7

Variational Principles for an Ideal Truss

7–1. GENERAL

The formulation of the governing equations for an ideal truss described in Chapter 6 involved three steps:

1. The elongation of a bar was related to the translations of the joints at the end of the bar.
2. Next, the bar force was expressed in terms of the elongation and then in terms of the joint translations.
3. Finally, the equilibrium conditions for the joints were enforced, resulting in equations relating the external joint loads and internal bar forces.

The system equations were obtained by generalizing the member force-displacement and joint force equilibrium equations and required defining only two additional transformation matrices ($\mathscr{A}$, $\mathscr{B}$). Later, in Chapter 10, we shall follow essentially the same approach to establish the governing equations for an elastic solid.

In this chapter, we develop two variational principles and illustrate their application to an ideal truss. The principle of virtual displacements is treated first. This principle is just an alternate statement of force equilibrium. Next, we discuss the principle of virtual forces and show that it is basically a geometrical compatibility relation. Both principles are then identified as the stationary requirements for certain functions. For this step, we utilize the material presented in Chapter 3, which treats relative extremas of a function. Finally, we discuss the question of stability of an elastic system and develop the stability criterion for an ideal truss.

Why bother with variational principles when the derivation of the governing equations for an ideal truss is straightforward? Our objective in discussing them at this time is primarily to expose the reader to this point of view. Also, we can illustrate these principles quite easily with the truss. Later, we shall

use these principles, particularly the principle of virtual forces, to construct approximate formulations for a member.

7–2. PRINCIPLE OF VIRTUAL DISPLACEMENTS

The principle of virtual displacements is basically an alternate statement of force equilibrium. We will establish its form by treating first a single particle and then extending the result to a system of particles interconnected with internal restraints. The principle utilizes the concept of incremental work and, for completeness, we review briefly the definition of work before starting with the derivation.

Let v be the displacement of the point of application of a force F in the direction of F. The work done by F (see Fig. 7–1) is defined as

$$W = W_0 + \int_{v_0}^{v} F\, dv = W(v) \tag{7–1}$$

where v_0 is an arbitrary reference displacement. Since W is a function of v, the increment in W due to an increment Δv can be expressed in terms of the differentials of W when F is a continuous function of v:†

$$\begin{aligned} \Delta W &= dW + \tfrac{1}{2}d^2W + \cdots \\ dW &= \frac{dW}{dv}\Delta v = F\,\Delta v \\ d^2W &= d(dW) = \frac{dF}{dv}(\Delta v)^2 \end{aligned} \tag{7–2}$$

We refer to dW as the *first-order work*. Similarly, we call d^2W the *second-order work*. If dF/dv is discontinuous, as in inelastic behavior, we must use the value of dF/dv corresponding to the sense of Δv. This is illustrated in

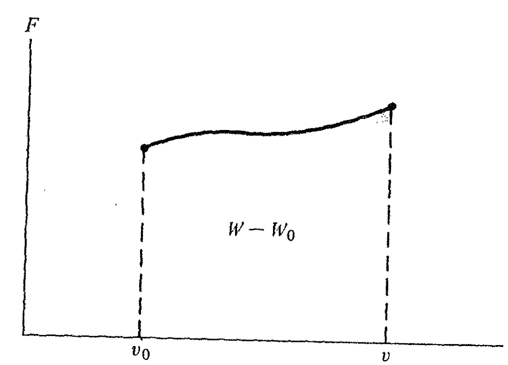


Fig. 7–1. Work integral for the one-dimensional force-displacement relation.

† Differential notation is introduced in Sec. 3–1.

Fig. 7–2. We use $dF/dv = +k_1$ for $\Delta v > 0$, and $dF/dv = -k_2$ for $\Delta v < 0$. Note that W is *not* a single-valued function of v when there is a reversal in the F-v curve.

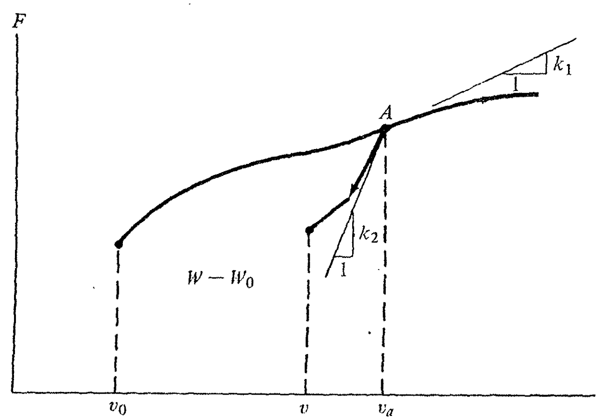


Fig. 7–2. Work integral for direction-dependent force.

We consider first a single mass particle subjected to a system of forces (see Fig. 7–3). Let $\vec{R}$ be the resultant force vector. By definition, the particle is in equilibrium when $\vec{R} = \vec{0}$. We visualize the particle experiencing a displacement increment $\Delta\vec{u}$ from the initial position. The first-order work is†

$$dW = \vec{R} \cdot \Delta\vec{u} \tag{7-3}$$

If the initial position is an equilibrium position, $dW = 0$ for arbitrary $\Delta\vec{u}$ since $\vec{R} = \vec{0}$. Therefore, an alternate statement of the equilibrium requirement is:

The first-order work is zero for an arbitrary displacement of a particle from an equilibrium position. (7–4)

The incremental displacement $\Delta\vec{u}$ is called a *virtual displacement*; this statement is the definition of the principle of virtual displacements.

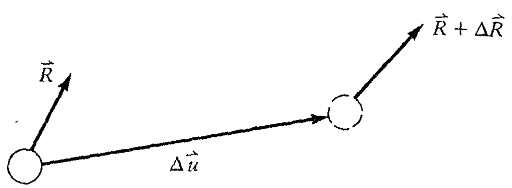


Fig. 7–3. Virtual displacement of a single mass particle.

One can readily generalize (7–4) for the case of S particles. Let dW_q be the first-order work associated with the forces acting on particle q and $\Delta\vec{u}_q$ the

† We consider the forces to be continuous functions of $\Delta\vec{u}$.

corresponding virtual-displacement vector. If particle q is in equilibrium, $dW_q = 0$ for arbitrary $\Delta\vec{u}_q$. It follows that the scalar force-equilibrium equations for the system are equivalent to the general requirement,

$$dW = \sum_{q=1}^{S} dW_q = 0 \qquad \text{for arbitrary } \Delta\vec{u}_q \qquad q = 1, 2, \ldots, S \tag{7-5}$$

Equation (7–5) is the definition of the principle of virtual work for a system of particles.

In general, some of the forces acting on the particles will be due to internal restraints. We define dW_E as the first-order work done by the external forces and dW_I as the work done by the internal restraint forces acting *on the particles*. Substituting for dW, (7–5) becomes

$$dW_E + dW_I = 0 \qquad \text{for arbitrary } \Delta\vec{u}_q \qquad q = 1, 2, \ldots, S \tag{a}$$

Now, let dW_D be the work done by the internal restraint forces acting *on the restraints*. We use the subscript D for this term since it involves the

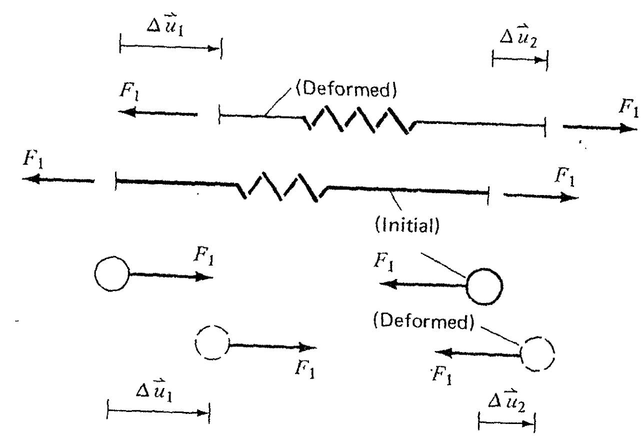


Fig. 7–4. Work done on the mass particles and internal restraints.

deformation of the restraints. The restraint force acting on a particle is equal in magnitude, but opposite in sense, to the reaction of the particle on the restraint. Since the points of application coincide, it follows that

$$dW_D = -dW_I \tag{b}$$

As an illustration, consider the simple system shown in Fig. 7–4. For this case, we have

$$dW_D = -F_1\,\Delta u_1 + F_1\,\Delta u_2$$
$$dW_I = F_1\,\Delta u_1 - F_1\,\Delta u_2$$

Using (b), we can write (a) as:

$$dW_E = dW_D \quad \text{for arbitrary } \Delta\tilde{u}_q \qquad q = 1, 2, \ldots, S \tag{7-6}$$

Also, the general principle of virtual displacements can be expressed as follows:

> The first-order work done by the external forces is equal to the first-order work done by the internal forces acting on the restraints for any arbitrary virtual displacement of a system of particles from an equilibrium position.

