INTEGER PROGRAMMING

1.224J/ESD.204J
TRANSPORTATION OPERATIONS,
PLANNING AND CONTROL:
CARRIER SYSTEMS

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IP OVERVIEW

Sources:

-Introduction to linear optimization (Bertsimas, Tsitsiklis)- Chap 1

- Slides 1.224 Fall 2000

Outline

- When to use Integer Programming (IP)
- Binary Choices
  - Example: Warehouse Location
  - Example: Warehouse Location 2
- Restricted range of values
- Guidelines for strong formulation
- Set Partitioning models
- Solving the IP
  - Linear Programming relaxation
  - Branch-and bound
  - Example

When to use IP Formulation?

- IP (Integer Programming) vs. MIP (Mixed Integer Programming)
  - Binary integer program
- Greater modeling power than LP
- Allows to model:
  - Binary choices
  - Forcing constraints
  - Restricted range of values
  - Piecewise linear cost functions

Example: Warehouse Location

A company is considering opening warehouses in four cities: New York, Los Angeles, Chicago, and Atlanta. Each warehouse can ship 100 units per week. The weekly fixed cost of keeping each warehouse open is $400 for New York, $500 for LA, $300 for Chicago, and $150 for Atlanta. Region 1 requires 80 units per week, region 2 requires 70 units per week, and Region 3 requires 40 units per week. The shipping costs are shown below.

Formulate the problem to meet weekly demand at minimum cost.

From/To	Region 1	Region 2	Region 3
New York	20	40	50
Los Angeles	48	15	26
Chicago	26	35	18
Atlanta	24	50	35

Warehouse Location- Approach

- What are the decision variables?
  - Variables to represent whether or not to open a given warehouse (y_i=0 or 1)
  - Variables to track flows between warehouses and regions: x_{ij}
- What is the objective function?
  - Minimize (fixed costs+shipping costs)
- What are the constraints?
  - Constraint on flow out of each warehouse
  - Constraint on demand at each region
  - Constraint ensuring that flow out of a closed warehouse is 0.

Warehouse Location- Formulation

- Let y_i be the binary variable representing whether we open a warehouse *i* (y_i=1) or not (y_i=0).
- x_{ij} represents the flow from warehouse *i* to region *j*
- c_i= weekly cost of operating warehouse *i*
- t_{ij}= unit transportation cost from *i* to *j*
- *W* = the set of warehouses; *R* = the set of regions

$$MIN\left(\sum_{i\in W} c_i . y_i + \sum_{i\in W}\sum_{j\in R} t_{ij} . x_{ij}\right)$$

s.t.

