

## Quiz 2

12/14/2005

### Problem 1

(a) The longest path on the tree is 13 units. Therefore, for the 3-center problem,  $13/6$  is the smallest maximum distance we can hope for. This can indeed be achieved by placing three centers on the BC edge: one is a distance of  $7/6$  from B; the second is a distance of  $(7/6) + 2(13/6) = 33/6 = 5.5$  from B; and the third is a distance of  $(33/6) + 2(13/6) = 59/6$  from B (or  $1/6$  to the left of C). This is also the unique optimal solution to the 3-center problem.

(b) Now we also have to worry about the “geometry” of the network. Some thought will convince you that the minimum maximum distance we can hope for is 2 units. To achieve this we must definitely place one facility at C. The other three facilities must be placed to the left of the point which is 8 units away from B on BC (or 2 units away from C on BC). For example one possible solution would place one of the three other facilities at B, a second facility 4 units away from B on BC, and the third facility 7 units away from B on BC.

(c) We are now looking for the 1-absolute-median of the network, because all we want to do is minimize the expected travel time from the facility to a random demand.

The easy way to find this 1-median is by using the majority theorem, thus requiring that we must have as much length to the “left” of the facility as to its “right”. (Length is the same as weight in this case, since the demands are distributed uniformly on the edges of the network.) Then, by inspection, the optimal location is at the point which is 6 units away from B on BC. (Note that we then have 8 units of length to the left and 8 to the right.)

A more formal way is by assuming that the optimal location is at a point which is  $x$  units away from B on BC. We then have for the expected travel distance:

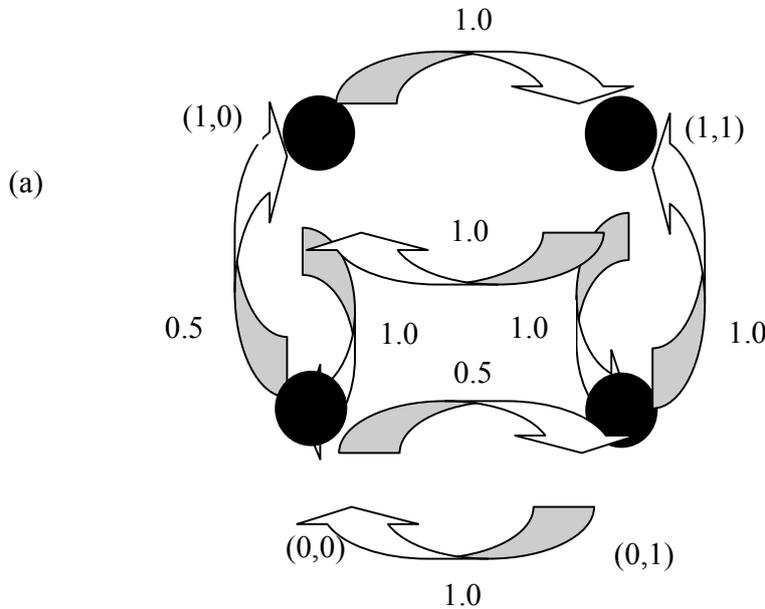
$$E[D] = \frac{x}{2} \cdot \frac{x}{16} + \frac{2}{16} \cdot \left(x + \frac{1}{2}\right) + \frac{10-x}{16} \cdot \frac{10-x}{2} + \frac{4}{16} \cdot (11-x) = \frac{x^2 - 12x + 95}{16} \quad (1)$$

and, setting  $\frac{d(E[D])}{dx} = 0$ , we obtain  $2x=12$ , or  $x=6$ , as above.

(d) As discussed in lecture (see distributed notes) the Stochastic Queue Median (SQM) coincides with the (Hakimi) median, when the demand rate reaches the highest level at which the average response time is still finite, i.e., at which the system is still stable. This means that the optimal location for  $\lambda_{max}$  is still at the point we found in part (c), namely the point which is 6 units away from B on BC.

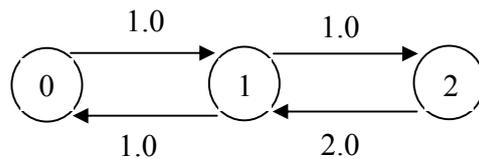
At that point, the expected (one-way) travel distance (and travel time) can be found by setting  $x=6$  in (1) and is equal to  $59/16$ . Then, the expected service time is equal to  $2(59/16)+1= 134/16$  and, to maintain stability,  $\lambda_{max}$  must be less than  $16/134= 0.1194$  demands per unit of time.

**Problem 2**



(b) By symmetry we note that each ambulance has the same performance statistics. Thus we can use the reduced state space M/M./2/0 model.

$$P_0 + P_0 + 0.5P_0 = 1.0 \quad P_0 = 0.4 = P_1 \quad P_2 = 0.2 \quad \rho_1 = \rho_2 = 0.5P_1 + P_2 = 0.4$$



(c) By symmetry, mean response distance to a random customer equals the mean response distance traveled by ambulance A to a random customer.

Mean A-to-A response distance is  $(2/3)\{(1/2)(2\pi/8)(1/4) + (1/2)(3*2\pi/8)(3/4)\} + (1/3)(1/2)(2\pi/4) = 5\pi/24 + \pi/12 = 7\pi/24$ .