We emphasize again that (7–6) is just an alternate statement of the force equilibrium conditions for the system. Some authors refer to (7–6) as the work equation.

To apply the principle of virtual displacements to an ideal truss, we consider the joints to be mass points and the bars to be internal restraints. We have defined $\mathscr{P}$ and $\mathscr{U}$ as the column matrices of external joint loads and corresponding joint displacements. Then,

$$dW_E = \mathscr{P}^T \Delta\mathscr{U} \tag{a}$$

where $\Delta\mathscr{U}$ contains the virtual joint displacements. The first-order work done by the restraint forces acting on bar n due to the virtual displacements is †

$$(dW_D)_n = F_n\, de_n \tag{b}$$

Generalizing (b), we have

$$dW_D = \mathbf{F}^T\, d\mathbf{e} \tag{c}$$

Finally, the work equation for an ideal truss has the form

$$\mathscr{P}^T \Delta\mathscr{U} = \mathbf{F}^T\, d\mathbf{e} \qquad \text{for arbitrary } \Delta\mathscr{U} \tag{7-7}$$

The scalar force-equilibrium equations are obtained by substituting for $d\mathbf{e}$ in terms of $\Delta\mathscr{U}$.

It is convenient to first establish the expression for the differential elongation of an individual bar and then assemble $d\mathbf{e}$. Operating on e_n,

$$e_n = \boldsymbol{\alpha}_n(\mathbf{u}_{n_+} - \mathbf{u}_{n_-}) + \frac{1}{2L_n}(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})^T(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})$$

and noting the definition of $\boldsymbol{\beta}_n$ (see (6–22)), we obtain

$$\begin{aligned} de_n &= \left[\boldsymbol{\alpha}_n + \frac{1}{L_n}(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})^T\right](\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-}) \\ &= \boldsymbol{\beta}_n(\Delta\mathbf{u}_{n_+} - \Delta\mathbf{u}_{n_-}) \end{aligned} \tag{7-8}$$

† $W_d = \int_{e_n} F_n\, de_n = W_d(e_n)$. We must use the rules for forming the differentials of a compound function since e_n depends on the joint displacements. Using (3–17), we can write

$$dW_d = \frac{dW_d}{de_n}\, de_n = F_n\, de_n$$

$$d^2W_d = d(F_n\, de_n) = \frac{dF_n}{de_n}(de_n)^2 + F_n\, d^2e_n$$

The assembled form follows from (6–25). We just have to replace $\boldsymbol{\gamma}_n$ with $\boldsymbol{\beta}_n$:

$$d\mathbf{e} = \mathscr{B}^T \Delta\mathscr{U} \tag{7-9}$$

Substituting for $d\mathbf{e}$ in (7–7),

$$\mathscr{P}^T \Delta\mathscr{U} = \mathbf{F}^T\mathscr{B}^T \Delta\mathscr{U} \tag{a}$$

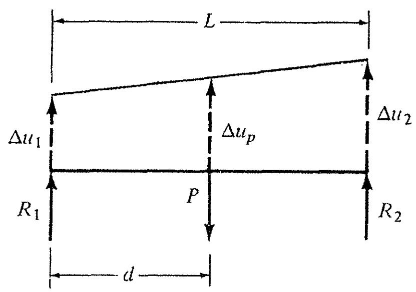
and requiring (a) to be satisfied for arbitrary $\Delta\mathscr{U}$ results in the joint force force equilibrium equations.

For the geometrically linear case, $\mathbf{e} = \mathscr{A}\mathscr{U}$ where $\mathscr{A}$ is *constant* and $d\mathbf{e} = \mathscr{A}\,\Delta\mathscr{U}$ follows directly from $\mathbf{e}$. We have treated the geometrically nonlinear case here to show that the principle of virtual displacements leads to force-equilibrium equations which are consistent with the geometrical assumptions associated with the deformation-displacement relations.

Example 7–1

We consider a rigid member subjected to a prescribed force, P, and reactions R_1, R_2, as in the diagram. There is no internal work since the body is rigid. Introducing the virtual

Fig. E7–1



displacements shown above, and evaluating the first-order work,

$$dW = dW_E = R_1\,\Delta u_1 + R_2\,\Delta u_2 - P\,\Delta u_p \tag{a}$$

Now, Δu_p is not independent:

$$\Delta u_p = \Delta u_1\left(1 - \frac{d}{L}\right) + \Delta u_2\left(\frac{d}{L}\right) \tag{b}$$

Then,

$$dW = \Delta u_1\left\{R_1 - P\left(1 - \frac{d}{L}\right)\right\} + \Delta u_2\left\{R_2 - P\left(\frac{d}{L}\right)\right\} = 0 \tag{c}$$

Requiring (c) to be satisfied for arbitrary Δu_1, Δu_2 leads to

$$\begin{aligned} R_1 &= P\left(1 - \frac{d}{L}\right) \\ R_2 &= P\frac{d}{L} \end{aligned} \tag{d}$$

which are the force and moment equilibrium equations, in that order.

Example 7–2

We consider the outside bars to be rigid (see sketch). To obtain the force equilibrium equation relating P and the internal bar forces F_1, F_2, we introduce a virtual displacement, Δu_1, of the point of application of P:

$$\begin{aligned} dW_E &= P\,\Delta u_1 \\ dW_D &= F_1\,de_1 + F_2\,de_2 \end{aligned} \tag{a}$$

The first-order increments in the elongations are

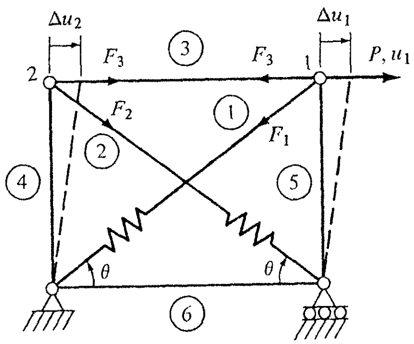
$$de_1 = \Delta u_1 \cos\theta \qquad de_2 = -\Delta u_2 \cos\theta = -\Delta u_1 \cos\theta \tag{b}$$

where θ defines the initial position. Then, equating dW_E and dW_D,

$$\begin{aligned} dW_E = dW_D \qquad &\text{for arbitrary } \Delta u_1 \\ &\Downarrow \\ P = (F_1 - F_2)&\cos\theta \end{aligned} \tag{c}$$

The force in bar 3 does not appear explicitly in the equilibrium equation, (c). It is possible

Fig. E7–2



Bars 3, 4, 5, 6 are rigid

to include F_3 even though bar 3 is rigid by treating it as a Lagrange multiplier.† We consider Δu_2 as independent in the work equation:

$$P\,\Delta u_1 - (F_1\cos\theta)\Delta u_1 + (F_2\cos\theta)\Delta u_2 = 0 \tag{d}$$

Now,

$$\Delta u_1 - \Delta u_2 = 0 \tag{e}$$

Multiplying the constraint relation by $-\lambda$, adding the result to (d), and collecting terms, we obtain

$$\Delta u_1(P - F_1\cos\theta - \lambda) + \Delta u_2(F_2\cos\theta + \lambda) = 0 \tag{f}$$

Finally, we require (f) to be satisfied for arbitrary Δu_1 and Δu_2. The equilibrium equations

† See Sec. 3–3.

are

$$\begin{aligned} P &= F_1\cos\theta + \lambda \\ F_2\cos\theta + \lambda &= 0 \end{aligned} \tag{g}$$

and we recognize λ as the force in bar 3.

7–3. PRINCIPLE OF VIRTUAL FORCES

The principle of virtual forces is basically an alternate statement of geometrical compatibility. We develop it here by operating on the elongation–joint displacement relations. Later, in Chapter 10, we generalize the principle for a three-dimensional solid and describe an alternate derivation.

