Problem Set #2

Issued: September 25, 2006
Due: October 4, 2006

Problem 1

(i). In this problem, we should consider random incidence.

There are three different interval lengths: 4, 5 or 6 minutes.

Let A_i be the event that he arrives in interval of length i, then $P(A_i) = \frac{i}{4+5+6} = \frac{i}{15}$.

Let T be the number of minutes he waits.

We have: $P(T \in [4,5] \mid A_i) = \begin{cases} 0 & \text{for } i = 4; \\ \frac{5-4}{i} = \frac{1}{i} & \text{for } i = 5 \text{ or } 6. \end{cases}$

Therefore:

$$P(T \in [4,5]) = \sum_{i=4}^{6} P(T \in [4,5] \mid A_i) \cdot P(A_i)$$

$$P(T \in [4,5]) = 0 * \frac{1}{4} + \frac{1}{5} * \frac{5}{15} + \frac{1}{6} * \frac{6}{15}$$

$$P(T \in [4,5]) = \frac{2}{15}$$

ii). If the intervals between trains were exactly 5 minutes, the probability for Mendel to arrive in an interval of length 5 would be 1, but his probability to wait between 4 and 5 minutes would remain the same, i.e. $\frac{1}{5}$.

Thus, $P(T \in [4,5]) = \frac{1}{5}$ (using the same notations as in the previous question). The probability increases.

From the previous question, we have the result:
$P(T \in [4,5] \mid A_5) \succ P(T \in [4,5] \mid A_6) \succ P(T \in [4,5] \mid A_4)$. Therefore, if the probability of arriving in an interval of length 6 or 4 decreases in favor of the probability of arriving in an interval of 5, the chances of waiting between 4 and 5 minutes increases. However, the probability of waiting more than 5 minutes is now zero.

Problem 2

a)

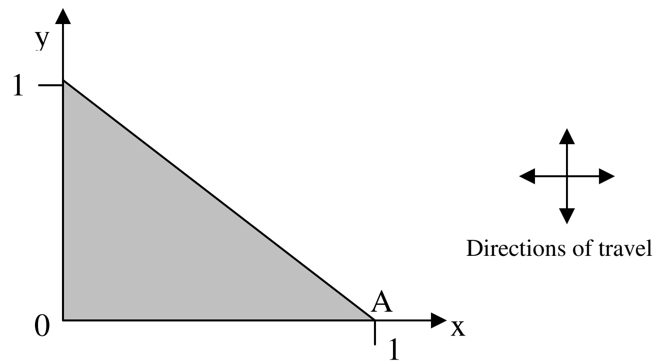
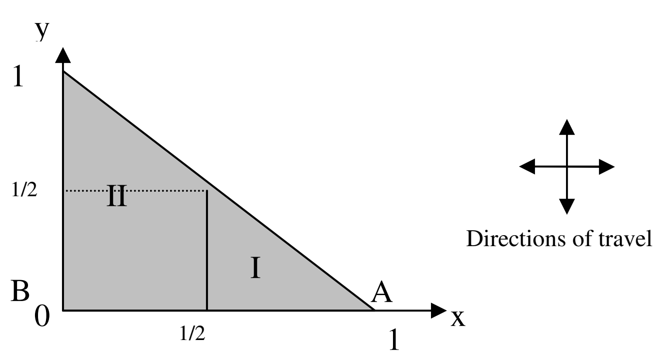


Fig. 1: Urban Area

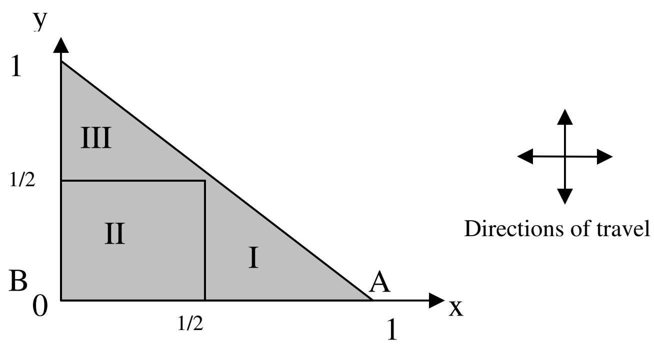
The area of the triangular region is $\frac{1}{2}$. Since the demand is uniformly distributed of the region, the joint pdf is $f_{X,Y}(x, y) = 2$.

Let D be the travel distance from a point in the region to the facility at A, and D_x and D_y the distance from A to demand along x axis and y axis: $E[D] = E[D_x] + E[D_y] = \frac{2}{3} + \frac{1}{3} = 1$

b) There are two different areas. If the demand is in area I, it travels to A. If the demand is in area II, it uses facility at B.



However, it is easier to divide the region in three area in order to compute the expected travel distance to the two facilities.



