Urban OR **Fall 2006**

Problem Set 3

(Due: **Thursday**, October 19, 2006)

[*Please note:* Despite including seven problems, this assignment is not a long one. Most of the problems, with the possible exception of Problem 3, do not require much in terms of mathematical manipulation. They mostly emphasize modeling.]

Problem 1

Alvin is walking toward a M/M/1 queueing system that operates in a big building according to a FIFO discipline. The customer arrival rate at that system is 0.4 per minute, and the mean service time is 1 minute per customer. Bo is walking ahead of Alvin on the way to the building and reaches it 2 minutes before Alvin does.

At the moment he enters the building, Alvin knows that nobody arrived there after Bo and before Alvin. Alvin, however, has no idea how many customers were present at the system when Bo arrived. Nor does he know anything about the customer departure process: when services end, customers leave the building from a different door that Alvin cannot see.

Given this information:

(a) What is the probability that Bo did not have to wait at all for the service to start, but Alvin did have to wait?

(b) What is the probability that Bo arrived to find 8 other customers present, but that, upon entering the building, Alvin found only one customer in the system, i.e., Bo?

Problem 2

Problem 4.1 in the text. [Hint: The key here is to model the situation correctly, i.e., to identify correctly the states of this system. Note, for example, that we cannot have a positive number of passengers *and* of taxis at the taxi station simultaneously.]

Problem 3

Parts (a) and (c) of Problem 4.2 in the textbook. [Hint: Part (a) is straightforward. In part (c), after you compute *L*, make sure to use Little's Law correctly to compute *W*: note that the arrival rate is a function of the state of the system and thus you need to compute the mean arrival rate $\bar{\lambda}$ by taking the expectation over all possible states of the system.]

Problem 4

Problem 4.3 in the textbook

Problem 5

The ABC Automobile Rental Company must choose between operating one of two types of maintenance shops for its cars. It estimates that cars will arrive at the maintenance facility in a Poisson manner at the rate of one every 40 minutes. In the first type of shop, there is a single server, which can service a car in 15 minutes on average, with a negative exponential service time. In the second type, there are two identical servers operating in parallel; each can service a car in 30 minutes on average with a service time described again by a negative exponential pdf. Assume that each minute a car must spend in the shop costs ABC 1 monetary unit. Let C1 and C2 be the cost per minute of operating the first and second type facilities, respectively. Find the value of the difference between C1 and C2, such that ABC would be indifferent between operating the two types of facilities. Which of C1 and C2 is the larger at this critical value?

Problem 6

Consider a M/M/2 queueing facility with no queue allowed (i.e., arrivals when both servers are busy do not receive service and are lost). Arrivals occur at the rate of λ per hour while each server provides service at the rate of μ per hour. Let $\rho = (\lambda/\mu)$ and assume that each server is operated by one man.

(a) With the system in steady state, what is the expected number of men who are busy serving a customer at any given time? Your answer should be solely in terms of ρ.

Each man working at the facility records the number of hours required to service each customer and the number of customers he serves. The following data were collected over a long period of time: During 10,000 hours of the facility's operation, 40,000 customers received service and 8,000 man hours of service were recorded. (This last number is the total number of hours when the two men were busy, not necessarily simultaneously, serving customers.) Assume the sample is large enough to provide reliable statistical information.

(b) Estimate the number of customers lost during these 10,000 hours.

Problem 7

Consider a single server queueing system whose customers arrive in a Poisson manner at a rate of λ customers per minute. There is infinite queue capacity and customers are served in a FCFS manner. This system is similar to our familiar M/M/1 system, but with one important difference:

The FIRST customer to arrive in a busy period (i.e., every customer who arrives to find the system in the idle state) has a service time, which is Erlang of order 2, with mean service time equal to $2/\mu$ minutes. In other words, the server gets the busy period off to a *"slow start," requiring an average of twice the time to serve the customer who starts the busy period. All service times are independent.*

Note that the service times of all the other customers (i.e., of those who are not the first to arrive after a busy period) have the more usual negative exponential probability density function with mean $1/\mu$. Note, as well, that for the first customer in a busy period, one may think of the service being carried out sequentially in two phases: each phase has a duration drawn from a negative exponential probability density function with mean $1/\mu$ and the two phases are independent.

(a) If the queueing system had zero queue capacity, meaning that all approaching customers who arrived when the server was busy would be "lost" or "turned away," then every customer who is served would initiate a busy period and have Erlang order 2 service time density. Argue carefully and briefly that in steady state one can solve this system using a state-transition diagram having three states, one for server idle and two for server busy. Draw the 3-state transition diagram and define the states of the system. (Hint: Think in terms of remaining "phases" of work.) Use this model to find an expression for the server utilization in this case, i.e., for the fraction of time the server is busy (= the probability that the server is busy at any random time).

For the remainder of this problem, we assume the more usual infinite capacity, first-come, first-served system, but with the important difference from M/M/1 noted earlier.

(b) Generalize the idea of part (a) to carefully draw and label the state-transition diagram of the infinite capacity queue system that we have described. Note that with the exception of the first three states, your diagram will look very similar to that of the M/M/1 model, BUT the state of the system is NOT the number of customers in the system, but rather the number of service phases (each with mean equal to $1/\mu$) in the system.

(c) What condition must be satisfied for this system to reach steady state? [Hint: The condition is a very simple one.) Derive an expression for the server utilization in this case (assuming steady state exists).

Assume steady state conditions from here on.

(d) What fraction of customers experience the "Erlang order 2" service time?

(e) What is the expected service time for a random customer utilizing this system?

[FOR EXTRA CREDIT]. Find L, L_q, W and W_q. You will find it easiest to begin with W_q.