We restrict this discussion to *geometric linearity*. The governing equations are

$$\mathscr{P} = \mathscr{B}\mathbf{F} \tag{a}$$

$$\mathbf{e} = \mathscr{A}\mathscr{U} = \mathscr{B}^T\mathscr{U} \tag{b}$$

Now, we visualize a set of bar forces $\Delta\mathbf{F}$, and joint loads, $\Delta\mathscr{P}$, which satisfy the force-equilibrium equations:

$$\Delta\mathscr{P} = \mathscr{B}\,\Delta\mathbf{F} \tag{c}$$

A force system which satisfies the equations of static equilibrium is said to be *statically permissible.* Equation (b) relates the actual elongations and joint displacements. If we multiply the equation for e_k by ΔF_k, sum over the bars, and note (c), we obtain the result

$$\begin{aligned} \Delta\mathbf{F}^T\mathbf{e} &= \Delta\mathbf{F}^T(\mathscr{B}^T\mathscr{U}) \\ &= \Delta\mathscr{P}^T\mathscr{U} \end{aligned} \tag{d}$$

which is the definition of the principle of virtual forces:

> The actual elongations and joint displacements satisfy the condition
> $$\Delta\mathbf{F}^T\mathbf{e} - \Delta\mathscr{P}^T\mathscr{U} = 0 \tag{7–10}$$
> for any statically permissible system of bar forces and joint loads.

The principle of virtual forces is independent of material behavior but *is* restricted to the geometrically linear case. The statically permissible system $(\Delta\mathbf{F}, \Delta\mathscr{P})$ is called a virtual-force system.

To illustrate the application of this principle, we express $\mathscr{U}$ and $\Delta\mathscr{P}$ in partitioned form,

$$\mathscr{U} \Rightarrow \mathbf{U} = \begin{bmatrix} \mathbf{U}_1 \\ \hline \bar{\mathbf{U}}_2 \end{bmatrix} \qquad \Delta\mathscr{P} \Rightarrow \Delta\mathbf{P} = \begin{bmatrix} \Delta\mathbf{P}_1 \\ \hline \Delta\mathbf{P}_2 \end{bmatrix} \tag{a}$$

where $\bar{\mathbf{U}}_2$ contains the prescribed support movements. Using (a), (7–10) takes the form:

$$\Delta\mathbf{F}^T\mathbf{e} - \Delta\mathbf{P}_2^T\bar{\mathbf{U}}_2 = \Delta\mathbf{P}_1^T\mathbf{U}_1 \tag{b}$$

If the elongations are known, we can determine the unknown displacements by specializing $\Delta\mathbf{P}_1$. To determine a particular displacement component, say u_{kj}, we generate a force system consisting of a unit value of p_{kj} and a set of bar forces and reactions which equilibrate $p_{kj} = 1$.

$$\begin{aligned}\Delta\mathbf{F} &= \mathbf{F}\big|_{p_{kj}=1} \\ \Delta\mathbf{P}_2 &= \mathbf{P}_2\big|_{p_{kj}=1}\end{aligned} \tag{7-11}$$

The internal bar forces and reactions are obtain from an equilibrium analysis of a statically determinate structure. Since only one element of $\Delta\mathbf{P}_1$ is finite,

$$\Delta\mathbf{P}_1^T\mathbf{U}_1 \Rightarrow (1)u_{kj} \tag{c}$$

and (b) reduces to

$$u_{kj} = \mathbf{e}^T\mathbf{F}\big|_{p_{kj}=1} - \mathbf{U}_2{}^T\mathbf{P}_2\big|_{p_{kj}=1} \tag{7-12}$$

The principle of virtual forces is also used to establish geometric compatibility relations required in the force method which is discussed in Chapters 9 and 17. We outline the approach here for completeness. One works with self-equilibrating virtual-force systems, i.e., statically permissible force systems which involve only bar forces and reactions.

By definition, a self-equilibrating force system $\mathbf{F}^*$, $\mathbf{P}^*$ satisfies

$$\begin{aligned}\mathbf{B}_1\mathbf{F}^* &= \mathbf{P}_1^* = \mathbf{0} \\ \mathbf{P}_2^* &= \mathbf{B}_2\mathbf{F}^*\end{aligned} \tag{7-13}$$

For this case, (b) reduces to

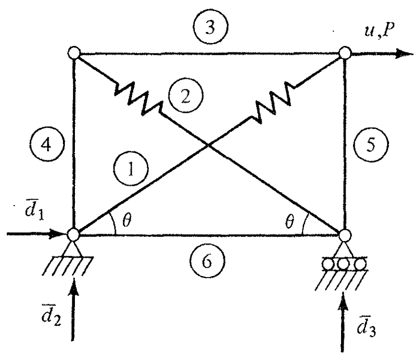
$$\mathbf{e}^T\,\Delta\mathbf{F} - \bar{\mathbf{U}}_2^T\,\Delta\mathbf{P}_2 = 0 \tag{7-14}$$

Equation (7-14) represents a restriction on the elongations and is called a *geometric compatibility* equation.

Example 7-3

The truss shown (Fig. E7-3A) has support movements $(\bar{d}_1, \bar{d}_2, \bar{d}_3)$ and is subjected to a loading which results in elongations (e_1, e_2) in the diagonal bars. We are considering the outside bars to be rigid.

Fig. E7-3A



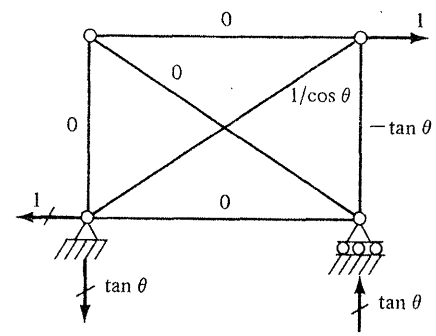
Bars 3, 4, 5, 6 are rigid

To determine the translation, u, we select a statically determinate force system consisting of a unit force in the direction of u and a set of bar forces and reactions required to equilibrate the force. One possible choice is shown in Fig. E7-3B. Evaluating (7-12) leads to

$$u = \frac{e_1}{\cos\theta} + \bar{d}_1 - \tan\theta(\bar{d}_3 - \bar{d}_2)$$

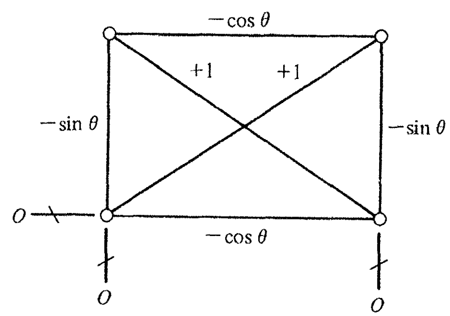
This truss is statically indeterminate to the first degree. A convenient choice of force redundant is one of the diagonal bar forces, say F_2. The equation which determines F_2 is

Fig. E7-3B



derived from the geometric compatibility relation, which, in turn, is obtained by taking a self-equilibrating force system consisting of $F_2 = +1$ and a set of bar forces and reactions required for equilibrium. The forces are shown in Fig. E7-3C.

Fig. E7-3C



Evaluating (7-14), we obtain

$$e_1 + e_2 = 0 \tag{a}$$

To show that (a) represents a geometrical compatibility requirement, we note that the elongation-displacement relations for the diagonal bars are

$$e_1 \doteq u\cos\theta \qquad e_2 = -u\cos\theta \tag{b}$$

Specifying e_1 determines u and also e_2. We could have arrived at Equation (a) starting from Equation (b) rather than (7-14). However, (7-14) is more convenient since it does not involve any algebraic manipulation. We discuss this topic in depth later in Chapter 9.

7–4. STRAIN ENERGY; PRINCIPLE OF STATIONARY POTENTIAL ENERGY

In this section, we specialize the principle of virtual displacements for elastic behavior and establish from it a variational principle for the joint displacements.

