^{(n)}} \tag{8–36}$$

The iteration involves only evaluation of $\Delta\mathcal{P}_g^*$ and back-substitution once $\mathcal{K}_t^*$ is transformed to a triangular matrix. The factor method is particularly convenient since $\mathcal{K}_t^*$ is symmetrical. With this method,

$$\mathcal{K}_t^* \Rightarrow \mathbf{S}^T\mathbf{S} \tag{8–37}$$

where $\mathbf{S}$ is an upper triangular matrix. We replace (8–36) with

$$\begin{aligned} \mathbf{S}\,\Delta\mathcal{U}^{(n+1)} &= \mathbf{Q} \\ \mathbf{S}^T\mathbf{Q} &= \Delta\mathcal{P}_N^* - \Delta\mathcal{P}_g^*\big|_{\Delta\mathcal{U}^{(n)}} \end{aligned} \tag{8–38}$$

In linearized incremental analysis, we delete $\Delta\mathcal{P}_g^*$ in (8–35) and take the solution of

$$\mathcal{K}_t^* \,\Delta\mathcal{U}^J = \Delta\mathcal{P}_N^* \tag{8–39}$$

as the "actual" displacement increment. One can interpret this scheme as one cycle successive substitution.

The solution degenerates when the tangent stiffness matrix becomes singular. To investigate the behavior in the neighborhood of this point, we apply the classical stability criterion developed in Sec. 7–6. The appropriate form for a truss is given by (7–41):

$$d^2W_D = \sum_{n=1}^{m} (F_n\, d^2e_n + dF_n\, de_n) > 0 \qquad \text{for arbitrary } \Delta\mathbf{U}_1 \text{ with } \Delta\mathbf{U}_2 = 0 \quad \text{(a)}$$

We have already evaluated the above terms. Using (8–26), (8–27), and (8–29) with $\Delta e_0 = 0$,

$$F_n\, d^2e_n + dF_n\, de_n \Rightarrow (\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-})^T\mathbf{k}_{t,\,n}(\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-}) \tag{b}$$

and (a) can be written as

$$d^2W_D = \Delta\mathbf{U}_1^T\mathbf{K}_{t,\,11}\,\Delta\mathbf{U}_1 > 0 \qquad \text{for arbitrary } \Delta\mathbf{U}_1 \tag{8–40}$$

It follows that $\mathbf{K}_{t,\,11}$ must be *positive definite* for a stable equilibrium position.

† Iterative techniques are discussed in greater detail in Secs. 18–7, 18–8, 18–9.

But $\mathbf{K}_{t,\,11}$ and $\mathcal{K}_t^*$ are related by

$$\mathcal{K}_t^* = \mathbf{\Pi}^T \begin{bmatrix} \mathbf{K}_{t,\,11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix} \mathbf{\Pi} \tag{8–41}$$

where $\mathbf{\Pi}$ is a nonsingular permutation matrix, which rearranges the elements of $\Delta\mathcal{U}^J$ according to

$$\Delta\mathbf{U} = \mathbf{\Pi}\,\Delta\mathcal{U}^J \tag{8–42}$$

Then, $\mathcal{K}_t^*$ and $\mathbf{K}_{t,\,11}$ have the *same* definiteness property.† Finally, we can classify the stability of an equilibrium position in terms of the determinant of the tangent stiffness matrix:‡

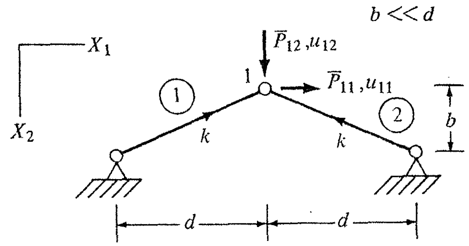
$$D = |\mathcal{K}_t^*| = |\mathbf{K}_{t,\,11}|$$

$$\begin{aligned} &\text{stable} && D > 0 \\ &\text{neutral} && D = 0 \\ &\text{unstable} && D < 0 \end{aligned} \tag{8–43}$$

Example 8–5

We illustrate the application of both the total (8–13) and incremental (8–34) formulations to the truss shown in Fig. E8-5A. To simplify the analysis, we suppose the material is linearly elastic, $k_1 = k_2 = k$, and there are no initial elongations or support movement.

Fig. E8–5A



The initial direction cosines for the bars are

$$\boldsymbol{\alpha}_1 = \frac{1}{L}[d \quad -b] \qquad \boldsymbol{\alpha}_2 = \frac{1}{L}[-d \quad -b] \tag{a}$$

The deformed geometric measures are defined by (8–25). They reduce to

$$\begin{aligned} \boldsymbol{\beta}_n &= \boldsymbol{\alpha}_n + \mathbf{u}_1^T\mathbf{g}_n \\ e_n &= \boldsymbol{\gamma}_n\mathbf{u}_1 \qquad\qquad n = 1, 2 \\ \boldsymbol{\gamma}_n &= \boldsymbol{\alpha}_n + \tfrac{1}{2}\mathbf{u}_1^T\mathbf{g}_n \end{aligned} \tag{b}$$

for this example. Since $b \ll d$, we can neglect the nonlinear terms due to u_{11}, i.e., we can take

$$\mathbf{g}_n \approx \frac{1}{L}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{c}$$

† See Prob. 2–13 for a proof.
‡ See Sec. 2–5.