$$\sum_j x_{ij} \leq 100 . y_i , \forall i \in W$$

Forcing constraint

$$\sum_i x_{ij} = b_j , \forall j \in R$$

$$x_{ij} \in Z^+ , y_i \in \{0,1\}$$

Warehouse Location- Additional Constraints

- If the New York warehouse is opened, the LA warehouse must be opened

$$y_{NYC} \leq y_{LA}$$

Relationship constraint

yNYC	yLA
1 →	1
0 →	1 or 0

- At most 2 warehouses can be opened

$$\sum_i y_i \leq 2$$

Relationship constraint

- Either Atlanta or LA warehouse must be opened, but not both

$$y_{LA} + y_{ATL} = 1$$

Relationship constraint

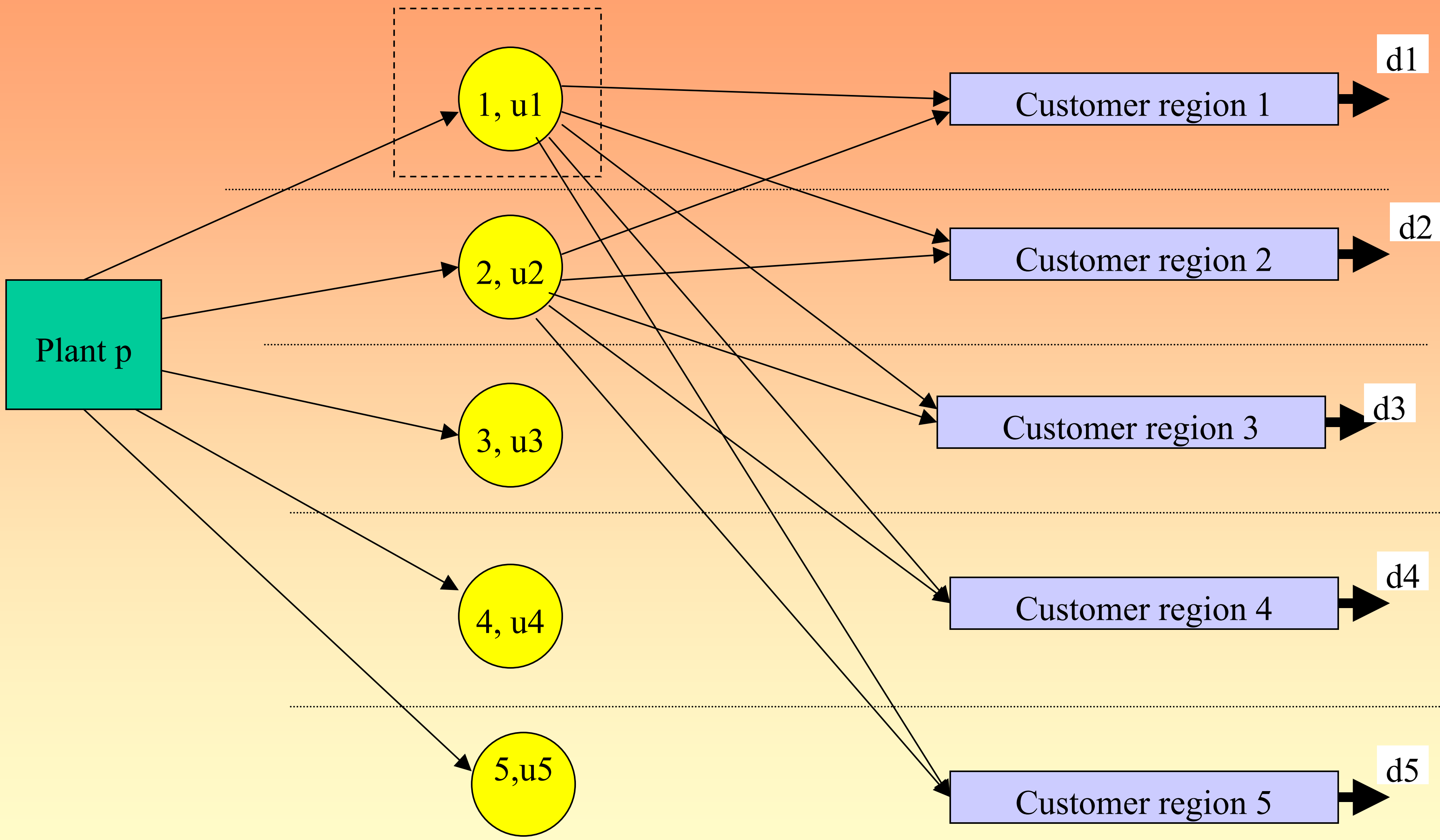
Binary Choices

- Model choice between 2 alternatives (open or closed, chosen or not, etc)
  - Set x=0 or x=1 depending on the chosen alternative
- Can model fixed or set-up costs for a warehouse
- Forcing flow constraints
  - if warehouse is not open, no flow can come out of it
- Can model relationships

Example: Warehouse Location 2

- A company is looking at adding one or more warehouses somewhere in the R regions which they serve. Each warehouse costs $\$c_w$ per month to operate and can deliver a total of u_w units per month. It costs $\$c_{ij}$ to transport a unit from the plant in region *i* to the warehouse in region *j*. Furthermore, the delivery costs from a warehouse in region *j* to consumers in region *j* is zero. Warehouses can service other regions, but the company must pay additional transportation costs of \$t per unit per additional region crossed. So to deliver 1 unit from a warehouse in region 2 to a customer in region 4 would cost $\$(2 \cdot t)$. Note that the cost to transport a good from warehouse 0 to warehouse R is $\$(R \cdot t)$, not \$t. All units must travel through a warehouse on their way to the customer. Finally, there is a monthly demand for d_j units of the product in region *j*. Formulate the problem to <u>determine where to locate the new warehouses</u> so as to <u>minimize the total cost each month</u> if the plant is located in region p.