We start with the general form developed in Sec. 7–2,

$$\mathbf{F}^T\, d\mathbf{e} = \mathscr{P}^T\, \Delta\mathscr{U} \qquad \text{for arbitrary } \Delta\mathscr{U} \tag{a}$$

If we consider all the elements of $\mathscr{U}$ to be arbitrary, i.e., unrestrained,

$$d\mathbf{e} = \mathscr{B}^T\, \Delta\mathscr{U} \tag{b}$$

and (a) leads to the complete set of force-equilibrium equations in unpartitioned form,

$$\mathscr{P} = \mathscr{B}\mathbf{F} \tag{c}$$

We can obtain the equation for $\bar{\mathbf{P}}_1$ by rearranging (c) or by starting with the partitioned form of $\mathscr{P}^T\, \Delta\mathscr{U}$,

$$\mathscr{P}^T\, \Delta\mathscr{U} \Rightarrow \mathbf{P}^T\, \Delta\mathbf{U} = \bar{\mathbf{P}}_1^T\, \Delta\mathbf{U}_1 + \mathbf{P}_2^T\, \Delta\bar{\mathbf{U}}_2 \tag{d}$$

and noting that $\Delta\mathbf{U}_2 = \mathbf{0}$ since $\bar{\mathbf{U}}_2$ is prescribed. The reduced form is

$$\mathbf{F}^T\, d\mathbf{e} - \bar{\mathbf{P}}_1^T\, \Delta\mathbf{U}_1 = 0 \qquad \text{for arbitrary } \Delta\mathbf{U}_1 \tag{7–15}$$

where now

$$d\mathbf{e} = \mathbf{B}_1^T\, \Delta\mathbf{U}_1 + \mathbf{B}_2^T\, \Delta\bar{\mathbf{U}}_2 \Rightarrow \mathbf{B}_1^T\, \Delta\mathbf{U}_1$$

In what follows, we will work with (7–15).

Our objective is to interpret (7–15) as the stationary requirement for a function of $\mathbf{U}_1$. We consider $\mathbf{F}$ to be a function of $\mathbf{e}$, where $\mathbf{e} = e(\mathbf{U}_1)$. The form of $\mathbf{F} = F(\mathbf{e})$ depends on the material behavior.† We could express $\mathbf{F}$ in terms of $\mathbf{U}_1$ but it is more convenient to consider $\mathbf{F}$ as a compound function of $\mathbf{e}$. The essential step involves defining a function, $V_T = V_T(\mathbf{e})$, according to

$$\mathbf{F}^T\, d\mathbf{e} = \sum F_j\, de_j \equiv dV_T \tag{7–16}$$

With this definition, and letting

$$\Pi_P = V_T - \bar{\mathbf{P}}_1^T\mathbf{U}_1 = \Pi_P(\mathbf{U}_1) \tag{7–17}$$

we can write (7–15) as

$$d\Pi_P = 0 \qquad \text{for arbitrary } \Delta\mathbf{U}_1 \tag{7–18}$$

We call V_T the total strain energy function and Π_P the total potential energy. One should note that V_T exists only when $\mathbf{F}$ is a *continuous single-valued* function of $\mathbf{e}$. This requirement is satisfied when the material is *elastic.*

Equation (7–18) states that the joint force-equilibrium equations ($\bar{\mathbf{P}}_1 = \mathbf{B}_1\mathbf{F}$) expressed in terms of the unknown displacements are the Euler equations for the

† See Secs. 6–4, 6–5.

total potential energy. It follows that the actual displacements, i.e., the displacements which satisfy the equilibrium equations, correspond to a stationary value of Π_p.

It remains to discuss how one generates the strain-energy function. By definition,

$$dV_T = \sum dV_j \tag{a}$$

and

$$dV_j = F_j\, de_j \tag{b}$$

where V_j is the strain energy for bar j. Since we are considering V_j to be a compound function of e_j, Equation (b) is equivalent to

$$F_j(e_j) = \frac{d}{de_j} V_j \tag{7–19}$$

That is, the strain energy function for a bar has the property that its derivative with respect to the elongation is the bar force expressed in terms of the elongation. Finally, we can express V_j as

$$V_j = \int_{e_{0,j}}^{e_j} F_j\, de_j \tag{7–20}$$

where e_0 is the initial elongation, i.e., the elongation not associated with the force. Actually, the lower limit can be taken arbitrarily. This choice corresponds to taking V_j as the area between the F-e curve and the e axis, as shown in Fig. 7–5.

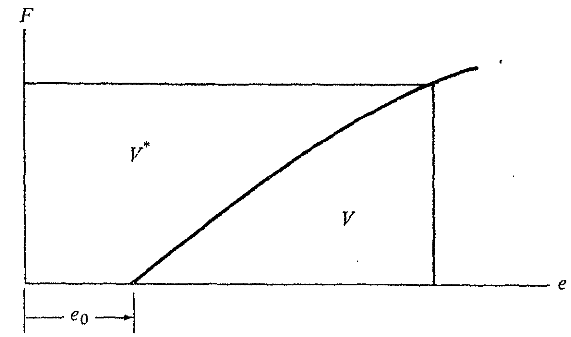


Fig. 7–5. Graphical representation of strain energy and complementary energy.

We consider the linearly elastic case. Using (6–30),

$$F_j = k_j(e_j - e_{0,j}) \tag{a}$$

Then

$$V_j = \tfrac{1}{2}k_j(e_j - e_{0,j})^2 \tag{7–21}$$

The total strain energy is obtained by summing over the bars. We can express V_T as

$$V_T = \sum_{j=1}^{m} V_j = \tfrac{1}{2}(\mathbf{e} - \mathbf{e}_0)^T\mathbf{k}(\mathbf{e} - \mathbf{e}_0) \tag{7–22}$$

Finally, we substitute for $\mathbf{e}$ in terms of $\mathbf{U}_1$, $\bar{\mathbf{U}}_2$, using

$$\mathbf{e} = \mathbf{A}_1\mathbf{U}_1 + \mathbf{A}_2\bar{\mathbf{U}}_2 \tag{7-23}$$

When the geometry is linear, $\mathbf{A}_1$, $\mathbf{A}_2$ are constant and V_T is a quadratic function. If the geometry is nonlinear, V_T is a fourth degree function of the displacements.

Up to this point, we have shown that the displacements defining an equilibrium position correspond to a stationary value of the potential energy function. To determine the character (relative maximum, relative minimum, indifferent, neutral) of the stationary point, we must examine the behavior of the second differential, $d^2\Pi_p$, in the neighborhood of the stationary point.

Operating on $d\Pi_p$, and noting that $\Delta\bar{P}_1 = \mathbf{0}$ leads to

$$\begin{aligned} d^2\Pi_p &= d(d\Pi_p) = d^2 V_T \\ d^2 V_T &= \sum_j (dF_j\, de_j + F_j d^2 e_j) \end{aligned} \tag{7-24}$$

The next step involves expressing $d^2 V_T$ as a quadratic form in $\Delta\mathbf{U}_1$. We restrict this discussion to linear behavior (both physical and geometrical). The general nonlinear case is discussed in Sec. 17.6 When the geometry is linear, we can operate directly on (7–23) to generate the differentials of $\mathbf{e}$,

$$\begin{aligned} d\mathbf{e} &= \mathbf{A}_1\Delta\mathbf{U}_1 \\ d^2\mathbf{e} &= \mathbf{0} \end{aligned} \tag{a}$$

since $\mathbf{A}_1$ is constant. When the material is linear,

$$d\mathbf{F} = \mathbf{k}\, d\mathbf{e} \tag{b}$$

where $\mathbf{k}$ is a diagonal matrix containing the stiffness factors (AE/L) for the bars. Then, $d^2 V_T$ reduces to

$$\begin{aligned} d^2 V_T &= d\mathbf{F}^T d\mathbf{e} = d\mathbf{e}^T\mathbf{k}\, d\mathbf{e} \\ &= \Delta\mathbf{U}_1^T(\mathbf{A}_1^T\mathbf{k}\mathbf{A}_1)\Delta\mathbf{U}_1 \end{aligned} \tag{7-25}$$

If $d\mathbf{e} \neq 0$ for all nontrivial $\Delta\mathbf{U}_1$, $d^2 V_T$ is positive definite and the stationary point is a relative minimum. This criterion is satisfied when the system is initially stable, since $d\mathbf{e} = \mathbf{0}$ for $\Delta U_1 \neq \mathbf{0}$ would require that

$$\mathbf{A}_1\,\Delta\mathbf{U}_1 = \mathbf{0}\ (m \text{ equations in } n_d \text{ unknowns}) \tag{a}$$

have a nontrivial solution. But a nontrivial solution of (a) is possible only when $r(\mathbf{A}_1) < n_d$. However, $\mathbf{A}_1 = \mathbf{B}_1^T$ for the geometrically linear case and $r(\mathbf{B}_1) = n_d$ when the system is initially stable. Therefore, it follows that the displacements defining the equilibrium position for a stable linear system correspond to an absolute *minimum* value of the potential energy.