Using (c),

$$\begin{aligned}
\gamma_1 &\approx \frac{1}{L}[d \mid -b + \tfrac{1}{2}u_{12}] \\
\boldsymbol{\beta}_1 &\approx \frac{1}{L}[d \mid -b + u_{12}] \\
\gamma_2 &\approx \frac{1}{L}[-d \mid -b + \tfrac{1}{2}u_{12}] \\
\boldsymbol{\beta}_2 &\approx \frac{1}{L}[-d \mid -b + u_{12}]
\end{aligned} \tag{d}$$

Continuing, the bar force-displacement relations are

$$F_n = ke_n = k\gamma_n\mathbf{u}_1 \qquad n = 1, 2 \tag{e}$$

Finally, the force-equilibrium equation for joint 1 follows by applying (8–6) to both bars.

$$\bar{\mathbf{p}}_1 = (\mathbf{k}_1 + \mathbf{k}_2)\mathbf{u}_1 = k(\boldsymbol{\beta}_1^T\gamma_1 + \boldsymbol{\beta}_2^T\gamma_2)\mathbf{u}_1 \tag{f}$$

Equations (e) and (f) expand to

$$\begin{Bmatrix} \bar{p}_{11} \\ \bar{p}_{12} \end{Bmatrix} = \left[\begin{array}{c|c} 2k\left(\dfrac{d}{L}\right)^2 & 0 \\ \hline 0 & \dfrac{2k}{L^2}(b - u_{12})(b - \tfrac{1}{2}u_{12}) \end{array}\right] \begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} \tag{g}$$

and

$$\begin{aligned}
F_1 &= \frac{k}{L}(du_{11} - (b - \tfrac{1}{2}u_{12})u_{12}) \\
F_2 &= \frac{k}{L}(-du_{11} - (b - \tfrac{1}{2}u_{12})u_{12})
\end{aligned} \tag{h}$$

The diagonal form of the coefficient matrix is due to the fact that we neglected u_{11} in the expressions for γ and $\boldsymbol{\beta}$. This approximation uncouples the equations. Note that (g) is the first equation in (8–13) with $\mathbf{U}_2$ and $\mathbf{P}_{0,1}$ set to $\mathbf{0}$.

Solving the first equation† in (g), we obtain

$$u_{11} = \frac{1}{2k}\left(\frac{L}{d}\right)^2 \bar{p}_{11} \tag{i}$$

The corresponding bar forces are

$$\begin{aligned}
F_1 &= \frac{1}{2}\left(\frac{L}{d}\right)\bar{p}_{11} \\
F_2 &= -F_1
\end{aligned} \tag{j}$$

This result is actually the solution for the linear geometric case.

The expression for u_{12} and the corresponding bar forces follow from the second equation in (g).

$$\bar{p}_{12} = \left[\frac{2k}{L^2}(b - u_{12})(b - \tfrac{1}{2}u_{12})\right]u_{12} \tag{k}$$

† Equation (g) is (8–1) with $F_i = 0$.

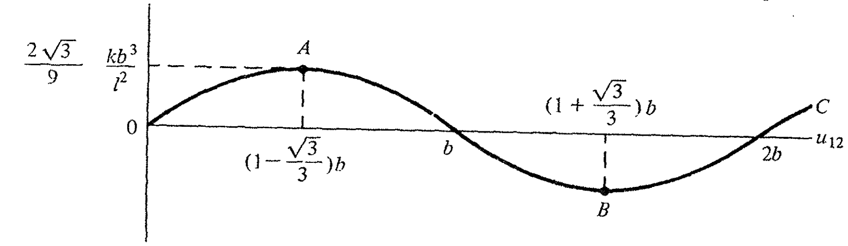
$$F_1 = F_2 = -\frac{k}{L}(b - \tfrac{1}{2}u_{12})u_{12} = -\frac{1}{2}\frac{\bar{p}_{12}}{\dfrac{b - u_{12}}{L}} \tag{l}$$

We can write (*k*) as

$$u_{12} = \frac{L^2}{2k}\frac{\bar{p}_{12}}{(b - u_{12})(b - \tfrac{1}{2}u_{12})} \tag{m}$$

and solve (m) by iteration. Alternatively, one can specify u_{12} and evaluate $\bar{p}_{12}$ from (k). The latter approach works only when there is one variable. The solution is plotted in Fig. E8–5B.

Fig. E8–5B



We describe next the generation of the incremental equations which follow from (8–26)–(8–32). Applying (8–26), (8–27) to (b)–(d) results in

$$\begin{aligned}
\Delta\boldsymbol{\beta}_n &= d\boldsymbol{\beta}_n = \Delta\mathbf{u}_1^T\mathbf{g}_n = \left[0 \quad \frac{\Delta u_{12}}{L}\right] \\
de_n &= \boldsymbol{\beta}_n\,\Delta\mathbf{u}_1 = \alpha_{n1}\,\Delta u_{11} + \frac{1}{L}(-b + u_{12})\Delta u_{12} \\
d^2e_n &= \Delta\mathbf{u}_1^T\mathbf{g}_n\,\Delta\mathbf{u}_1 = \frac{1}{L}(\Delta u_{12})^2 \\
\alpha_{n1} &= +\frac{d}{L} \qquad \alpha_{n2} = -\frac{d}{L} \\
n &= 1, 2
\end{aligned} \tag{n}$$

We are assuming no initial elongation. Then,

$$\Delta F_n = k\,\Delta e_n = k(de_n + \tfrac{1}{2}d^2e_n) \tag{o}$$

The tangent stiffness matrix and incremental geometric load term are defined by (8–31). Using (n), we obtain

$$\mathbf{k}_{t,n} = \begin{bmatrix} k\alpha_{n1}^2 & \dfrac{k\alpha_{n1}}{L}(-b + u_{12}) \\ \text{Sym} & \dfrac{k}{L^2}(-b + u_{12})^2 + \dfrac{F_n}{L} \end{bmatrix} \tag{p}$$

$$\Delta\mathbf{p}_{g,n} = k\left\{\begin{array}{c} \dfrac{\alpha_{n1}}{2L}(\Delta u_{12})^2 \\ \hline \dfrac{\alpha_{n1}}{L}\Delta u_{11}\,\Delta u_{12} + \dfrac{3}{2}\left(\dfrac{\Delta u_{12}}{L}\right)^2(-b + u_{12}) + \dfrac{1}{2}\Delta u_{12}\left(\dfrac{\Delta u_{12}}{L}\right)^2 \end{array}\right\} \tag{q}$$

