

Problem Set 4

(Due: Wednesday, November 8, 2006)

Problem 1 [This is a review problem on M/G/1 queues. We have already done the first four parts in class, as an example.]

Consider a single-server queueing system with infinite queue capacity. Arrivals of customers to this system occur in a Poisson manner at the rate of 36 per hour. The service times, S , of customers are mutually independent and their duration is uniformly distributed between 1 and 2 minutes, in other words, $f_S(t) = U[1, 2]$ in units of minutes.

- (1) Compute the expected waiting time in queue, W_q , and the expected number of customers in the queue, L_q , when this queueing system is in steady state.
[Numerical answers are expected.]
- (2) Assume now that service times have a negative exponential probability density function with $f_S(t) = \frac{2}{3} \cdot e^{-\frac{2}{3}t}$ for $t \geq 0$, in units of minutes. Repeat part (1), i.e., compute numerically the values of W_q and of L_q for this case.
- (3) Assume now that service times are constant and all have duration equal to 1.5 minutes. Repeat part (1), i.e., compute numerically the values of W_q and of L_q for this case.
- (4) Using the G/G/1 bounds discussed in class, compute numerically upper and lower bounds for W_q and for L_q for the case of Part (1).
- (5) Return to the original case (uniform service times) from here until the end of the problem. Suppose that immediately after the completion of a service, exactly 2 customers are left in the system. Carefully write an expression for the probability that after the end of the next service, exactly 4 customers will be left in the system. Do NOT evaluate numerically this expression.
- (6) Repeat part (4) under the supposition that after the completion of a service, exactly 0 customers are left in the system. Carefully write an expression for the probability that after the end of the next service, exactly 4 customers will be left in the system. Do NOT evaluate numerically this expression.
- (7) Consider again part (5) where after the completion of a service, exactly 2 customers are left in the system. Carefully write an expression for the probability that TWO service completions later, exactly 4 customers will be left in the system. Do NOT evaluate numerically this expression.

- (8) Suppose that George begins observing the queueing system at a random time (assume the system is in steady state). [George is not a customer – he is just an observer.] What is the expected time until George observes the first completion of a service? A numerical answer is expected in this case.
- (9) In this part assume that the system's customers are of two types: Type 1, who constitute 40% of the customers, and Type 2, who constitute 60%. Assume that Type 1 customers enjoy *non-preemptive priority* over the ones of Type 2. Assume also that the arrivals of Type 1 and Type 2 customers are randomly intermingled. Finally note that the service times of both types of customers are uniformly distributed between 1 and 2 minutes. Compute numerically the expected waiting time, W_q , for each type of customer and for *all* the customers using the system.
- (10) Assume we now separate the customers of the system into two classes, A and B. Class A customers constitute 40% of the total and have $f_{SA}(t) = U[1, 1.4]$. Class B customers constitute 60% of the total and have $f_{SB}(t) = U[1.4, 2]$. Suppose Class A customers have non-preemptive priority over Class B customers. Compute numerically the value of W_q for each class of customer. Compute, as well the value of W_q for *all* the customers using the system and compare with the answer to Parts (1) and (9). Why are the answers different or the same?
- (11) Repeat Part (10) but with Class B having non-preemptive priority over Class A.

Problem 2

Problem 4.13 in L + O, with the following important change: assume the queue capacity in front of Q.S. 3 is 0 (zero), not 1. This will significantly reduce the number of states you need to worry about. To do this problem, try first to identify all the possible states and then draw the state transition diagram. You will need some extra paper because the first time you'll try to draw the diagram, a rather messy picture will emerge.

Problem 3

Airline DeeDee has only three airplanes. (There are many small airlines of this size around the world.) DeeDee has signed a contract with the ABC Corporation under which ABC will provide unscheduled, non-routine repair services to DeeDee when aircraft suffer mechanical breakdowns. The service times to aircraft at ABC's repair facility have a negative exponential pdf with a mean value of 12 hours. However, it has been observed that, with probability 0.6, an aircraft will need a second, supplementary repair operation after the first repair is completed. (With probability 0.4, the first repair is satisfactory and the aircraft goes back to flying.) The service times for the second repair, if needed, also have a negative exponential pdf with a mean of 12 hours. After the second repair

operation, the aircraft is always in a “satisfactory” state for flying. The repair facility services only one aircraft at a time and will perform all repair operations on an aircraft (one or two), before processing the next aircraft on a first come, first served basis. Finally, it is known that the time between the instant when repair service to an aircraft is completed and the instant when that aircraft will break down again for the first time has a negative exponential pdf with a mean of 3 days. For the sake of simplicity, assume, as well, that both DeeDee and ABC operate continuously, on a 24-hours a day basis.

- (a) Draw a state-transition diagram for this queueing system, making sure to define carefully its states.
- (b) What is the expected number of DeeDee aircraft in ABC’s repair facility in steady state? This includes aircraft being repaired and aircraft waiting for repairs.

Problem 4

Consider a queueing system with two parallel servers and two spaces for queueing (in addition to the two servers).

This facility serves two types of customers. Type 1 customers are of the conventional type. They arrive in a Poisson manner at a rate of λ_1 per minute. The service time to these customers has a negative exponential pdf with a rate of μ_1 per minute for each server. Any arriving Type 1 customers who find the system full (i.e., 4 customers in the system) are lost.

Type 2 customers are unusual. They, too, arrive in a Poisson manner at a rate of λ_2 per minute, but they arrive IN PAIRS. [Think of a restaurant, where some of the customers (Type 1) arrive individually, and others (Type 2) arrive in pairs.] Moreover, when service to each one of these pairs begins, the pair occupies simultaneously TWO servers (i.e., both of the servers). The servers work together on each of these Type 2 pairs. The service time to the pair has a negative exponential pdf with a rate of μ_2 per minute. (Note that this means that the two servers begin and end service to each Type 2 pair simultaneously and, together, can serve μ_2 Type 2 pairs per minute if working continuously on Type 2 pairs.) Type 2 customers who do not find at least TWO available spaces upon arrival are lost.

[Please note: A Type 2 pair cannot occupy the servers, unless both servers are available. Thus, in the case where one Type 1 customer is in service and the only customers waiting are one Type 2 pair, the Type 2 pair must still wait in queue and the second server remains idle.]

- (a) Please draw carefully a state transition diagram that describes this queueing system. Please make sure to define clearly the states of the system.
- (b) Suppose there are currently four Type 1 customers in the system. Write an expression for the probability that *exactly* 3 state transitions later, there will be exactly two Type 1 customers *and* one Type 2 pair in the system. Your

expression should be solely in terms of λ_1 , μ_1 , λ_2 and μ_2 . [A state transition takes place whenever there is a new actual arrival (not a lost arrival) to the system or whenever there is a service completion. The arrival of a new Type 2 pair counts as *one* transition and so does the completion of a service to a Type 2 pair.]

Problem 5

We have solved in class Example 3 of pages 246 and 247 in Larson and Odoni, assuming distinct service rates μ_1 and μ_2 at the two queueing facilities. We shall now look at the case in which the first queueing system (the one on the left) has two independent and identical servers, each with service rate equal to μ_1 . There is still no waiting space in front of either of the two facilities.

- (a) Draw carefully the state-transition diagram (i.e., analogous to Figure 4.20) for this case. Define clearly the states of the system. Do NOT do any further calculations.
- (b) Assume that you have computed the steady-state probabilities associated with the states in the state-transition diagram you drew in part (a). Write an expression for L , the expected number of customers in this “network” (i.e., this two-queueing-facility system) in terms of the steady-state probabilities.