Example 7–4

We establish the total potential energy function for the truss considered in Example 7–2. For convenience, we assume no initial elongation or support movement. The strain energy is

$$V_T = \tfrac{1}{2}(k_1 e_1^2 + k_2 e_2^2) \tag{a}$$

Substituting for the elongations in terms of the displacement,

$$e_1 = u_1\cos\theta \qquad e_2 = -u_2\cos\theta = -u_1\cos\theta \tag{b}$$

results in

$$V_T = \tfrac{1}{2}(k_1 + k_2)u_1^2\cos^2\theta \tag{c}$$

and finally

$$\Pi_p = \tfrac{1}{2}(k_1 + k_2)u_1^2\cos^2\theta - P_1 u_1 \tag{d}$$

The first differential of Π_p is

$$d\Pi_p = \{[(k_1 + k_2)\cos^2\theta]u_1 - P_1\}\Delta u_1 \tag{e}$$

Requiring Π_p to be stationary leads to the Euler equation,

$$P_1 = [(k_1 + k_2)\cos^2\theta]u_1 \tag{f}$$

which is just the force-equilibrium equation

$$P_1 = (F_1 - F_2)\cos\theta \tag{g}$$

with the bar forces expressed in terms of the displacement using

$$F_1 = k_1 e_1 = k u_1\cos\theta \qquad F_2 = k_2 e_2 = -k_2 u_1\cos\theta \tag{h}$$

The second differential of Π_p is

$$d^2\Pi_p = [(k_1 + k_2)\cos^2\theta](\Delta u_1)^2 \tag{i}$$

and we see that the solution,

$$u_1 = \frac{P_1}{(k_1 + k_2)\cos^2\theta} \tag{j}$$

corresponds to an absolute minimum value of Π_p when $\theta \neq 0$. The truss is initially unstable when $\theta = 0$.

7–5. COMPLEMENTARY ENERGY; PRINCIPLE OF STATIONARY COMPLEMENTARY ENERGY

The principle of virtual forces can be transformed to a variational principle for the force redundants. We describe in this section how one effects the transformation and utilize the principle later in Chapter 9. This discussion is restricted to linear geometry.

We start with Equations (7–13) and (7–14), which we list below for convenience:

$$\mathbf{e}^T\,\Delta\mathbf{F} - \bar{\mathbf{U}}_2^T\,\Delta\mathbf{P}_2 = 0 \tag{a}$$

where $\Delta\mathbf{F}$, $\Delta\mathbf{P}_2$ represent a self-equilibrating force- system, i.e., they satisfy the the following constraint relations:

$$\mathbf{B}_1\,\Delta\mathbf{F} = \mathbf{0} \tag{b}$$

$$\Delta\mathbf{P}_2 = \mathbf{B}_2\,\Delta\mathbf{F} \tag{c}$$

Our objective is to establish a function of **F**, whose Euler equations are (a) and (b). We cannot work only with (a) since **F** is not arbitrary but is constrained by the force-equilibrium equations,

$$\overline{P}_1 = \mathbf{B}_1\mathbf{F} \qquad (n_d \text{ equations in } m \text{ variables}) \tag{d}$$

We interpret $\mathbf{e}^T\,\Delta\mathbf{F}$ as the first differential of a function $V_T^* = V_T^*(\mathbf{F})$,

$$dV_T^* = \mathbf{e}^T\,\Delta\mathbf{F} = \sum dV_j^* \tag{7-26}$$

and call V_j^* the complementary energy function for bar j. By definition,

$$e_j(F_j) = \frac{d}{dF_j}V_j^* \tag{7-27}$$

That is, the complementary energy function for a bar has the property that its derivative with respect to the bar force is the elongation expressed in terms of the force. We express V_j^* as

$$V_j^* = \int_0^{F_j} e_j\,dF_j \tag{7-28}$$

This definition corresponds to taking V_j^* as the area bounded by the F-e curve and the F axis as shown in Fig. 7-5. Also, the strain and complementary energy functions are related by

$$V_j + V_j^* = F_j e_j \tag{7-29}$$

When the material is linear elastic,

$$\begin{aligned} e_j &= e_{0,j} + f_jF_j \\ V_j &= e_{0,j}F_j + \tfrac{1}{2}f_jF_j^2 \\ V_T &= \mathbf{e}_0^T\mathbf{F} + \tfrac{1}{2}\mathbf{F}^T\mathbf{f}\mathbf{F} \end{aligned} \tag{7-30}$$

Next, we define Π_c as:

$$\begin{aligned} \Pi_c &= V_T^* - \overline{\mathbf{U}}_2^T\mathbf{P}_2 \\ &= V_T^* - \overline{\mathbf{U}}_2^T\mathbf{B}_2\mathbf{F} \end{aligned} \tag{7-31}$$

We call $\Pi_c(\mathbf{F})$ the total complementary energy function. With these definitions, Equations (a), (b), and (c) can be interpreted as

$$d\Pi_c = 0 \tag{e}$$

subject to the constraint condition

$$d(\overline{\mathbf{P}}_1 - \mathbf{B}_1\mathbf{F}) = \mathbf{0} \tag{f}$$

We can combine (e) and (f) into a single equation by introducing Lagrange multipliers. Following the procedure described in Sec. 3-3, we add to (7-31) the joint force equilibrium equations and write the result as:

$$\Pi_c' = \Pi_c + (\overline{\mathbf{P}}_1 - \mathbf{B}_1\mathbf{F})^T\boldsymbol{\alpha} \tag{7-32}$$

where $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_{nd}\}$ contains the Lagrange multipliers. The Euler equations for Π_c' treating **F** and $\boldsymbol{\alpha}$ as independent variables are

$$d\Pi_c' = 0 \qquad \text{for } \Delta\mathbf{F}, \Delta\boldsymbol{\alpha} \text{ arbitrary}$$

$$\Downarrow$$

$$\begin{aligned} \mathbf{e}(\mathbf{F}) &= \mathbf{B}_1^T\boldsymbol{\alpha} + \mathbf{B}_2^T\overline{\mathbf{U}}_2 \\ \mathbf{B}_1\mathbf{F} &= \overline{\mathbf{P}}_1 \end{aligned} \tag{7-33}$$

We recognize the first equation in (7-33) as the member force-displacement relation, and it follows that $\boldsymbol{\alpha} = \mathbf{U}_1$.

An alternate approach involves first solving the force-equilibrium equation, (d). There are n_d equations in m variables. Since $\mathbf{B}_1$ is of rank n_d when the system is initially stable, we can solve for n_d bar forces in terms of $\overline{\mathbf{P}}_1$ and the remaining $(m - n_d)$ bar forces. One can also work with a combination of bar forces and reactions as force unknowns. We let

$$\begin{aligned} q &= m - n_d && = \text{number of redundant forces} \\ \mathbf{X} &= \{X_1, X_2, \ldots, X_q\} && = \text{matrix of force redundants} \end{aligned} \tag{7-34}$$

and write the solution of the force-equilibrium equations as

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_x\mathbf{X} \\ P_2 &= \mathbf{P}_{2,0} + \mathbf{P}_{2,x}\mathbf{X} \end{aligned} \tag{7-35}$$

The force system corresponding to X_j is self-equilibrating, i.e.,

$$(\mathbf{B}_1\mathbf{F}_x)\mathbf{X} = \mathbf{0} \qquad \text{for arbitrary } \mathbf{X} \tag{7-36}$$

We substitute for **F** in (7-31) and transform $\Pi_c(\mathbf{F})$ to $\Pi_c(\mathbf{X})$. Then,

$$\begin{aligned} d\Pi_c(\mathbf{X}) &= \mathbf{e}^T\,\Delta\mathbf{F} - \overline{\mathbf{U}}_2^T\,\Delta\mathbf{P}_2 \\ &= (\mathbf{e}^T\mathbf{F}_x - \overline{\mathbf{U}}_2^T\mathbf{P}_{2,x})\Delta\mathbf{X} \end{aligned} \tag{g}$$

and the Euler equations are

$$\mathbf{e}^T\mathbf{F}_x - \overline{\mathbf{U}}_2^T\mathbf{P}_{2,x} = \mathbf{0} \tag{7-37}$$

Note that (7-37) is just a reduced form of (7-33). Also, we could have obtained this result by substituting directly in (a).