Finally, we assemble the incremental equilibrium equations for joint 1 using (8–30).

$$\Delta \mathbf{p}_1 = (\mathbf{k}_{t,1} + \mathbf{k}_{t,2})\Delta \mathbf{u}_1 + \Delta \mathbf{p}_{g,1} + \Delta \mathbf{p}_{g,2} \tag{r}$$

$$\Downarrow$$

$$\begin{bmatrix} 2k\left(\frac{d}{L}\right)^2 & 0 \\ 0 & \frac{2k}{L^2}(-b+u_{12})^2 + \frac{1}{L}(F_1+F_2)\end{bmatrix}\begin{Bmatrix}\Delta u_{11}\\ \Delta u_{12}\end{Bmatrix} = \begin{Bmatrix}\Delta p_{11}\\ \Delta p_{12}\end{Bmatrix} - \begin{Bmatrix} 0 \\ k\left(\frac{\Delta u_{12}}{L}\right)^2[3(-b+u_{12})+\Delta u_{12}]\end{Bmatrix} \tag{s}$$

Note that (s) is (8–34). Also, the incremental equations are uncoupled.

We restrict the analysis to only p_{12} loading. Setting $F_1 = F_2$ in (s) results in

$$\left[\frac{2}{L}\left(F + \frac{k}{L}(-b+u_{12})^2\right)\right]\Delta u_{12} = \Delta \bar{p}_{12} - k\left(\frac{\Delta u_{12}}{L}\right)^2[\Delta u_{12} + 3(-b+u_{12})] \tag{t}$$

where F is determined from (e). The coefficient of Δu_{12} is the tangent stiffness with respect to u_{12}.

$$\frac{dp_{12}}{du_{12}} = \frac{2}{L}\left(F + \frac{k}{L}(-b+u_{12})^2\right) \tag{u}$$

Applying the classical stability criterion (8–43) to (t), we see that

$$\frac{dp_{12}}{du_{12}} \quad \begin{matrix} > 0 & \text{stable} \\ = 0 & \text{neutral} \\ < 0 & \text{unstable}\end{matrix} \tag{v}$$

Points A, B are stability transition points and the segment A-B is unstable.

If $k = 0$, the truss is neutral with respect to Δu_{11}. Now there is a discontinuity in k at $F = -F_{eb}$, the pin-ended Euler load, when the material is linearly elastic:

$$\begin{aligned} |F| < F_{eb} \qquad & k = \frac{AE}{L} \\ F = -F_{eb} \qquad & k = 0 \\ F_{eb} = \frac{\pi^2 E I_3}{L^2} & \end{aligned} \tag{w}$$

To determine whether the members buckle before point A is reached, we compare F_A with $-F_{eb}$. Using (u),

$$F_A = -\frac{AE}{L^2}\left(-b + u_{12}\Big|_A\right)^2 = -\frac{AEb^2}{3L^2} \tag{x}$$

Then, for system instability rather than member instability to occur, b must satisfy

$$b < \left(\frac{3\pi^2 I_3}{A}\right)^{1/2} = \sqrt{3}\pi\rho \tag{y}$$

where ρ is the radius of gyration of the section.

Lastly, we outline how one applies the method of successive substitution to (t). For convenience, we drop the subscripts and write (t) as

$$k_t \, \Delta u = \Delta \bar{p} - \Delta \bar{p}_g \tag{z}$$

In the first step, we take $\Delta p_g = 0$.

$$\Delta u^{(1)} = \frac{\Delta \bar{p}}{k_t} \tag{aa}$$

The second estimate is determined from

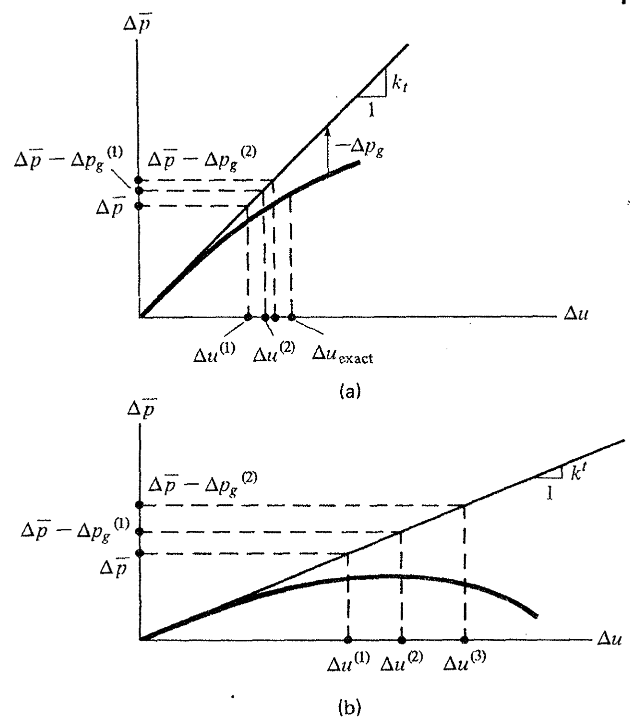
$$\Delta u^{(2)} = \frac{1}{k_t}(\Delta \bar{p} - \Delta p_g^{(1)}) \tag{bb}$$

Generalizing (bb),

$$\Delta u^{(n+1)} = \frac{1}{k_t}(\Delta \bar{p} - \Delta p_g^{(n)}) \tag{cc}$$

The convergence is illustrated in Fig. E8–5C. Case (b) shows how the scheme diverges

Fig. E8–5C



in the vicinity of a neutral point ($k_t = 0$). Convergence generally degenerates as $k_t \to 0$ and one has to resort to an alternate method.