Example 2: Network Representation



Example 2: Approach

- Decision Variables?
  - y_i= whether or not we open a warehouse in region *i*
  - z_{ij}=flow from warehouse *i* to region *j*
  - x_{pj}=flow from plant *p* to warehouse *j*.
- Objective Function?
  - MIN (fixed costs+transportation costs from plant to warehouse+transportation costs from warehouse to region)
- Constraints?
  - balance constraints at each warehouse
  - demand constraints for each region
  - capacity constraints at each warehouse.
- Let a_{ij}=cost of delivering a unit from warehouse *i* to region *j*, $a_{ij}=t.|j-i|$
- Let c_{pj}=cost of transporting one unit from the plant to warehouse *j*

Example 2: Formulation

$$\text{Min } \sum_{i \in R} c_w . y_i + \sum_{j \in R} c_{pj} . x_{pj} + \sum_{i \in R} \sum_{j \in R} a_{ij} . z_{ij}$$

s.t.

$$\sum_{i \in R} z_{ij} = d_j, \forall j \in R$$

$$\sum_{j \in R} z_{ij} - x_{pi} = 0, \forall i \in R$$

$$x_{pi} \leq u_w y_i, \forall i \in R$$

$$x_{ij}, z_{ij} \in Z^+ \forall i, j \in R; y_i \in \{0,1\} \forall i \in R$$

Example 2: Additional Constraints

- At most 3 warehouses can be opened

$$\sum_{i \in R} y_i \leq 3$$

- If you open a warehouse in some region r_{w1} or r_{w2}, you must also open a warehouse in region r_{w3}