Thus:

$$E[D] = P(I) \cdot E[D|I] + P(II) \cdot E[D|II] + P(III) \cdot E[D \mid III]$$

Area I only represents one fourth of the total region, therefore, the probability that the demand is in A is:

$P(I) = \frac{1}{4}$. And we have $P(II) = \frac{1}{2}$ and $P(III) = P(I)$.

$$E[D] = \frac{1}{4} \cdot \left(\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \right) + \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2} \right) + \frac{1}{4} \cdot \left(\frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \right)$$

$$E[D] = \frac{1}{8} + \frac{1}{4} + \frac{5}{24} = \frac{7}{12}$$

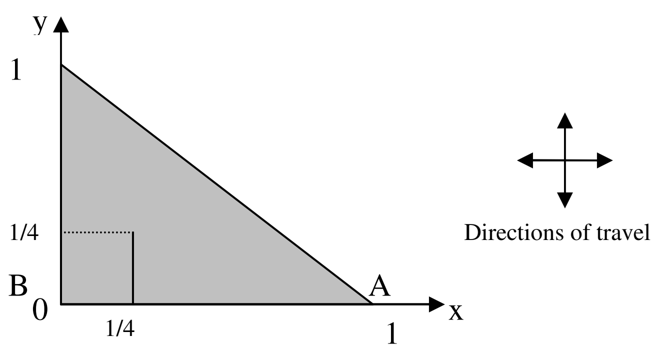
c) Please not that D_x is <u>not</u> independent of D_y, therefore, we cannot simply add $\sigma^2_{D_x}$ and $\sigma^2_{D_y}$ to find the total variance. Instead, we must derive σ^2_D from the definition.

$$E[D] = E[D_x] + E[D_y] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

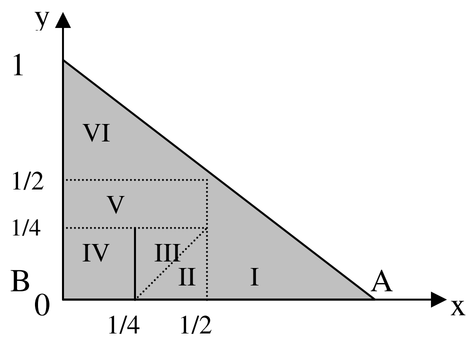
$$E[D^2] = \int_0^1 \int_0^{1-x} (x+y)^2 f_{X,Y}(x,y) dx dy = \frac{1}{2}$$

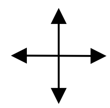
and $\sigma^2_D = \frac{9}{18} - \left(\frac{2}{3} \right)^2 = \frac{1}{18}$

d)



The region should be divided into 6 areas, as shown below:





Demands in I and II use A, demands in III, IV, V, and VI use B.
Note: only demands in II and III are affected by the presence of the barrier.

For each region, let's compute the probability for a demand to originate from that region, and the expected distance to the closest facility. Here is a summary of the results:

Region	**Probability**	**E[D] (to closest facility)**
I	¼	$\frac{2}{3}\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{2}\right)=\frac{1}{2}$
II	$\frac{1}{16}$	$\frac{1}{2}+\frac{1}{3}\left(\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}\right)=\frac{2}{3}$
III	$\frac{1}{16}$	$\frac{1}{2}+\frac{1}{3}\left(\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}\right)=\frac{2}{3}$
IV	$\frac{1}{8}$	$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$
V	¼	$\frac{1}{4}+\frac{1}{4}+\frac{1}{8}=\frac{51}{8}$
VI	¼	$\frac{1}{2}+\frac{1}{3}\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{2}\right)=\frac{5}{6}$

Multiplying out: $E[D]=\frac{29}{48}$

Note: we could have estimated that the extra distance caused by the barrier is equal to 1/48 and, therefore, $E[D]=\frac{7}{12}+\frac{1}{48}=\frac{29}{48}$

Problem 3

a). The random variable X has a binomial distribution: $F_X(x)=\begin{cases}\binom{5}{x}\cdot p^x\cdot(1-p)^{5-x} & for\ x=0,1,2,\cdots 5\\ 0 & otherwise\end{cases}$

where p is the probability for one helicopter to be within one mile of Mendel's house.

Since the position of any helicopter is uniformly distributed over the city, we have: $p=\frac{\pi r^2}{5\times 2}=\frac{\pi}{10}$.

b). The probability for a helicopter to be within half mile of Mendel's house is: $p = \frac{\pi r^2}{5\times 2} = \frac{\pi}{40}$.