8–5. LINEARIZED STABILITY ANALYSIS

In the previous section, we illustrated the behavior of a geometrically nonlinear system. The analysis involves first solving the *nonlinear* equilibrium equations for the displacements and then applying the classical stability criterion to determine the stability of a particular equilibrium position. Once the nonlinear equilibrium equations are solved, the stability can be readily determined. Now, if a geometrically *nonlinear* system is loaded in such a way that it behaves as if it were geometrically *linear*, we can neglect the displacement terms in $\mathbf{K}_{t,11}$, that is, we can take $\boldsymbol{\beta} = \boldsymbol{\alpha}$ in the expression for $\mathbf{k}_t$. This approximation is quite convenient since we have only to solve the linear problem in order to apply the stability criterion. We refer to this procedure as *linearized stability analysis*.

According to (8–40), an equilibrium position is stable (neutral, unstable) when the tangent stiffness matrix is positive definite (positive semi-definite, indifferent). We generate $\mathscr{K}_t$, transform to $\mathscr{K}_t^*$ and test $\mathscr{K}_t^*$ for positive definiteness. We have shown that $\mathscr{K}_t^*$ and $\mathbf{K}_{t,11}$ have the same definiteness property, i.e., $\mathbf{K}_{t,11}$ is positive definite if $\mathscr{K}_t^*$ is positive definite. Working with $\mathscr{K}_t^*$ rather than $\mathbf{K}_{t,11}$ avoids having to permute the rows and columns.

In linearized stability analysis, we approximate $\mathbf{k}_{t,n}$ with

$$\mathbf{k}_{t,n} = \mathbf{k}_n \boldsymbol{\alpha}_n^T \boldsymbol{\alpha}_n + F_n \mathbf{g}_n \tag{8–44}$$

The first term is the *linear* stiffness matrix. We interpret the second term as a *geometric* stiffness. The bar forces are determined from a linear analysis of the truss. If the loading is defined in terms of a single load parameter, λ, we can write (8–44) as

$$\begin{aligned} F_n &= \lambda \bar{F}_n \\ \mathbf{k}_{t,n} &= \mathbf{k}_{\ell,n} + \lambda \mathbf{k}_{g,n} \end{aligned} \tag{8–45}$$

The tangent stiffness matrix is generated by applying the Direct Stiffness Method to each term in (8–45). We express the *actual* and *modified* matrices as

$$\mathbf{K}_{t,11} = \mathbf{K}_{\ell,11} + \mathbf{K}_{g,11} \tag{8–46}$$

and

$$\mathscr{K}_t^* = \mathscr{K}_\ell^* + \lambda \mathscr{K}_g^* \tag{8–47}$$

where $\mathbf{K}_\ell$ is the system stiffness matrix for *linear* behavior. It is symmetrical and positive definite when the system is initially stable. The geometrical stiffness, $\mathbf{K}_g$, is also symmetrical but it may *not* be positive definite.

Equation (8–46) shows that the tangent stiffness matrix varies linearly with the load parameter. If the system is initially stable, $\mathbf{K}_{t,11}$ is positive definite for $\lambda = 0$. As λ is increased, a transition from stable to neutral equilibrium may occur at some load level, say λ_{cr}. To determine λ_{cr}, we note that neutral equilibrium (see (8–43)) corresponds to $|\mathbf{K}_{t,11}| = 0$ which, in turn, can be interpreted as the existence of a non-trivial solution of

$$\mathbf{K}_{t,11}\,\Delta\mathbf{U}_1 = \mathbf{0} \tag{a}$$

Substituting for $\mathbf{K}_{t,11}$ transforms (a) to a characteristic value problem,

$$\mathbf{K}_{\ell,11}\,\Delta\mathbf{U}_1 = -\lambda \mathbf{K}_{g,11}\,\Delta\mathbf{U}_1 \tag{8–48}$$

and λ_{cr} is the smallest eigenvalue† of (8–48).

Since $|\mathbf{K}_{t,11}| = |\mathscr{K}_t^*|$, we can work with

$$\begin{gathered} \mathscr{K}_t^*\,\Delta\mathscr{U}^J = \mathbf{0} \\ \Downarrow \\ \mathscr{K}^*\,\Delta\mathscr{U}^J = -\lambda \mathscr{K}_g^*\,\Delta\mathscr{U}^J \end{gathered} \tag{8–49}$$

instead of (8–48). Both equations lead to the same value of λ_{cr}. However, (8–49) has r additional characteristic values equal to -1 since we have added r dummy equations. To show this, we substitute for $\mathscr{K}_t^*$ using (8–41) and note (8–42).

$$\boldsymbol{\Pi}^T \begin{bmatrix} \mathbf{K}_{\ell,11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{U}_1 \\ \Delta\bar{\mathbf{U}}_2 \end{Bmatrix} = -\lambda \boldsymbol{\Pi}^T \begin{bmatrix} \mathbf{K}_{g,11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{U}_1 \\ \Delta\bar{\mathbf{U}}_2 \end{Bmatrix} \tag{a}$$

Premultiplying by $\boldsymbol{\Pi} = (\boldsymbol{\Pi}^T)^{-1}$, (a) becomes

$$\mathbf{K}_{\ell,11}\,\Delta\mathbf{U}_1 = -\lambda \mathbf{K}_{g,11}\,\Delta\mathbf{U}_1 \qquad (n_d \text{ eqs.}) \tag{b}$$

$$\Delta\bar{\mathbf{U}}_2 = -\lambda\,\Delta\bar{\mathbf{U}}_2 \qquad (r \text{ eqs.}) \tag{c}$$

The solution of (c) is

$$\lambda_1 = \lambda_2 = \cdots = \lambda_r = -1$$

$$\Delta\mathbf{U}_2 = C_1 \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} + C_2 \begin{Bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{Bmatrix} + \cdots + C_r \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{Bmatrix} \tag{d}$$

This solution must be disregarded since $\Delta\bar{\mathbf{U}}_2$ is actually a null matrix.