Up to this point, we have shown that the force redundants which satisfy the geometric compatibility equations correspond to a stationary value of the total complementary energy. To investigate the character of the stationary point, we evaluate the second differential. Operating on (g),

$$d^2\Pi_c = d\mathbf{e}^T\mathbf{F}_x\,\Delta\mathbf{X} = d^2V_T^* \tag{h}$$

If $d^2V_T^*$ is positive definite with regard to $\Delta\mathbf{X}$, the stationary point is a relative minimum. This requirement is satisfied for the *linear elastic* case. To show this, we note that

$$\begin{aligned} d\mathbf{e} &= \mathbf{f}\,\Delta\mathbf{F} = \mathbf{f}\mathbf{F}_x\,\Delta\mathbf{X} \\ d^2V_T^* &= \sum f_j(\Delta F_j)^2 = \Delta\mathbf{X}^T(\mathbf{F}_x^T\mathbf{f}\mathbf{F}_x)\Delta\mathbf{X} \end{aligned} \tag{i}$$

Since **f** contains only positive elements, dV_T^* is positive definite with regard to $\Delta\mathbf{X}$ provided that there does not exist a nontrivial solution of

$$\mathbf{F}_x \, \Delta\mathbf{X} = \Delta\mathbf{F} = 0 \tag{j}$$

For (j) to have a nontrivial solution, there must be at least one relation between the columns of $\mathbf{F}_x$. But this would correspond to taking force redundants which are not independent, and the solution scheme would degenerate. Therefore, we can state that the actual force redundants correspond to an absolute minimum value of Π_c for the linear elastic case.

Example 7–5

We consider the truss treated in Example 7–3. It is statically indeterminate to the first degree with respect to the bars (statically determinate with respect to the reactions) and we take

$$X = F_2 \tag{a}$$

The force influence matrices defined by (7–35) follow from the force results listed on the sketches:

$$\begin{aligned} \mathbf{F}_0 &= P\{1/\cos\theta; 0; 0; 0; -\tan\theta; 0\} \\ \mathbf{F}_x &= \{+1; +1; -\cos\theta; -\sin\theta; -\sin\theta; -\cos\theta\} \\ \mathbf{P}_{2,0} &= P\{-1; -\tan\theta; +\tan\theta\} \\ \mathbf{P}_{2,x} &= \mathbf{0} \end{aligned} \tag{b}$$

Assuming a bar is rigid is equivalent to setting $f = 0$ for the bar. Then, the complementary energy is due only to the diagonal bars:

$$\begin{aligned} V_T^* &= V_1^* + V_2^* \\ &= e_{0,1}F_1 + e_{0,2}F_2 + \tfrac{1}{2}(f_1F_1^2 + f_2F_2^2) \end{aligned} \tag{c}$$

We convert V_T^* to a function of X by substituting

$$\begin{aligned} F_1 &= \frac{P}{\cos\theta} + X \\ F_2 &= +X \end{aligned} \tag{d}$$

Finally, $\Pi_c(X)$ has the form

$$\begin{aligned} \Pi_c(X) &= e_{0,1}\frac{P}{\cos\theta} + (\bar{d}_1 + \bar{d}_2\tan\theta - \bar{d}_3\tan\theta)P + \tfrac{1}{2}f_1\left(\frac{P}{\cos\theta}\right)^2 \\ &+ \left(e_{0,1} + e_{0,2} + f_1\frac{P}{\cos\theta}\right)X + \tfrac{1}{2}(f_1 + f_2)X^2 \end{aligned} \tag{e}$$

Differentiating (e) leads to

$$d\Pi_c = \left\{\left[e_{0,1} + e_{0,2} + f_1\frac{P}{\cos\theta}\right] + (f_1 + f_2)X\right\}\Delta X \tag{f}$$

$$d^2\Pi_c = (f_1 + f_2)(\Delta X)^2 \tag{g}$$

The Euler equation follows from (f):

$$e_{0,1} + e_{0,2} + f_1\frac{P}{\cos\theta} + (f_1 + f_2)X = 0 \tag{h}$$

Comparing (h) with (a) of Example 7–3, we see that the Euler equation for $\Pi_c(X)$ is the geometric compatibility equation expressed in terms of the force redundant.

7–6. STABILITY CRITERIA

Section 6–9 dealt with initial stability, i.e., stability of a system under infinitesimal load. We showed there that initial stability is related to rigid body motion. A system is said to be initially unstable when the displacement restraints are insufficient to prevent rigid body motion. In this section, we develop criteria for stability of a system under finite loading. If a linear system is initially stable, it is also stable under a finite loading. However, a nonlinear (either physical or geometrical) system can become unstable under a finite load.

We consider first a single mass particle subjected to a system of forces which are in equilibrium. Let $\bar{u}$ be the displacement vector defining the equilibrium position. We introduce a differential displacement $\Delta\bar{u}$, and let ΔW be the work done by the forces during the displacement $\Delta\bar{u}$. If $\Delta W > 0$, the particle energy is increased and motion would ensue. It follows that the equilibrium position ($\bar{u}$) is *stable* only when $\Delta W < 0$ for arbitrary $\Delta\bar{u}$.

We consider next a system of particles interconnected by internal restraints. Let ΔW_E be the incremental work done by the external forces and ΔW_I the incremental work done by the internal restraint forces acting *on* the particles. The total work, ΔW, is given by

$$\Delta W = \Delta W_E + \Delta W_I \tag{a}$$

The system is stable when $\Delta W < 0$ for all arbitrary permissible displacement increments, that is, for arbitrary increments of the *variable* displacements. Now, we let ΔW_D be the work done by the internal restraint forces acting on the restraints. Since $\Delta W_D = -\Delta W_I$, we can express the stability requirement as

$$\Delta W_D - \Delta W_E > 0 \tag{7–38}$$

One can interpret ΔW_D as the work *required* to deform the system to the alternate position and ΔW_E as the *actual* work done on the system.

When the behavior is continuous, we can express ΔW_D and ΔW_E as Taylor series expansions in terms of the displacement increments (see (7–2)):

$$\begin{aligned} \Delta W_E &= dW_E + \tfrac{1}{2}d^2W_E + \cdots \\ \Delta W_D &= dW_D + \tfrac{1}{2}d^2W_D + \cdots \end{aligned} \tag{b}$$

We have shown that the first-order work is zero at an equilibrium position:

$$dW_D - dW_E = 0 \tag{c}$$

If we retain only the first two terms in (b), the general stability condition reduces to

$$d^2W_D - d^2W_E > 0 \qquad \text{for all arbitrary permissible displacement increments} \tag{7-39}$$

Equation (7–39) is called the "classical stability criterion." Retaining only the first two differentials corresponds to considering only *infinitesimal* displacement increments. If (7–39) is satisfied, the equilibrium position is stable with respect to an infinitesimal disturbance. In order to determine whether it is stable with respect to a finite disturbance, one must use (7–38).

If

$$d^2W_D = d^2W_E \tag{7-40}$$

for a particular set of displacement increments, the equilibrium position is said to be *neutral*, and there exists an alternate equilibrium position infinitesimally displaced from the first position. One can interpret (7–40) as the necessary condition for a bifurcation of equilibrium positions.

To show this, suppose $\mathbf{U}$ and $\hat{\mathbf{U}}$ represent the displacement components for the two possible equilibrium positions of a system where

$$\hat{\mathbf{U}} = \mathbf{U} + \Delta\mathbf{U} \tag{a}$$

Also, let $\mathbf{R}$ and $\hat{\mathbf{R}}$ represent the resultant forces corresponding to $\mathbf{U}$ and $\hat{\mathbf{U}}$. We can express $\hat{\mathbf{R}}$ as

$$\hat{\mathbf{R}} = \mathbf{R} + d\mathbf{R} + \tfrac{1}{2}d^2\mathbf{R} + \cdots \tag{b}$$

Now, the second-order work for the initial equilibrium position is given by

$$d^2W = d^2W_E - d^2W_D = \Delta\mathbf{U}^T\, d\mathbf{R} \tag{c}$$

If $d^2W = 0$ for some finite $\Delta\mathbf{U}$, it follows that

$$d\mathbf{R} = \mathbf{R}_{,U}\,\Delta\mathbf{U} = \mathbf{0} \tag{d}$$

The condition

$$|\mathbf{R}_{,U}| = 0 \tag{e}$$

is equivalent to (7–40). Finally, if we consider $\Delta\mathbf{U}$ to be infinitesimal,

$$\mathbf{R} = \mathbf{R} + d\mathbf{R} \tag{f}$$

and (7–40) implies $\hat{\mathbf{R}} = \mathbf{0}$.