Mean A-to-B response distance is  $(2/3)\{2\pi/8 + 2\pi/4\} + (1/3)\{2\pi/4 + 2\pi/8\} = 3\pi/4$

Utilizing Equations (5.16) in the text, we find that  $f_{11} = f_{22} = 3/8$  and  $f_{12} = f_{21} = 1/8$ .

Therefore 3/4 of ambulance A's customers are in primary response area **A** and 1/4 take it to primary response area **B**. Thus, the desired answer is

$$\text{Mean response distance} = (3/4) 7\pi/24 + (1/4) 3\pi/4 = 13\pi/32. (=1.276)$$

(d) The state-rate-transition diagram remains unchanged from part (a). The only possible changes are from state (00), involving the two upward transition rates. But we know that these rates must sum to one and that there is symmetry between the units. You may think of it this way: For every customer that ambulance A loses due to ambulance B being closer, ambulance B loses due to ambulance A being closer,

(e) Answers the same as in part (b) above, since we can still shrink the state space to the same M/M/2/0 model and invoke symmetry, or simply note that the 2-dimensional Hypercube models are identical with regard to workloads.

(f) We divide and conquer, very carefully. First, since  $P_{00} = P_{01} + P_{10} = 0.4$ , 50% of answered customers find the system in state (0,0) and 50% of the answered customers find the system with one server free and one busy. (Customers who arrive when both are busy are rejected and do not arise in the mean travel distance calculations.)

$$\begin{aligned} \text{So, Answer} &= (\text{Conditional Mean Travel Distance When both are Free}) * 0.5 \\ &\quad + (\text{Conditional Mean Travel Distance When one is Free}) * 0.5 \\ &= E[D|2 \text{ free}] * 0.5 + E[D|1 \text{ free}] * 0.5. \end{aligned}$$

First consider,  $E[D|1 \text{ free}]$ . We can immediately see that  $E[D|1 \text{ free}] = (1/4) * 2\pi$ . This is because a customer arriving when only one ambulance is free is uniformly distributed over the circle of circumference  $2\pi$ . The free ambulance will take the shorter of the two circumferential paths to the customer, yielding a mean travel distance of  $(1/2) * (1/2 \text{ of the circumference}) = (1/4) * 2\pi$ .

Now consider  $E[D|2 \text{ free}]$ . For this we must divide and conquer over all possible combinations of real time locations of ambulances A and B. In examining the circular total system, we see that there are three different configurations of interest:

1. Ambulances A and B aligned at same latitude: (A1, B1), (A2, B2), (A3, B3)  
Probability = 3/9
2. Ambulances A and B at 'diagonals': (A1, B3), (A3, B1)  
Probability = 2/9
3. 1 ambulance in center & the other not: (A1, B2), (A3, B2), (A2, B1), (A2, B3).  
Probability = 4/9.

For condition 1, each ambulance is closer to all points in its primary response area, so the real time vehicle locations information makes no difference. The conditional mean travel distance is the same as the A-to-A mean distance in part (c) and equals  $\pi$ .

For condition 2, each ambulance serves the half circle nearest it. Thus the corresponding conditional mean travel distance is  $\pi/4$ .

For condition 3, suppose we have (A1, B2). Then ambulance A covers the first  $\pi/8$  miles of the 'northern' part of response area B and ambulance B covers the first  $\pi/8$  miles of the southern part of response area A. The conditional mean travel distance is  $(3/8) \cdot (3\pi/8) \cdot 0.5 + (5/8) \cdot (5\pi/8) \cdot 0.5 = 19\pi/4$

Thus  $E[D] = (1/4)2\pi(1/2) + \{(3/9)(7\pi/24) + (2/9)(\pi/4) + (4/9)(19\pi/128)\}(1/2)$   
 Or,  $E[D] = \pi/4 + \{7\pi/32\}(1/2) = 23\pi/64 \approx 1.129$ , approximately a 12% reduction compared to part (c) with no real time vehicle location information.

### Problem 3

(a) We wish to minimize  $\sum_{j=1}^{16} (w_1 \cdot h_{1,j} + w_2 \cdot h_{2,j}) \cdot d(x, j) = \sum \frac{h_{1,j} + h_{2,j}}{2} \cdot d(x, j)$ .

Therefore, this problem is just a 1-median problem in which the node weight of each node is equal to one-half of the sum of the node weights in each period.

To find this one median, we take advantage of the majority theorem (since the network is a tree) and, essentially by inspection, we find that the optimal location is at node 2/

(b) For node 16 to be the 1-median, it must be (using the majority theorem and approaching node 16 from the left):

- (1)  $13w_1 + 4w_2 < 12.5$  (if this were not true, then node 5 would be the 1 median) and
- (2)  $14w_1 + 5w_2 \geq 12.5$

But it is also true that  $w_1 = 1 - w_2$  and, substituting into (1) and (2) we obtain the required range  $\frac{1}{18} < w_2 \leq \frac{1}{6}$  for the node 16 to be the 1-median. For example,  $w_1 = 0.9$ ,  $w_2 = 0.1$ , gives node 16 as the optimal location.