Example 8–6

Consider the system shown. We suppose the bars are identical, the material is linearly elastic, and there is no support movement.

The geometry change is negligible under a vertical load and we can use the linearized stability criterion. Working with the undeformed geometry, we have

$$\begin{aligned} F_1 &= F_2 = -\frac{L}{2b}\lambda \\ \bar{F}_1 &= \bar{F}_2 = -\frac{L}{2b} \end{aligned} \tag{a}$$

† Matrix iteration (Ref. 1) is a convenient computational scheme for determining λ_{cr}. We apply it to

$$(-\mathbf{K}_{g,11})\Delta\mathbf{U}_1 = \left(\frac{1}{\lambda}\right)\mathbf{K}_{\ell,11}\,\Delta\mathbf{U}_1$$

which satisfies the restrictions on the method.

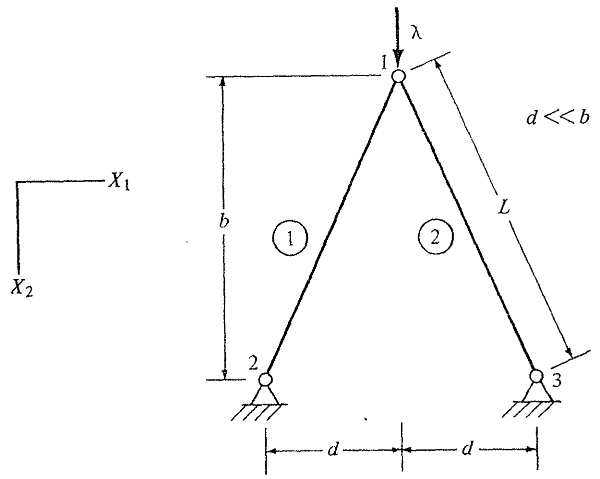


Fig. E8–6

We let $k_1 = k_2 = k$. The system stiffness matrices follow from (8–44) and (8–45).

$$\mathbf{K}_{\ell,11} = \mathbf{k}_{\ell,1} + \mathbf{k}_{\ell,2} = k(\boldsymbol{\alpha}_1^T\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2^T\boldsymbol{\alpha}_2)$$
$$= 2k\begin{bmatrix}\left(\frac{d}{L}\right)^2 & 0 \\ 0 & \left(\frac{b}{L}\right)^2\end{bmatrix} \tag{b}$$

and

$$\mathbf{K}_{g,11} = \mathbf{k}_{g,1} + \mathbf{k}_{g,2} = -\frac{L}{2b}(\mathbf{g}_1 + \mathbf{g}_2) \tag{c}$$

It remains to determine $\mathbf{g}_1$ and $\mathbf{g}_2$ which are defined by

$$\boldsymbol{\beta}_n = \boldsymbol{\alpha}_n + \mathbf{u}_1^T\mathbf{g}_n \tag{d}$$

We neglect u_{12} in the general expression for $\boldsymbol{\beta}_n$. This is reasonable when $d \ll b$. The approximate expressions for $\boldsymbol{\beta}_n$ and $\mathbf{g}_n$ are

$$\boldsymbol{\beta}_1 \approx \frac{1}{L}[d + u_{11} \quad -b] \qquad \boldsymbol{\beta}_2 \approx \frac{1}{L}[-d + u_{11} \quad -b]$$
$$\mathbf{g}_1 = \mathbf{g}_2 = \frac{1}{L}\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix} \tag{e}$$

Finally,

$$\mathbf{K}_{g,11} = -\frac{1}{b}\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix} \tag{f}$$

and

$$\mathbf{K}_{t,11} = \mathbf{K}_{\ell,11} + \lambda\mathbf{K}_{g,11} = 2k\begin{bmatrix}\left(\frac{d}{L}\right)^2 & 0 \\ 0 & \left(\frac{b}{L}\right)^2\end{bmatrix} - \lambda\begin{bmatrix}\frac{1}{b} & 0 \\ 0 & 0\end{bmatrix} \tag{g}$$

Neutral equilibrium ($\mathbf{K}_{t,11}$ is semidefinite) occurs at

$$\lambda_{cr} = 2kb\left(\frac{d}{L}\right)^2 = \frac{2AEb}{L}\left(\frac{d}{L}\right)^2 \tag{h}$$

Note that (g) has only *one* eigenvalue instead of *two*. This is a consequence of our using approximate expressions for Equations (e) instead of the exact expressions. At $\lambda = \lambda_{cr}$, the system is neutral with respect to Δu_{11}, i.e., the buckling mode is antisymmetric.