To apply the classical stability criterion to an ideal truss, we note that the first-order work terms have the form

$$\begin{aligned} dW_E &= \bar{\mathbf{P}}_1^T\,\Delta\mathbf{U}_1 \\ dW_D &= \sum F_j\, de_j \end{aligned} \tag{a}$$

where $\bar{\mathbf{U}}_2$, $\bar{\mathbf{P}}_1$ are prescribed. Operating on (a) yields

$$\begin{aligned} d^2W_E &= 0 \\ d^2W_D &= \sum [F_j\, d^2e_j + dF_j\, de_j] \end{aligned} \tag{7-41}$$

and the stability criterion reduces to

$$\begin{array}{lll} \text{stable} & d^2W_D > 0 & \text{for arbitrary nontrivial } \Delta\mathbf{U}_1 \\ \text{neutral} & d^2W_D = 0 & \text{for a particular nontrivial } \Delta\mathbf{U}_1 \\ \text{unstable} & d^2W_D < 0 & \text{for a particular nontrivial } \Delta\mathbf{U}_1 \end{array} \tag{7-42}$$

where d^2W_D is a quadratic form in $\Delta\mathbf{U}_1$. We postpone discussing how one transforms (7–41) to a quadratic form in $\Delta\mathbf{U}_1$ until the next chapter.

When the material is elastic, we can identify (7–39) as the requirement that Π_p be a relative minimum. By definition,

$$d\Pi_p = dV_T - dW_E \tag{a}$$

For elastic behavior,

$$dV_T = dW_D$$

and it follows that

$$d^2W_D - d^2W_E = d^2\Pi_p \tag{7-43}$$

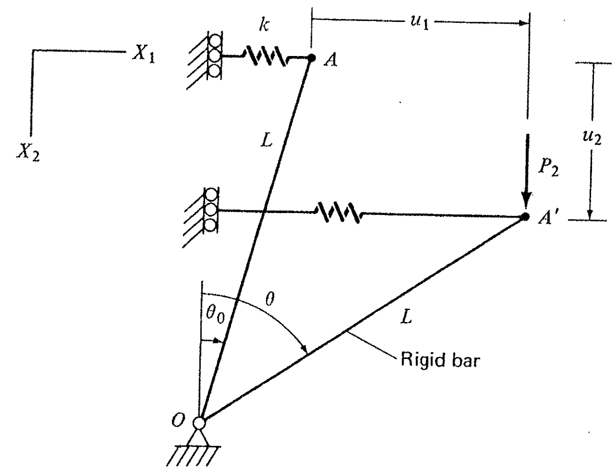
Finally, we can state:

> An equilibrium position for an elastic system is stable (neutral, unstable) if it corresponds to a relative minimum (neutral, indifferent) stationary point of the total potential energy.

Example 7–6

The system shown in Fig. E7–6A consists of a rigid bar restrained by a linear elastic spring which can translate freely in the x_2 direction. Points A and A' denote the initial and deformed positions. We will first employ the principle of virtual displacements to establish the equilibrium relations and then investigate the stability of the system.

Fig. E7–6A



The first-order work terms are

$$\begin{aligned} dW_D &= F\, de \\ dW_E &= P_2\, du_2 \end{aligned} \tag{a}$$

where F, e are the spring force and extension. Since the bar is rigid, the system has only one degree of freedom, i.e., only one displacement measure is required to define the configuration. It is convenient to take θ as the displacement measure. The deformation-displacement relations follow from the sketch:

$$\begin{aligned} e = u_1 &= L(\sin\theta - \sin\theta_0) \\ u_2 &= L(\cos\theta_0 - \cos\theta) \end{aligned} \tag{b}$$

Then,

$$F = ke = kL(\sin\theta - \sin\theta_0) \tag{c}$$

and

$$\begin{aligned} de &= (\cos\theta)L\,\Delta\theta \\ du_2 &= (\sin\theta)L\,\Delta\theta \end{aligned} \tag{d}$$

Using (a) and (d), the principle of virtual displacements takes the form

$$dW_D - dW_E = \{F\cos\theta - P_2\sin\theta\}\,(L\,\Delta\theta) = 0 \qquad \text{for arbitrary } \Delta\theta \tag{e}$$

Finally, (e) leads to the equilibrium relation,

$$F\cos\theta = P_2\sin\theta \tag{f}$$

which is just the moment equilibrium condition with respect to point O. We transform (f) to an equation for θ by substituting for F using (c). The result is

$$\sin\theta - \left(\frac{P_2}{kL}\right)\tan\theta = \sin\theta_0 \tag{g}$$

Since the system is elastic,

$$dW_D - dW_E \equiv d\Pi_P \tag{h}$$

and (e) is equivalent to

$$d\Pi_P = 0 \qquad \text{for arbitrary } \Delta\theta \tag{i}$$

The potential energy function for this system has the form

$$\begin{aligned} \Pi_P &= \tfrac{1}{2}ke^2 - P_2u_2 \\ &= \tfrac{1}{2}kL^2(\sin\theta - \sin\theta_0)^2 - P_2L(\cos\theta_0 - \cos\theta) \end{aligned} \tag{j}$$

and (g) can be interpreted as

$$\frac{d\Pi_P}{d\theta} = 0 \Rightarrow \text{Eq. (g)} \tag{k}$$

Curves of (P_2/kL) vs. θ for various values of θ_0 are plotted in Fig. E7–6B. The result for $\theta_0 = 0$ consists of two curves, defined by

$$\begin{aligned} \theta &= 0 && \text{for arbitrary } P_2/kL \\ \cos\theta &= P_2/k1 && \text{for } (P_2/kL) \leqslant 1 \end{aligned} \tag{l}$$

To investigate the stability of an equilibrium position, we have to evaluate the second-order work at the position. After some algebraic manipulation, we obtain

$$d^2\Pi_p = d^2W_D - d^2W_E = k(L\,\Delta\theta)^2\left[\frac{\cos^3\theta - P_2/kL}{\cos\theta}\right] \tag{m}$$

Let θ^* represent a solution of (g). Applying (m) to θ^* results in the following classification:

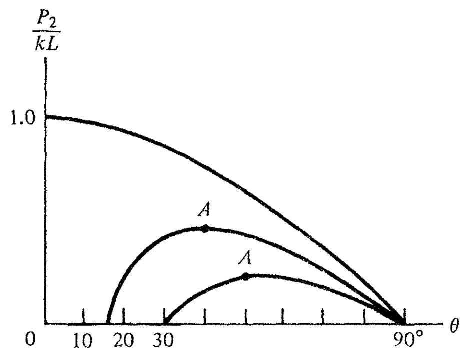
$$\begin{aligned} &\text{stable} && \cos^3\theta^* > \frac{P_2}{kL} \\ &\text{neutral} && \cos^3\theta^* = \frac{P_2}{kL} \\ &\text{unstable} && \cos^3\theta^* < \frac{P_2}{kL} \end{aligned} \tag{n}$$

One can show that (n) is equivalent to

$$\begin{aligned} &\text{stable} && \frac{dP_2}{d\theta} > 0 \\ &\text{neutral} && \frac{dP_2}{d\theta} = 0 \\ &\text{unstable} && \frac{dP_2}{d\theta} < 0 \end{aligned} \tag{o}$$

A transition from stable to unstable equilibrium occurs at point A, the peak of the load-deflection curve. The solution for $\theta_0 = 0$ is different in that its stable segment is the *linear* solution and the neutral equilibrium point ($P_2 = kL$) corresponds to a branch point. Both the linear and nonlinear branches are unstable.

Fig. E7–6B



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PROBLEMS

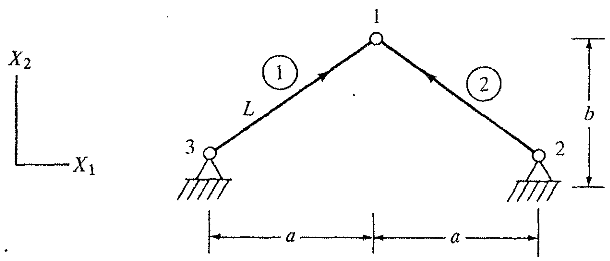
7–1. Consider the two-dimensional symmetrical truss shown. Assume $\mathbf{u}_2 = \mathbf{u}_3 = \mathbf{0}$.

(a) Determine the first two differentials of e_1 and e_2 by operating on the expanded expression (equation 6–19) for e.

(b) When $a \ll b$, we can neglect the nonlinear term involving u_{12} in the expressions for e *and* $\boldsymbol{\beta}$. Specialize (a) for this case.

