Problem 1
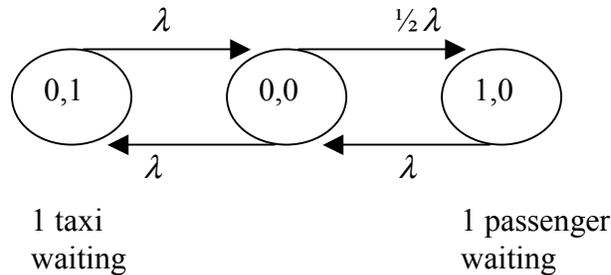
a)

$$\pi_{0,1} = \pi_{0,0}$$

$$\tfrac{1}{2}\pi_{0,0} = \pi_{1,0}$$

$$\pi_{0,1} + \pi_{0,0} + \pi_{1,0} = 1$$

$$\therefore \pi_{0,1} = \tfrac{2}{5}\,;\pi_{0,0} = \tfrac{2}{5}\,;\pi_{1,0} = \tfrac{1}{5}$$

b)

The chance that a passenger arriving at the taxi stand actually leaves it in a cab is:

$$\pi_{0,1} + \tfrac{1}{2}\pi_{0,0} = \tfrac{3}{5}$$

c)

The chance that a taxi stops is:

$$\pi_{0,0} + \pi_{1,0} = \tfrac{3}{5}$$

This is the same as b) because $\lambda_T = \lambda_P$ which implies that taxis and customers reach the stand at the same rate. Because, over the long run, the number of cabs that leave with customers is the same as the number of customers who leave in cabs, the FRACTION of arrivals that leave happily must be the same for taxis and customers.

d)

To find the average waiting time for passengers who leave by taxi, determine the expected waiting time for passengers and divide by the chance that an arriving passenger leaves by taxi:

$$\frac{(0)\pi_{0,1} + \left(\frac{1}{\lambda}\right)\pi_{1,0}}{\pi_{0,1} + \pi_{1,0}} = \frac{\left(\frac{1}{\lambda}\right)\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3\lambda}$$

e)

For two passengers in a row to leave by taxi and have zero waits, we must be in state (0,1) when the first passenger arrives. Then, another taxi must arrive before the next passenger, in which case the taxi will wait for the passenger and the passenger's wait will be zero. The chance of the first of these events is 2/5; the conditional probability of the second given the first is 1/2 (because lambda is the same for taxis and passengers) :

$$\pi_{0,1}\left(\frac{\lambda}{\lambda+\lambda}\right) = \frac{2}{5}\left(\frac{1}{2}\right) = \frac{1}{5}$$
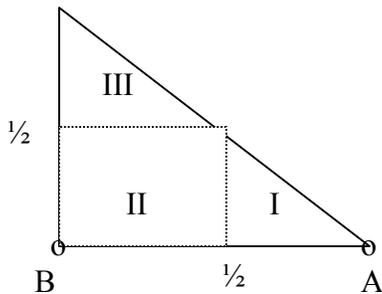
Problem 2:

a)
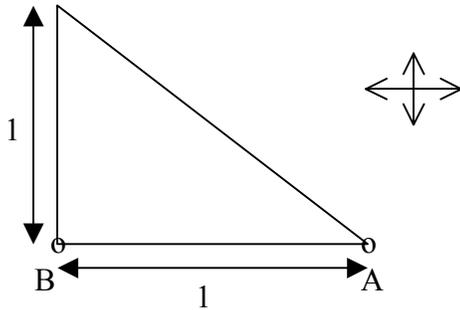
Note: Within triangle $f_{X,Y}(x, y) = 2$

For one facility at A:

$$E[D] = E[D_x] + E[D_y] = \frac{2}{3} + \frac{1}{3} = 1$$

b)

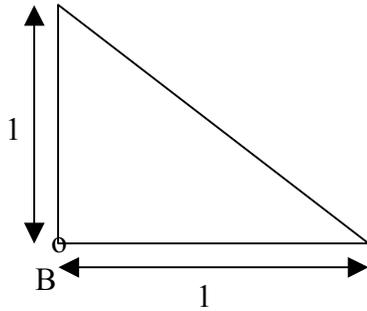Demands in I use facility A, demands in II and III use B.
We have:

$$E[D] = P[I] \cdot E[D|I] + P[II] \cdot E[D|II] + P[III] \cdot E[D|III]$$

$$E[D] = \frac{1}{4} \cdot \left(\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \right) + \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2} \right) + \frac{1}{4} \cdot \left(\frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \right)$$

$$E[D] = \frac{1}{8} + \frac{1}{4} + \frac{5}{24} = \frac{7}{12}$$

c)

Note: D_x (the distance from B to demand along x axis) is not independent of D_y (the distance along y axis), therefore, we cannot simply add $\sigma_{D_x}^2$ and $\sigma_{D_y}^2$ to find the total variance. Instead, we must derive σ_D^2 from the first principle.
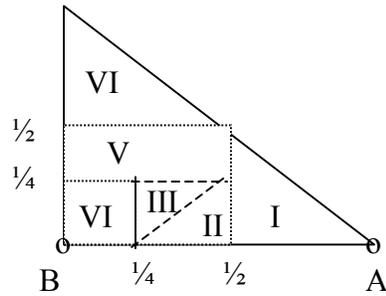
$$E[D] = E[D_x] + E[D_y] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$E[D^2] = \int\limits_0^1 \int\limits_0^{1-x} (x+y)^2 f_{X,Y}(x,y)dxdy = \frac{1}{2}$$

and

$$\sigma_D^2 = \frac{9}{18} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

d)

Demands in I and II use A, demands in III, IV, V, and VI use B.
<u>Note:</u> only demands in II and III are affected by the presence of the barrier.

Region	**Probability**	**E[D] (to closest facility)**
I	¼	$\frac{2}{3}\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{2}\right)=\frac{1}{2}$
II	¹⁄₁₆	$\frac{1}{2}+\frac{1}{3}\left(\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}\right)=\frac{2}{3}$
III	¹⁄₁₆	$\frac{1}{2}+\frac{1}{3}\left(\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}\right)=\frac{2}{3}$
IV	⅛	$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$
V	¼	$\frac{1}{4}+\frac{1}{4}+\frac{1}{8}=\frac{51}{8}$
VI	¼	$\frac{1}{2}+\frac{1}{3}\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{2}\right)=\frac{5}{6}$

Multiplying out: $E[D]=\dfrac{29}{48}$

Alternatively, we could have estimated that the extra distance caused by the barrier is equal to 1/48 and, therefore, $E[D]=\dfrac{7}{12}+\dfrac{1}{48}=\dfrac{29}{48}$