Neutral equilibrium also occurs when the bars either buckle or yield. The value of λ for Euler buckling of the bars is

$$\lambda_{cr,eb} = \frac{2b}{L}F_{eb} = \frac{2b}{L}\left[\frac{\pi^2 EI}{L^2}\right] = \frac{2AEb}{L}\left(\frac{\pi\rho}{L}\right)^2 \tag{i}$$

Comparing (h) and (i), we see that Euler buckling of the bars controls when

$$d > \pi\rho \tag{j}$$

The exact expression for $\mathbf{g}_n$ is

$$g_n = \frac{1}{L_n}\mathbf{I}_i \tag{k}$$

If we work with (k),

$$\mathbf{K}_{g,11} = -\frac{1}{b}\mathbf{I}_2 \tag{l}$$

and

$$\mathbf{K}_{t,11} = \mathbf{K}_{\ell,11} + \lambda\mathbf{K}_{g,11} = 2k\begin{bmatrix}\left(\frac{d}{L}\right)^2 & 0 \\ 0 & \left(\frac{b}{L}\right)^2\end{bmatrix} - \lambda\begin{bmatrix}\frac{1}{b} & 0 \\ 0 & \frac{1}{b}\end{bmatrix} \tag{m}$$

In this case, there are two characteristic values and therefore two critical values of λ.

$$\lambda_{cr,1} = 2kb\left(\frac{d}{L}\right)^2$$
$$\lambda_{cr,2} = 2kb\left(\frac{b}{L}\right)^2 = \left(\frac{b}{d}\right)^2\lambda_{cr,1} \tag{n}$$

The second root corresponds to neutral equilibrium with respect to Δu_{12}. For this example, $d \ll b$, and the first root defines the critical load.

It is of interest to compare $\lambda_{cr,2}$ with the buckling load found in Example 8–5. There we considered $d \gg b$ and followed the nonlinear behavior up to the point at which the slope of the $p_{12}-u_{12}$ curve vanished (neutral with respect to Δu_{12}):

$$K_t = \frac{dp_{12}}{du_{12}} = 0 \Rightarrow p_{12}\Big|_{max} = \frac{\sqrt{3}}{9}\lambda_{cr,2} \tag{o}$$

The linearized result is significantly higher than the true buckling load. In general, the linear buckling load is an upper bound. How close it is to the actual value will depend on the geometry and loading. When $d \ll b$, it is quite close, while it considerably overestimates the true load for $d \gg b$.

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PROBLEMS

8–1. Consider $\mathbf{U}_2$ and $\mathbf{P}_1$ to be *prescribed* and the behavior to be physically *linear*.

(a) Express $\Pi_P = V_T - \bar{\mathbf{P}}_1^T\mathbf{U}_1$ in terms of $\mathbf{U}_1$, $\bar{\mathbf{U}}_2$. Use

$$V_T = \sum_{j=1}^{m} \tfrac{1}{2}k_j(e_j - e_{0,j})^2 = \tfrac{1}{2}(\mathbf{e} - \mathbf{e}_0)^T\mathbf{k}(\mathbf{e} - \mathbf{e}_0)$$

(b) Show that (8–1) are the Euler equations for $\Pi_P(\mathbf{U}_1)$. Note that

$$dV_T = \mathbf{F}^T\,\mathbf{de} \qquad \mathbf{de} = \mathbf{B}_1^T\,\Delta\mathbf{U}_1$$

(c) Express $d^2\Pi_p$ as a quadratic form in $\Delta\mathbf{U}_1$. *Hint*: Obtain $d^2\mathbf{e}$ by operating on (7–8).

8–2. For the structure sketched:
(a) Determine $\mathbf{K}_{11}$.
(b) Determine $\mathbf{u}_1$ and $\mathbf{F}$ due to a temperature increase of 100°F for all the bars. Assume no support movements at joints 2, 3, 4.

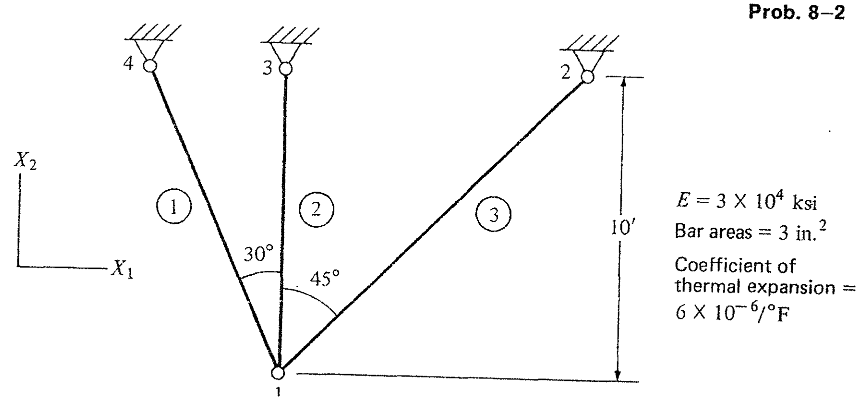
8–3. For the structure sketched:
Determine the displacements, bar forces, and reactions.

8–4. Refer to Example 8–2. Suppose we number the joints as shown. Develop the general form of $\mathscr{K}$ and compare with the result of Example 8–2.

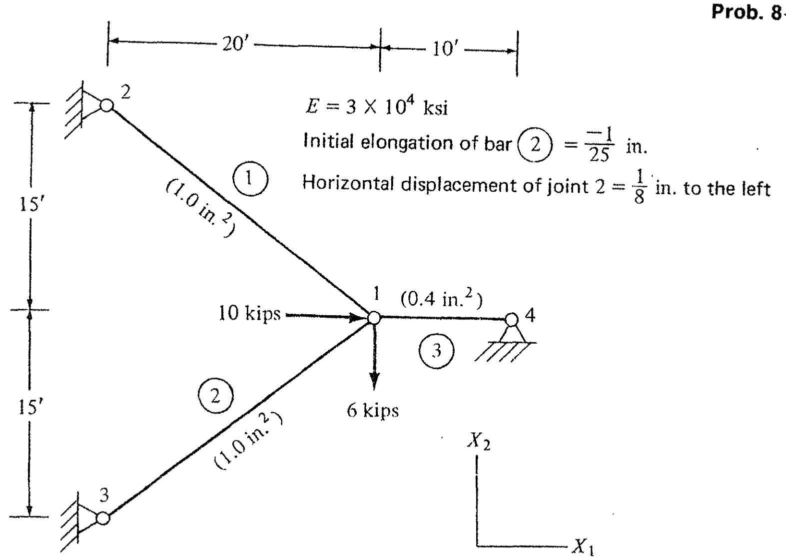
8–5. For the structure sketched, determine $\left(\dfrac{5}{Ea}\right)\mathscr{K}^*$.