(c) When $a \gg b$, we can neglect the nonlinear term involving u_{11} in the expressions for e *and* $\boldsymbol{\beta}$. Specialize (a) for this case.

Prob. 7–1



7–2. Refer to the figure of Prob. 7–1. Assume $\mathbf{u}_2 = \mathbf{u}_3 = \mathbf{0}$ and $a \gg b$. Using the principle of virtual displacements, determine the scalar force-equilibrium equations for joint 1.

7–3. Suppose a force F is expressed in terms of e,

$$F = C_1 e + \tfrac{1}{3} C_2 e^3 \tag{a}$$

where e is related to the independent variable u by

$$e = u + \tfrac{1}{2} u^2 \tag{b}$$

(a) Determine the first two differentials of the work function, $W = W(u)$, defined by

$$W = \int_0^e F\, de$$

(b) Suppose (a) applies for increasing e and

$$F = C_1(e - e_0^*) \tag{c}$$

for e decreasing from e^*. Determine d^2W at $e = e^*$.

7–4. Refer to Prob. 6–23. The $n - 1$ *independent* node equations relating the branch currents are represented by

$$\mathbf{A}^T\mathbf{i} = \mathbf{0} \tag{a}$$

Now, the branch potential differences, $\mathbf{e}$, are related to the $n - 1$ *independent* node potentials, $\mathbf{V}$, *by*

$$\mathbf{e} = \mathbf{A}\mathbf{V} \tag{b}$$

Deduce that the requirement,

$$\mathbf{i}^T\, d\mathbf{e} = 0 \qquad \text{for arbitrary } \Delta\mathbf{V} \tag{c}$$

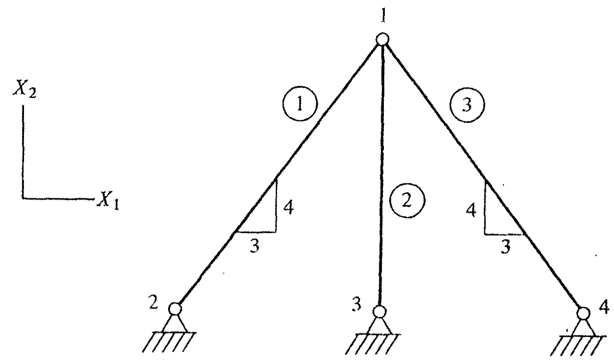
is equivalent to (a). Compare this principle with the principle of virtual displacements for an ideal truss.

7–5. Consider the two-dimensional truss shown. Assume $\mathbf{u}_2 = \mathbf{u}_4 = \mathbf{0}$.

(a) Using (7–14), obtain a relation between the elongations and $\bar{\mathbf{u}}_{32}$. Take the virtual-force system as ΔF_2 and the necessary bar forces and reactions required to equilibrate ΔF_2.

(b) Using (7–12), express u_{11}, u_{12} in terms of e_1, e_3. Note that bar 2 is not needed. One should always work with a statically determinate system when applying (7–12).

Prob. 7–5



7–6. Refer to Prob. 6–23. One can develop a variational principle similar to the principle of virtual forces by operating on the branch potential difference–node potential relations. Show that

$$\Delta \mathbf{i}^T \mathbf{e} = 0 \tag{a}$$

for any *permissible* set of current increments. Note that the currents must satisfy the node equations

$$\mathbf{A}^T\mathbf{i} = \mathbf{0}$$

Deduce Kirchhoff's law (the sum of the voltage drops around a closed loop must equal zero) by suitably specializing $\Delta\mathbf{i}$ in (a). Illustrate for the circuit shown in Prob. 6–6, using branches 1, 2, 4, and 6.

7–7. By definition, the first differential of the strain-energy function due to an increment in $\mathbf{U}_1$ has the form

$$dV_T = \sum_{n=1}^{m} dV_n = \sum_{n=1}^{m} F_n\, de_n \tag{a}$$

We work with V_T expressed as a compound function of $\mathbf{e} = e(\mathbf{U})$ since it is more convenient than expressing V directly in terms of $\mathbf{U}_1$. One can also write dV_T as

$$dV_T = V_{T,\mathrm{U}_1}\,\Delta\mathbf{U}_1 \tag{b}$$

(a) Using (b), show that the system of ij joint force-equilibrium equations expressed in terms of the joint displacements can be written as:

$$\frac{\partial V_T}{\partial u_{\ell k}} = p_{\ell k} \qquad \begin{matrix} \ell = 1, 2, \ldots, j \\ k = 1, 2, \ldots, i \end{matrix} \tag{c}$$

Equation c is called Castigliano's principle, part I.

(b) Show that an alternate form of (c) is

$$p_{\ell k} = \sum_{n=1}^{m} F_n \frac{\partial e_n}{\partial u_{\ell k}} \tag{d}$$

Note that (d) is just the expansion of (c). Rework Prob. 7–2, using (d).

7–8. Determine $V(e)$, dV, and d^2V for the case where the stress-strain relation has the form (see Prob. 6–10)

$$\sigma = E(\varepsilon - b\varepsilon^3)$$

7–9. Determine $V^*(F)$, dV^*, and d^2V^* for the case where the stress-strain relation has the form

$$\varepsilon = \frac{1}{E}(\sigma + c\sigma^3)$$

7–10. Show that (7–12) can be written as

$$u_{kj} = \frac{\partial}{\partial p_{kj}}\Pi_c$$

where $\Pi_c = \Pi_c(\mathbf{P}_1)$ is defined by (7–31). This result specialized for $\bar{\mathbf{U}}_2 = \mathbf{0}$ is called Castigliano's principle, part II. Apply it to Prob. 7–5, part b. Assume linear elastic material and $f_1 = f_2 = f_2 = f$.

7–11. The current and potential drop for a *linear* resistance are related by

$$e_j = e_{0,j} + R_j i_j \tag{a}$$

Inverting (a), we can express i_j as a function of e_j.

$$i_j = R_j^{-1}(e_j - e_{0,j}) \tag{b}$$

(a) Suppose we define a function, $W_j(e_j)$, which has the property that

$$\frac{dW_j}{de_j} = i_j(e_j) \tag{c}$$

Determine W_j corresponding to (b).

(b) Let $W = \sum_{j=1}^{b} W_j$ where b = total number of branches. Considering the branch potential drops to be functions of the node potentials, deduce that the actual node potentials $\mathbf{V}$ correspond to a stationary value of W. Use the results of Prob. 7–4. The Euler equations for $W = W(\mathbf{V})$ are the node current equilibrium equations expressed in terms of the node potentials.

(c) Suppose we define a function $W_j^*(i_j)$, which has the property that

$$\frac{dW_j^*}{di_j} = e(i_j) \tag{d}$$

Determine W_j^* corresponding to (a).

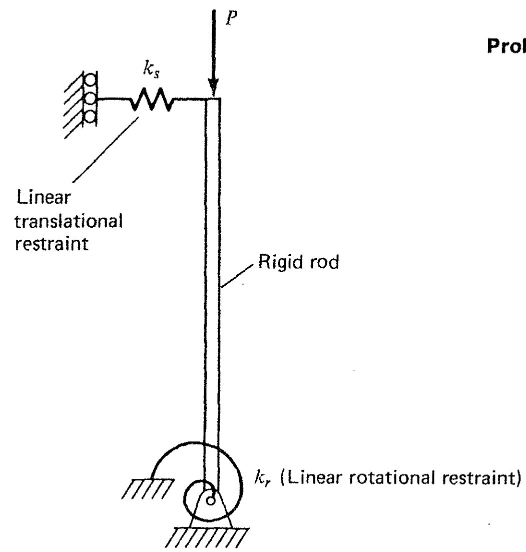
(d) Let $W^* = \sum_{j=1}^{b} W_j^*$. Show that the Euler equations for

$$\Pi = \mathbf{i}^T\mathbf{e} - W^* = \mathbf{i}^T(\mathbf{A}\mathbf{V}) - W^* = \Pi(\mathbf{i}, \mathbf{V}) \tag{e}$$

are the governing equations for a d-c network.

(e) Show that the actual currents correspond to a stationary value of W^*. One can either introduce the constraint condition, $\mathbf{A}^T\mathbf{i} = \mathbf{0}$, in (e) or use the result of Prob. 7–6.

7–12. Investigate the stability of the system shown below. Take $k_r = aL^2k_s$



Prob. 7–12

and consider a to range from 0 to 6.