$$y_{rw3} \geq y_{rw1}$$

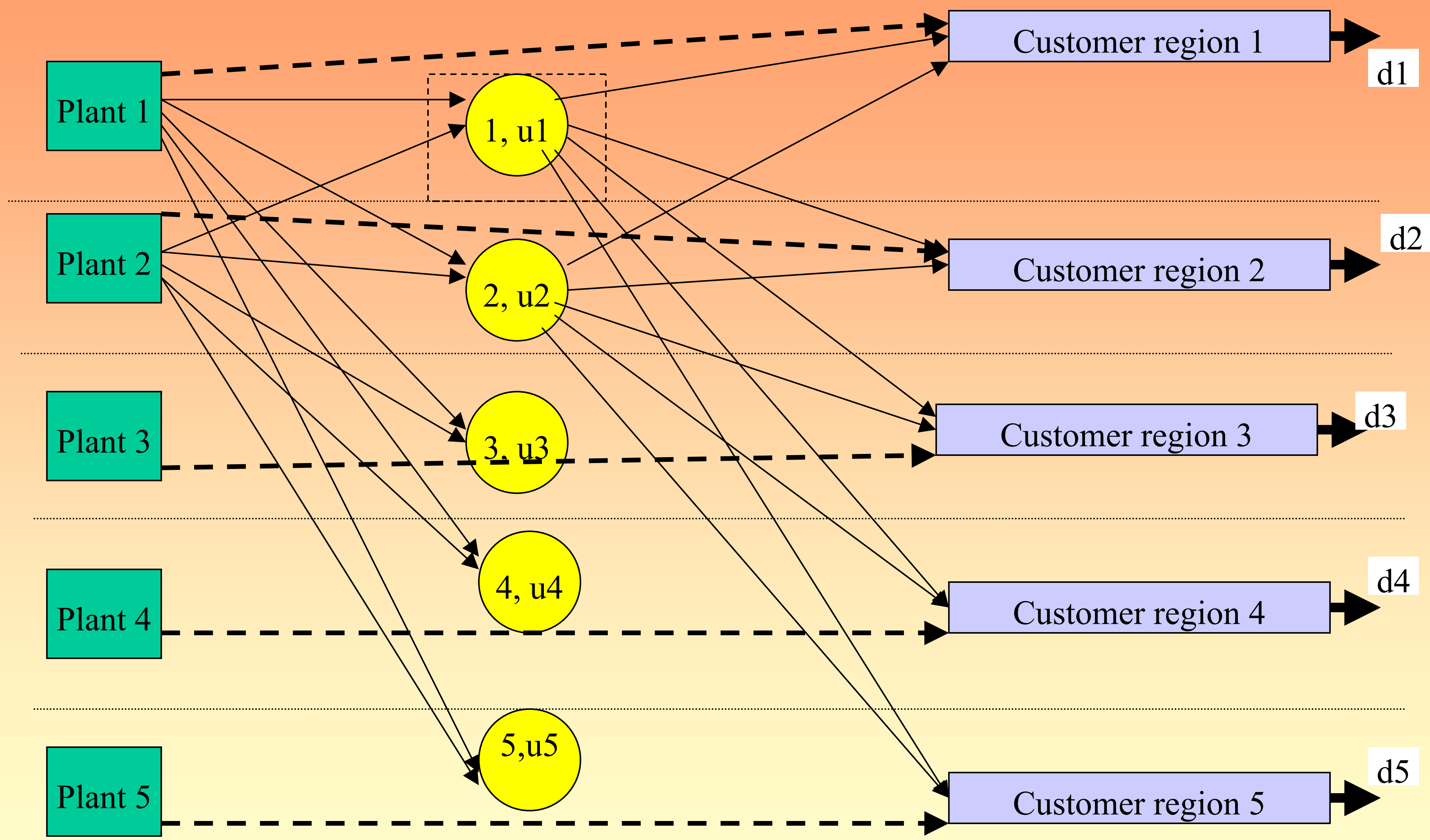
$$y_{rw3} \geq y_{rw2}$$

yrw1	yrw2	yrw3
1	0	1
0	1	1
1	1	1
0	0	0 or 1

Example 2: Additional Constraints

- A plant costs $\$c_p$ per month to operate and can output u_p units per month. In this case, a plant can deliver directly to customers in its region at no additional cost, however it cannot deliver directly to customers in other regions; all units traveling out of the plant's region must pass through a warehouse before their delivery to the customer. Formulate the problem to find the optimal distribution of plants and warehouses.
- Additional decision variables:
  - w_i= whether or not we open a plant in region *i*
  - u_i= amount of flow directly from plant *i* to region *i* (no cost)
- Objective Function
  - Additional term to account for the cost of the plants
- Revised constraints
  - Constraints range over all regions, not only region *p*
  - Add direct flow from plant to customers in same region
  - Add constraint that total flow leaving a plant is less than u_p

Example 2: Network Representation 2



Example 2: Formulation 2

$$MIN \sum_{i \in R} c_w . y_i + \boxed{\sum_{i \in R} c_p . w_i + \sum_{i \in R} \sum_{j \in R} c_{ij} . x_{ij}} + \sum_{i \in R} \sum_{j \in R} a_{ij} . z_{ij}$$

$$s.t$$

$$\sum_{i \in R} z_{ij} + \boxed{u_j} = d_j, \forall j \in R$$

$$\sum_{j \in R} z_{ij} - \boxed{\sum_{j \in R} x_{ji}} = 0, \forall i \in R$$

$$\sum_{j \in R} x_{ji} \le u_w . y_i, \forall i \in R$$

$$\boxed{\sum_{j \in R} x_{ij} + u_i \le u_p . w_i, \forall i \in R}$$

$$u_i, x_{ij}, z_{ij} \in Z^+ \forall i, j \in R; w_i, y_i \in \{0,1\} \forall i \in R$$

Restricted range of values

- Restrict a variable x to take values in a set $\{a_1, \ldots, a_m\}$
- Introduce *m* binary variables y_j, j=1..m and the constraints

$$X = \sum_{j=1..m} a_j y_j$$

$$s.t.$$

$$\sum_{j=1..m} y_j = 1$$

$$y_j \in \{0,1\}, \forall j$$

Guidelines for strong formulation

- Good formulation in LP: small number of variables (n) and constraints (m), because computational complexity of problem grows polynomially in n and m
- LP: choice of a formulation is important but does not critically affect ability to solve the problem
- IP: Choice of formulation is crucial!
- Example: aggregation of demand (Warehouse)

Set Partitioning models

- Very easy to write, often very hard to solve
- All rules, even non-linear, impractical rules can be respected
- Every object is in exactly one set
- Huge number of variables (all feasible combinations)

Linear Programming relaxation

- Relax the integrality constraint
- Examples:
  - X_j in Z^+ becomes $X_j \geq 0$
  - X_j in $\{0,1\}$ becomes $0 \leq X_j \leq 1$
- If an optimal solution to the relaxation is feasible for the MIP (i.e., X take on integer values in the optimal solution of the relaxation) => it is also the optimal solution to the MIP
- The LP relaxation provides a lower bound on the solution of the IP
- Good formulations provide a “tight” bound on the IP

Branch-and-Bounds: A solution approach for binary Integer programs

- Branch-and-bound is a smart enumeration strategy:
  - With branching, all possible solutions are enumerated (e.g. $2^{\text{number of binary variables}}$)
  - With bounding, only a (usually) small subset of possible solutions are evaluated before a provably optimal solution is found