8–6. For the structure sketched:
Develop the general form of $\mathscr{K}^*$. Indicate how you would obtain $\mathbf{K}_{11}$.

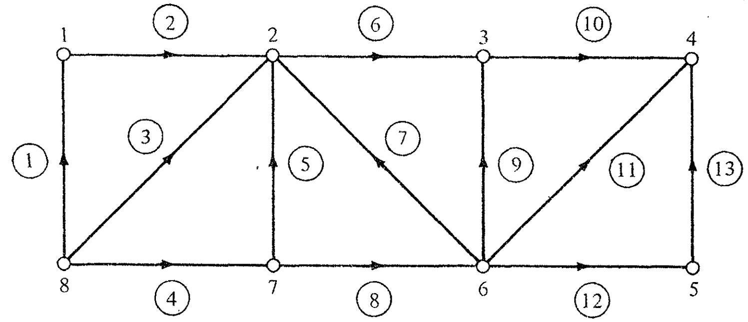
Prob. 8–2



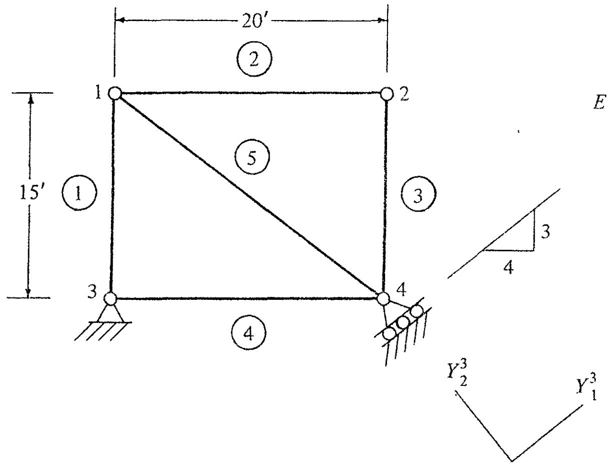
Prob. 8–3



Prob. 8–4



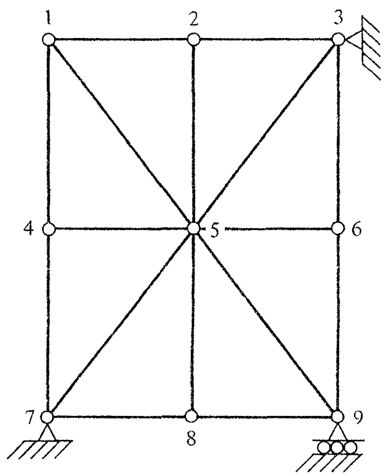
Prob. 8–5



E constant for all bars

Bar	Area
1	$3a$
2	$4a$
3	$3a$
4	$4a$
5	$2.5a$

Prob. 8–6

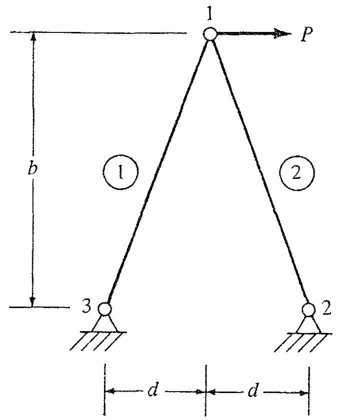


8–7. Determine the load-deflection relation for the system shown. Consider the material to be linearly elastic and the bars to be identical. Assume no initial elongation or support movement.

8–8. Investigate the elastic stability of the system shown. Assume the material is linearly elastic and no support movements. Use the linearized stability criterion and work with the exact expression for $\mathbf{g}_n$. Rework the problem, considering $d \ll b$ and using the corresponding approximate expression for $\mathbf{g}_n$.

8–9. Determine the lowest critical load for the truss shown. Assume the material is linearly elastic and all bars have the same stiffness.

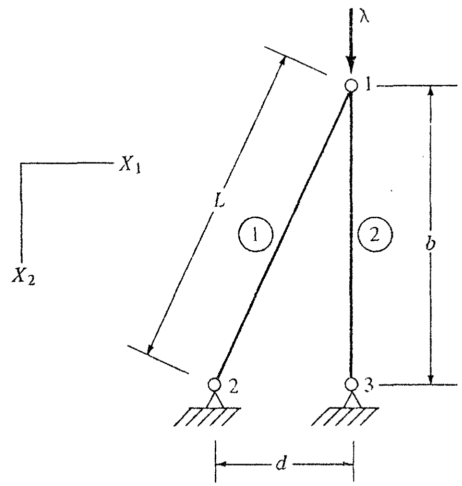
Prob. 8–7



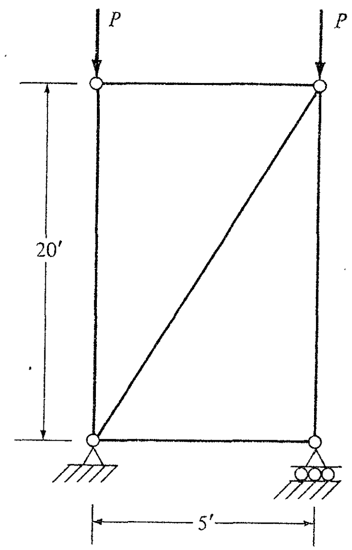
$d \ll b$

$k_1 = k_2 = \frac{AE}{L}$

Prob. 8–8



Prob. 8–9



8–10. The governing equations for geometrically nonlinear behavior of a linearly elastic discrete system such as a truss are nonlinear algebraic equations containing up to third-degree displacement terms. We have expressed them as

$$\mathscr{P} = \mathscr{P}_0 + \mathscr{K}\mathscr{U} = \mathscr{P}_0 + \mathscr{B}\mathbf{k}\mathscr{A}\mathscr{U} \tag{a}$$

where $\mathscr{A}$, $\mathscr{B}$ contain linear displacement terms. This form is dictated by our choice of matrix notation. In order to expand (a), we must shift from matrix to indicial notation.