And the third nearest helicopter to Mendel is within one-half mile there are 3 or more helicopters within half a mile of Mendel's house. Thus, according to the previous question:

$$P(3^{rd}\,nearest\ within\ \frac{1}{2}\ mile) = \sum_{x=3}^{5}\binom{5}{x}\cdot\left(\frac{\pi}{40}\right)^{x}\cdot\left(1-\frac{\pi}{40}\right)^{5-x}.$$

c). Let's find the probability that no helicopter is within one mile of Mendel's house. According to a), we have: $P(0\ within\ 1\ mile) = \binom{5}{0}\cdot\left(\frac{\pi}{10}\right)^{0}\left(1-\frac{\pi}{10}\right)^{5} = \left(1-\frac{\pi}{10}\right)^{5}$.

Therefore, the probability that Mendel's house is covered at a random time is:

$$P = 1-\left(1-\frac{\pi}{10}\right)^{5}.$$

d). The coverage probability is the lowest at the fours corners. The probability for one helicopter to be within one mile of a corner is only $p = \frac{\frac{1}{4}\pi}{10}$ because the helicopters do not cover the area that is not in the city.

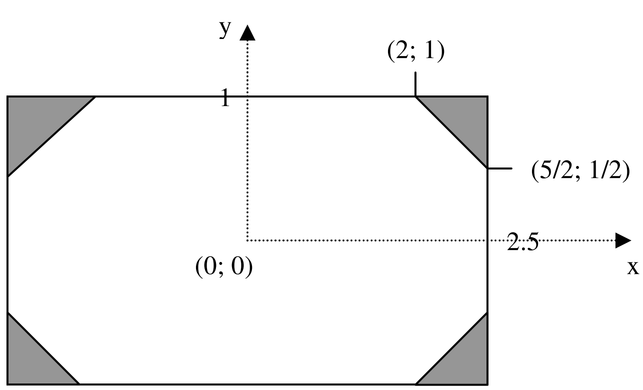
e). One point with the highest coverage probability is Mendel's house. From the previous questions c) and d), the ratio is: $R = \frac{1-\left(1-\frac{\pi}{10}\right)^{5}}{1-\left(1-\frac{\pi/4}{10}\right)^{5}}$.

f). The probability to be covered by a land-based vehicle is: $p = \frac{area\ of\ diamand}{total\ area} = \frac{4\cdot \frac{1}{2}\cdot 1\cdot 1}{10} = \frac{1}{5}$.

The reasoning is the same as in part a), therefore, the probability distribution of z is:

$$F_Z(z) = \begin{cases}\binom{4}{z}\cdot\frac{1}{5^z}\cdot\left(\frac{4}{5}\right)^{4-z} & for\ z = 0, 1, 2, 3, 4\\ 0 & otherwise\end{cases}$$

g). Let's draw the situation:



The shaded areas represent the points of the city that are more than 3 Manhattan miles away from Mendel's house. Thus, we are interested in the total area A of the shaded regions: $A_{shaded} = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$.

We know the total area. Thus, the probability that are no land-based vehicles within 3 Manhattan miles of Mendel's house is:

$$P = \left(\frac{A_{shaded}}{A_{total}}\right)^4 = \frac{1}{20^4}.$$

Problem 4

Let A be the event that $v_4 \succ 4v_1$, and B the event that $v_4 \succ 2v_2$. We are looking for $P(A \cap B) = P(B \mid A) \cdot P(A)$.

Without loss of generality, we can assume that v_4 is 1 mile away from the given point.
There are therefore 3 other vehicles within a circle of 1 mile from the given point.
$P(A)$ is the probability that there is at least one vehicle within a circle of radius $\frac{1}{4}$. Let's find $P(\overline{A})$. The probability for a vehicle of being outside a circle of radius $\frac{1}{4}$ is $\frac{\pi \cdot 1^2 - \pi(1/4)^2}{\pi \cdot 1^2} = \frac{15}{16}$. Thus

$$P(A) = 1 - \left(\frac{15}{16}\right)^3 = \frac{721}{4096}.$$

$P(B \mid A)$ is the probability that the second vehicle is in circle of radius $\frac{1}{2}$ given that the first one is in a radius of $\frac{1}{4}$. Let's fin $P(\overline{B} \mid A)$d $P(\overline{B} \mid A)$ again.

We will use Bayes' Theorem: $P(\overline{B} \mid A) = \frac{P(\overline{B} \cap A)}{P(A)}$.

$P(\overline{B} \cap A)$ is the probability that the two vehicles are outside of circle with radius $\frac{1}{2}$ but one vehicle is inside the circle of radius $\frac{1}{4}$.

$$P(\overline{B} \cap A) = \frac{3 \cdot \frac{1}{16} \cdot \left(\frac{3}{4}\right)^2}{\frac{721}{4096}} = \frac{432}{721}$$

So $P(B \mid A) = \frac{289}{721}$

And finally, $P(A \cap B) = \frac{721}{4096} \cdot \frac{289}{721} = \frac{289}{4096} \approx 0.071$