Branch-and-Bound Algorithm

Beginning with root node (minimization):

- Bound:
  - Solve the current LP with this and all restrictions along the (back) path to the root node enforced
- Prune
  - If optimal LP value is greater than or equal to the incumbent solution => Prune
  - If LP is infeasible => Prune
  - If LP is integral => Prune
- Branch
  - Set some variable to an integer value
- Repeat until all nodes pruned

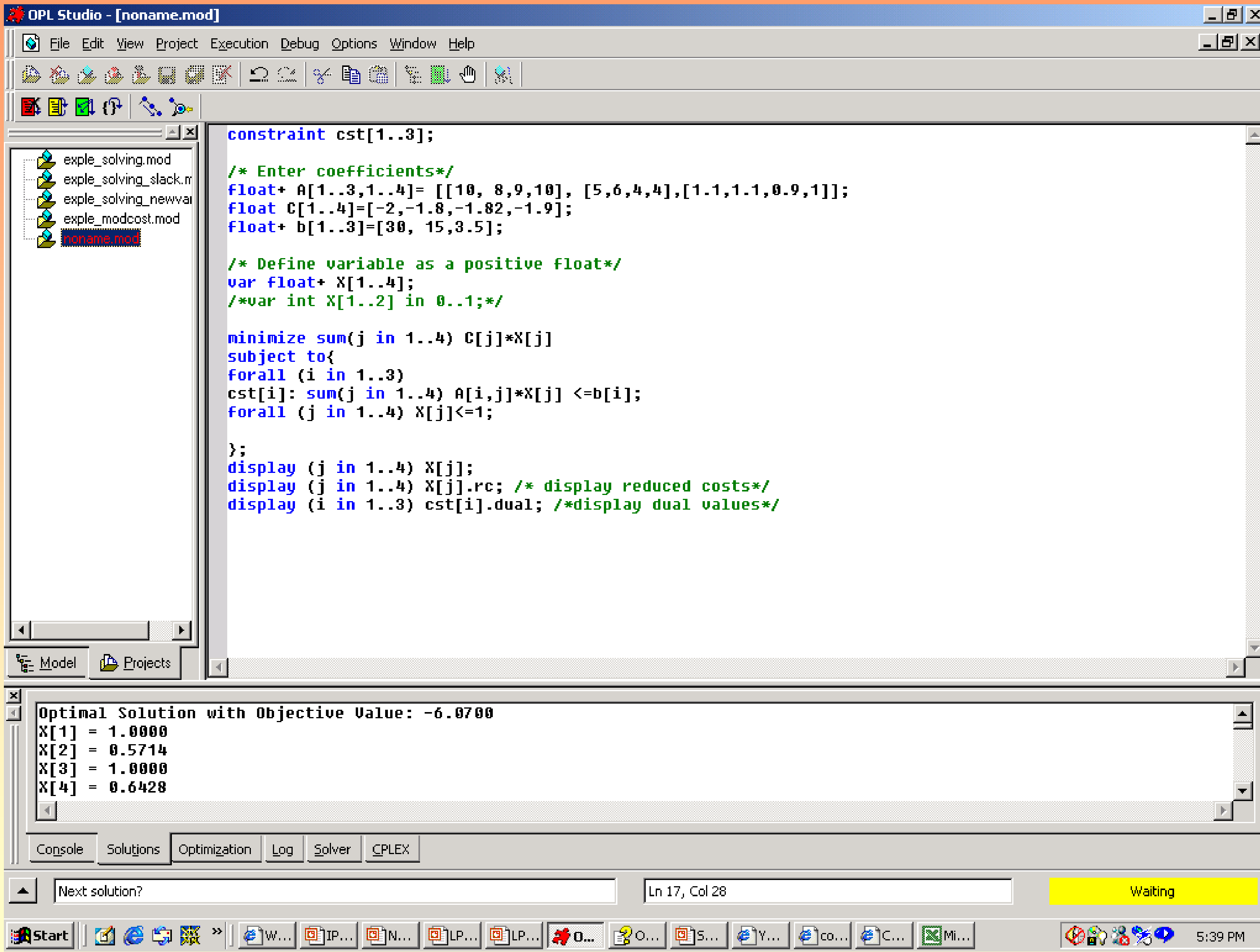
Example

Company XYZ produces products A, B, C and D. In order to manufacture these products, Company XYZ needs:

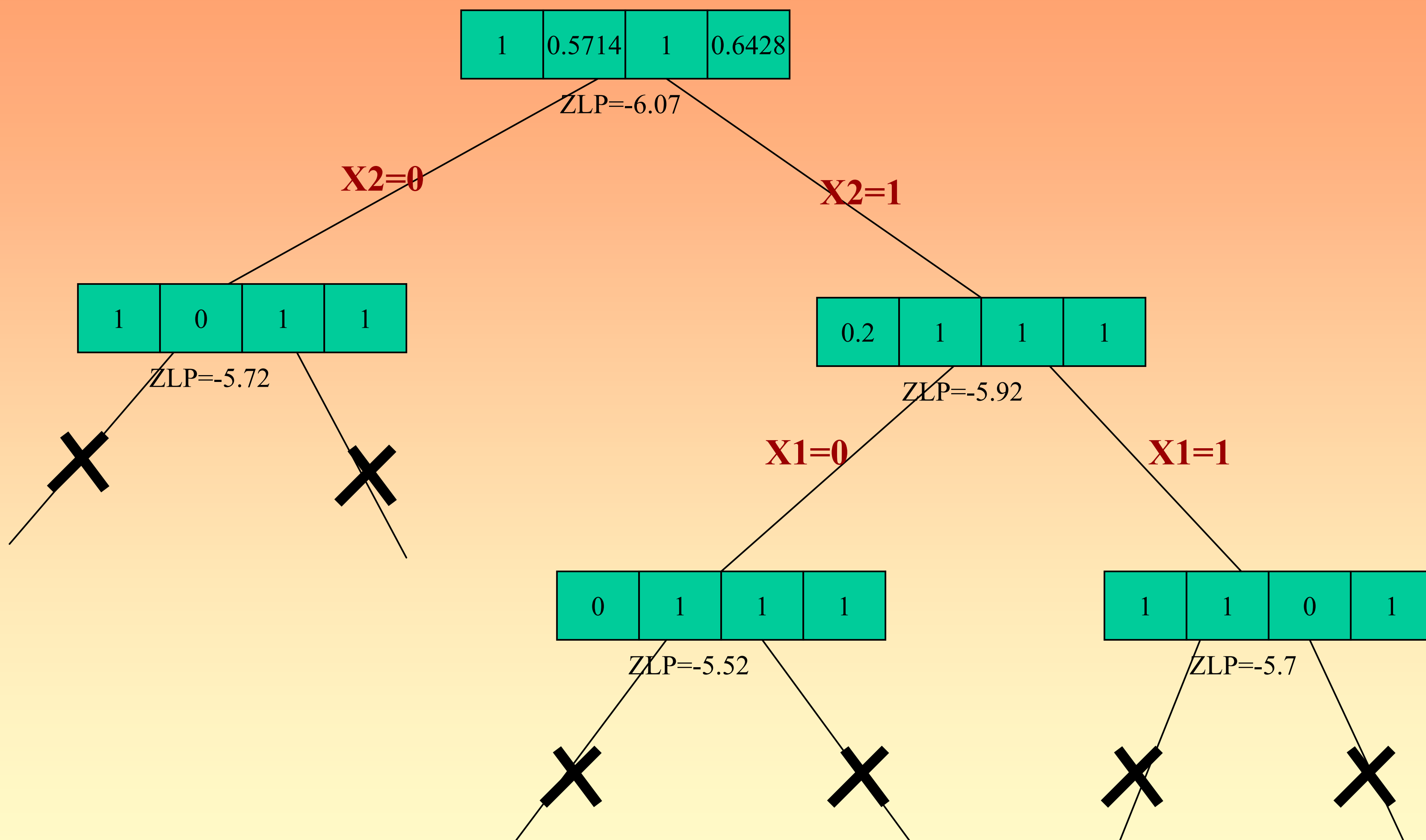
	A	B	C	D	
Profit	2	1.8	1.82	1.9	Availability
Nails	10	8	9	10	30
Screws	5	6	4	4	15
Glue	1.1	1.1	0.9	1	3.5

- Company XYZ wants to know which products it should manufacture.
- Let $X_P = 1$ if product P is manufactured, 0 otherwise

Solving the LP



Branch-and-Bound



RESULT: X_1=1; X_2=0; X_3=1; X_4=1 =>Obj. Value= -5.72