For convenience, we employ the summation convention. If a subscript is repeated in a term, it is understood the term is summed over the range of the repeated subscript. An example is

$$a_{ij}b_j \equiv \sum_{j=1}^{n} a_{ij}b_j \qquad (j = 1, 2, \ldots, n) \tag{b}$$

We write the ith equilibrium equation for the system as (this representation is suggested in Ref. 8–10):

$$(K_{ij} + K_{ijk}U_k + K_{ijk\ell}U_kU_\ell)U_j = \lambda\bar{P}_i - P_{0,i} \tag{c}$$

where i, j, k, ℓ range over the total number of unknowns, U_j is the total value of the jth displacement unknown, λ is a load parameter, $\bar{P}_i$ defines the load distribution, and the K's are constants which can be interpreted as second-, third, and fourth-order tensors. The second-order tensor, K_{ij}, is the *linear* stiffness matrix.

(a) We generate the system tensors by superimposing the contribution of each bar. The first step involves converting the matrix expressions

$$\mathbf{p}_{n+} = \boldsymbol{\beta}_n^T F_n \qquad \mathbf{p}_{n-} = -\mathbf{p}_{n+} \tag{d}$$

where

$$\begin{aligned} F_n &= k_n e_n + F_{0,n} \\ e_n &= \gamma_n(\mathbf{u}_{n+} - \mathbf{u}_{n-}) \\ \gamma_n &= \boldsymbol{\alpha}_n + \frac{1}{2L}(\mathbf{u}_{n+} - \mathbf{u}_{n-})^T \\ \boldsymbol{\beta}_n &= \boldsymbol{\alpha}_n + \frac{1}{L}(\mathbf{u}_{n+} - \mathbf{u}_{n-})^T \end{aligned} \tag{e}$$

over to indicial form. We drop the n subscript, define $\mathbf{p}$ and $\mathbf{u}$ as

$$\mathbf{p} = \begin{Bmatrix} \mathbf{p}_{n+} \\ \mathbf{p}_{n-} \end{Bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} \mathbf{u}_{n+} \\ \mathbf{u}_{n-} \end{Bmatrix} \tag{f}$$

and write (d) in the form

$$p_i = (k_{ij} + k_{ijk}u_k + k_{ijk\ell}u_ku_\ell)u_j + p_{0,i} \tag{g}$$

Show that

$$\begin{aligned} k_{ij} &= k\alpha_s\alpha_r c_{si}c_{rj} \\ k_{ijk} &= \frac{k}{L}\left[c_{si}c_{rj}(\tfrac{1}{2}c_{rk}\alpha_s + c_{sk}\alpha_r)\right] \\ k_{ijk\ell} &= \frac{k}{2L^2}c_{si}c_{s\ell}c_{rj}c_{rk} \end{aligned} \tag{h}$$

where $\mathbf{c}$ is defined by

$$\mathbf{u}_{n+} - \mathbf{u}_{n-} = [\mathbf{I} \mid -\mathbf{I}]\begin{Bmatrix} \mathbf{u}_{n+} \\ \mathbf{u}_{n-} \end{Bmatrix} = \mathbf{c}\mathbf{u} \tag{i}$$

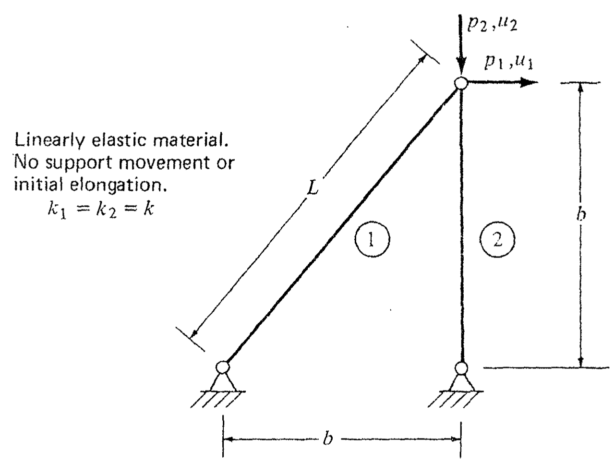
Discuss how you would locate the appropriate addresses for the bar stiffness tensors in the system tensors. What symmetry properties do the k's exhibit? Do these properties also apply for the system tensors?

(b) Develop the incremental equations relating Δu, $\Delta\lambda$ and compare with (8–30).

(c) Specialize the incremental equations for linearized stability analysis.

8–11. For the structure sketched:

Prob. 8–11



(a) Determine the nonlinear incremental equilibrium equations at the equilibrium position corresponding to $p_1 = 0$, $p_2 = p_{2,\text{cr}}$, the linearized critical load.

(b) Take $\Delta p_1 = 0$ and solve for Δp_2 as a function of Δu_1. Comment on how the system behaves when a small horizontal load, $p_1 = \pm\varepsilon p_2$, is applied in addition to p_2.